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# A Solution of Volterra-Hamerstain Integral Equation in Partially Ordered Sets

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**Abstract**

This paper is aimed at proving the existence of a solution to the Volterra-Hamerstain integral equations in partially-ordered spaces. To this end, some fixed point theorems are proved in this kind of spaces.

*Keywords* : Partially ordered set; Fixed point; Contractive mapping.

## 1 Introduction

Mathematical modeling has been used more and more in many areas such as in science, engineering, medicine, economics and social sciences. For example, electrical engineering deals with the manipulation of electrons and photons to produce products that benefit humanity. The design of these products is based on scientific principles and theories that are best described mathematically. Mathematics is thus the universal language of electrical engineering science.

In mathematics, an integral equation is an equation in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations, and some problems may be formulated either way. See, for example, Maxwell's equations. The most basic type of integral equation is a Fredholm equation of the first type. In mathematics, the Volterra integral equations are a special type of integral equations. The Volterra integral equations were introduced by Vito Volterra and then studied by Traian Lalescu in 1908. In 1911, Lalescu wrote the first book ever on integral equations. Volterra integral equations have developed application in demography, the study of viscoelastic materials, and in insurance mathematics through the renewal equation.

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In this paper we consider the Volterra-Hamerstain integral equation as follows:

$$x(t) = f(t) + \int_0^t k(t,s)g(s,x(s))ds.$$

Here, the existence of a solution to this equation is proved in the partially ordered spaces. In order to do this, we recall two definitions from [3, 4] as follows:

**Definition 1.1.** *A partial ordering on a nonempty set $X$ is a relation $R$ on $X$ with the following properties:*

(I) $xRx$ *for all* $x \in X$,

(II) *if* $xRy$ *and* $yRx$ *then* $x = y$,

(III) *if* $xRy$ *and* $yRz$ *then* $xRz$.

**Definition 1.2.** *Let $(X, \leq)$ be a partially ordered set and $f : X \to X$. $f$ is called monotone nondecreasing if*

$$x \leq y \Rightarrow f(x) \leq f(y),$$

*where $x, y \in X$.*

## 2 Existence of a solution to the Volterra-Hamerstain integral equation

In this section, a solution to Volterra-Hamerstain integral equations is presented.

**Theorem 2.1.** *Let $I$ be a closed interval in $\mathbb{R}$. Consider the following integral equation*

$$x(t) = \int_0^t k(t,s)g(s,x(s))ds + f(t), \tag{2.1}$$

*For $t \in I$ assume $g : I \times \mathbb{R} \to \mathbb{R}$ is increasing with respect to the second component and $f : I \to \mathbb{R}$, $k : I \times \mathbb{R} \to \mathbb{R}^+$ such that*

$$\sup_{t \in I} \int_0^t k(t,s)ds \leq 1$$

*Suppose there exists an increasing function $\varphi$ such that*

$(I)$ $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$,

$(II)$ $\varphi(t) < t$ *for all* $t > 0$,

$(III)$ $\varphi$ *is an upper semicontinuous function.*

*If $g$ satisfies*

$$|g(s,x) - g(s,y)| \leq \varphi(d(x,y))$$

*for every $s \in I$ and $x \leq y$ and there exists $x_0$ such that*

$$x_0(t) \leq \int_0^t k(t,s)g(s,x_0(s))ds + f(t).$$

*Then the integral equation (2.1) has a solution in $C(I,R)$.*

**Proof:** Let $X = C(I, R)$ and define $d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}$ for every $x, y \in X$. We define following order relation in $X$,

$$x \le y \text{ if and only if } x(t) \le y(t) \text{ for all } t \in I$$

where $x, y \in X$. It is obvious that $(X, \le)$ is a partially ordered set and it is easily seen that $(X, d)$ is a complete metric space. Define $T : X \to X$ by

$$Tx(t) := \int_0^t k(t,s)g(s,x(s))ds + f(t)$$

Since the mapping $g$ is increasing with respect to the second component, then for $x \le y$ we have

$$Tx(t) = \int_0^t k(t,s)g(s,x(s))ds + f(t) \le \int_0^t k(t,s)g(s,y(s))ds + f(t) = Ty(t)$$

Thus $T$ is nondecreasing. In addition, for $x \le y$

$$\begin{aligned} d(T(x), T(y)) = &\ \sup_{t\in I}|[Tx](t) - [Ty](t)| \\ \le &\ \sup_{t\in I} \textstyle\int_0^t k(t,s)|g(s,x(s)) - g(s,y(s))|ds \\ \le &\ \sup_{t\in I} \textstyle\int_0^t k(t,s)\varphi(|x(s) - y(s)|)ds \\ \le &\ \varphi(\sup|x-y|)\sup_{t\in I} \textstyle\int_0^t k(t,s)ds \\ = &\ \varphi(d(x,y)). \end{aligned}$$

Hence $d(T(x), T(y)) \le \varphi(d(x, y))$, for every $x, y \in X$.
Now, we show that $T$ has a fixed point and thus the integral equation has a solution in $X$. In order to prove this, Note that, if $x_0 = T(x_0)$ then the proof is finished. If $x_0 \ne T(x_0)$ then $x_0 \le T(x_0)$ and since $T$ is nondecreasing we have

$$x_0 \le T(x_0) \le T^2(x_0) \le T^3(x_0) \le \cdots \le T^n(x_0) \le T^{n+1}(x_0) \le \cdots .$$

Using $\varphi(t) < t$,

$$\begin{aligned} d(T^n(x_0, T^{n+1}(x_0)) \le &\ \varphi(d(T^{n-1}(x_0), T^n(x_0))) \\ < &\ d(T^{n-1}(x_0), T^n(x_0)). \end{aligned}$$

Thus $\{d(T^n(x_0), T^{n+1}(x_0))\}$ is a nonnegative decreasing sequence and then it converges to $a$ as $n \to \infty$. We show $a = 0$. If not, since $\varphi$ is upper semicontinuous, we have

$$\begin{aligned} a = &\ \lim d(T^n(x_0), T^{n+1}(x_0)) \\ \le &\ \varphi(d(T^n(x_0), T^{n+1}(x_0))) \\ \le &\ \varphi(a). \end{aligned}$$

This means $a \le \varphi(a)$, which is a contradiction. Now, we prove $\{T^n(x_0)\}$ is a Cauchy sequence. Suppose not, there exists $\epsilon > 0$ such that for all $r \in \mathbb{N}$, there exists $m_r > n_r \ge r$ such that

$$d(T^{m_r}(x_0), T^{n_r}(x_0)) \ge \epsilon. \tag{2.2}$$

We can suppose that for all $r$, $m_r$ is the smallest number greater than $n_r$ such that (2.2) holds. Therefore for such $k$ we have

$$\begin{aligned} \epsilon \leq \ & d(T^{m_r}(x_0), T^{n_r}(x_0)) \\ \leq \ & d(T^{m_r}(x_0), T^{m_{r-1}}(x_0)) + d(T^{m_{r-1}}(x_0), T^{n_r}(x_0)) \\ \leq \ & d(T^{m_r}(x_0), T^{m_{r-1}}(x_0)) + \epsilon. \end{aligned}$$

Hence

$$\lim_{r\to\infty} d(T^{m_r}(x_0), T^{n_r}(x_0)) = \epsilon^+,$$

thus we have

$$\begin{aligned} d(T^{m_r}(x_0), T^{n_r}(x_0)) & \leq d(T^{m_r}(x_0), T^{m_{r+1}}(x_0)) + d(T^{m_{r+1}}(x_0), T^{n_{r+1}}(x_0)) \\ & + d(T^{n_{r+1}}(x_0), T^{n_r}(x_0)) \\ & \leq d(T^{m_{r+1}}(x_0), T^{m_r}(x_0)) \\ & + d(T^{n_{r+1}}(x_0), T^{n_r}(x_0)) + \varphi(d(T^{m_r}(x_0), T^{n_r}(x_0))). \end{aligned}$$

It follows that $\epsilon \leq \varphi(\epsilon)$, and this a contradiction. Hence $\{T^n(x_0)\}$ is a Cauchy sequence. Since $X$ is complete, there exists a $q$ in $X$ such that $T^n(x_0) \to q$ as $n \to \infty$. We prove that $q \in X$ is a fixed point of $T$. Let $\epsilon > 0$. Using the continuity of $T$ at the point $q$, given $\epsilon/2 > 0$, there exists $\delta > 0$ such that $d(z, q) < \delta$ implies that $d(T(z), T(q)) < \epsilon/2$. Now, by the convergence of $\{T^n(x_0)\}$ to $q$, assume $\eta = \min\{\epsilon/2, \delta\}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d(T^n(x_0), q) < \eta.$$

Then

$$\begin{aligned} d(T(q), q) \leq \ & d(T(q), T^n(x_0)) + d(T^n(x_0), q) \\ \leq \ & \epsilon/2 + \eta = \epsilon. \end{aligned}$$

This proves that $d(T(q), q) = 0$, and $q$ is a fixed point of $f$.

## 3 Conclusion

In this paper, a solution to Volterra-Hamerstain integral equations in partially- ordered spaces, was presented, and to this end, some fixed point theorems were proved in this kind of spaces.

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