On Inclusion Relations Between Generalized Wiener Classes

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Abstract

In this paper we aim to study inclusion relations between the generalized Wiener classes $\text{BV}^{(p_n, \gamma p)}$. In particular, we give a sufficient condition for the inclusion $\text{BV}^{(p_n, \gamma p)} \subseteq \Gamma \text{BV}^{(\gamma, \gamma p)}$ which leads us to new results for other classes of functions previously considered. We also obtain a necessary and sufficient condition for equality of two distinct classes of this type. Furthermore, we extend and unify a number of results in the literature including an important theorem of Avdispahić about Waterman spaces.

Keywords: Generalized bounded variation; Modulus of variation; Generalized Wiener class; Waterman class.

1 Introduction

We commence this paper by recalling a generalization of the classical concept of bounded variation which is central to our work here. A nondecreasing sequence $\Lambda = \{\lambda_j\}$ of positive reals is said to be a $\Lambda$-sequence if $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$.

Definition 1.1 Let $\Lambda$ be a $\Lambda$-sequence and $\{p_n\}$ be a sequence of positive reals such that $1 \leq p_n \uparrow p < \infty$. A real function $f$ on an interval $[a, b] \subseteq \mathbb{R}$ is said to be of $p_n$-$\Lambda$-bounded variation if:

$$V_{\Lambda}(f; p_n \uparrow p) := \sup_{n \geq 1} \sup_{\{I_j\}} \left(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right)^{1/p_n} < \infty$$

where the $\{I_j\}_{j=1}^{s}$ are collections of nonoverlapping subintervals of $[a, b]$ such that $\inf_{I_j} |I_j| \geq \frac{b-a}{2^n}$.

The symbol $\text{BV}^{(p_n, \gamma p)}$ stands for the linear space of functions of $p_n$-$\Lambda$-bounded variation. This class was introduced by Vyas in [11] where, among other things, it is shown that $\text{BV}^{(p_n, \gamma p)}$ with pointwise operations and a suitable norm turns into a Banach algebra.

When $\lambda_j = 1$ for all $j$, we obtain the class $\text{BV}^{(p_n, \gamma p)}$—introduced by Kita and Yoneda [6]—which is a generalization of the well-known Wiener class $\text{BV}^p$. On the other hand, taking $p_n = p$ for all $n$, we obtain the class $\text{BV}^{(p, \gamma p)}$ [10]. If further we take $p = 1$, the Waterman class $\text{BV}$ is obtained. In the sequel, we suppose that $[a, b] = [0, 1]$.

The main purpose of this paper is extending and unifying a number of inclusion theorems in the literature. More specifically, in Section 3 we give a necessary and sufficient condition for equality of two distinct $\text{BV}^{(p_n, \gamma p)}$ classes, which extends the main result of [6]. We shall also present a sufficient condition for the inclusion $\text{BV}^{(p_n, \gamma p)} \subseteq \Gamma \text{BV}^{(\gamma, \gamma p)}$ for arbitrary $\Lambda, \Gamma, p_n$ and
Then an application of the Hölder inequality yields
\[
\sum_{j=1}^{n} x_j^q z_j = \sum_{j=1}^{n} (x_j z_j)^q z_j^{1-q} 
\leq \left( \sum_{j=1}^{n} x_j z_j \right)^{q} \left( \sum_{j=1}^{n} z_j \right)^{1-q}
\leq \left( \sum_{j=1}^{n} x_j y_j \right)^{q} \left( \sum_{j=1}^{n} z_j \right)^{1-q} \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} y_i \right)^{-q}
\leq \left( \sum_{j=1}^{n} x_j y_j \right)^{q} \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} z_i \right) \left( \sum_{i=1}^{k} y_i \right)^{-q}
\]
where the last two inequalities are due, respectively, to \(2.2\) and the fact that
\[
\left\{ \sum_{i=1}^{k} z_i / \sum_{i=1}^{k} y_i \right\}_k
\]
is nondecreasing.

We will also need the following inequality which is sometimes called the equimonotonic sequences inequality [5, Theorem 368]:
\[
\sum_{j=1}^{n} x_j y_{n+1-j} \leq \sum_{j=1}^{n} x_j y_j \leq \sum_{j=1}^{n} \bar{x}_j \bar{y}_j \tag{2.3}
\]
where \(\{x_j\}, \{y_j\}\) are sequences of real numbers, and \(\{\bar{x}_j\}, \{\bar{y}_j\}\) are their descending rearrangements, respectively.

### 3 Relationships between the generalized Wiener classes \(\Lambda BV^{(p_n \uparrow p)}\)

Now we turn our attention to the mutual relations between generalized Wiener classes \(\Lambda BV^{(p_n \uparrow p)}\).

The following result is a nontrivial extension of [6, Theorem 3.1]. For the necessity part of the proof we use a refinement of the method in [6].

Let \(1 \leq p \leq q \leq \infty, 1 \leq p_n \uparrow p\) and \(1 \leq q_n \uparrow q\). Then \(\Lambda BV^{(p_n \uparrow p)} = \Lambda BV^{(q_n \uparrow q)}\) if and only if
\[
\limsup_{n \to \infty} \left( \sum_{j=1}^{n} \frac{1}{\lambda_j} \right)^{1/p_n} - \frac{1}{1/q} < \infty \tag{3.4}
\]
that with inequality \((q)\), hence:

\[
\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \leq \left( \sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right)^{\frac{2n}{p_n}} \max_{1 \leq k \leq s} \left( \sum_{j=1}^{k} \frac{1}{\lambda_j} \right)^{1-\frac{2n}{p_n}}
\]

Taking suprema over all collections \(\{I_j\}_{j=1}^{s}\) as above, and over all \(n\) yields:

\[
V_\lambda(f; q_n \uparrow q) \leq V_\lambda(f; p_n \uparrow p) \sup_n \left( \sum_{j=1}^{2n} \frac{1}{\lambda_j} \right)^{\frac{1}{p_n} - \frac{1}{p_n}} < \infty
\]

which means that \(f \in \text{ABV}(q_n \uparrow q)\). Repeating a similar argument, we obtain the reverse inclusion as well.

\textit{Necessity.} To proceed by contraposition, suppose that (3.4) does not hold. We may, without loss of generality, assume that:

\[
\limsup_{n \rightarrow \infty} \left( \sum_{j=1}^{2n} \frac{1}{\lambda_j} \right)^{\frac{1}{p_n} - \frac{1}{p_n}} = \infty
\]

and

\[
d_n := \left( \sum_{j=1}^{2n} \frac{1}{\lambda_j} \right)^{\frac{1}{p_n}} \downarrow 0
\]

Then we define a sequence of functions \(\{f_n\}_{n=1}^{\infty}\) on the interval [0, 1] inductively. Let \(f_0\) be the identically zero function on [0, 1]. When \(f_{n-1}\) is defined, \(f_n\) will be defined to be the function whose graph on the interval

\[
\left[ \frac{j-1}{2n-1}, \frac{j}{2n-1} \right]; \quad j = 1, \ldots, 2^{n-1}
\]

consists of the two consecutive line segments connecting the points:

\[
\left( \frac{j-1}{2n-1}, f_{n-1}\left( \frac{j-1}{2n-1} \right) \right), \left( \frac{2j-1}{2n}, f_n\left( \frac{2j-1}{2n} \right) \right)
\]

and

\[
\left( \frac{j}{2n-1}, f_{n-1}\left( \frac{j}{2n-1} \right) \right)
\]

where:

\[
f_n\left( \frac{2j-1}{2n} \right) :=
\begin{cases}
\min \left\{ f_{n-1}\left( \frac{j-1}{2n-1} \right), f_{n-1}\left( \frac{j}{2n-1} \right) \right\} + d_n, & (n \text{ is odd}) \\
\max \left\{ f_{n-1}\left( \frac{j-1}{2n-1} \right), f_{n-1}\left( \frac{j}{2n-1} \right) \right\} - d_n, & (n \text{ is even})
\end{cases}
\]
Since \( \{p_n\} \) is increasing,

\[
\left( \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right) \]

\[
1 p^n \leq \left( \sum_{j=1}^{2^n+1} \frac{1}{\lambda_j} \right) \leq \left( \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right) \]

\[
1 p^n \leq 2^{\frac{1}{p^n}} \leq 2 \]

\[
1 p^n + \frac{d_n}{d_{n+1}} = \left( \sum_{j=1}^{2^{n+1}} \frac{1}{\lambda_j} \right) \leq \left( \sum_{j=1}^{2^n+1} \frac{1}{\lambda_j} \right) \leq \left( \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right) \]

\[
1 p^n \leq 2^{\frac{1}{p^n}} \leq 2 \]

that is,

\[
d_n \leq 2d_{n+1} \quad \text{for all } n \]

This, along with the fact that \( d_n \downarrow 0 \), implies:

\[
|f_n(x) - f_{n+m}(x)| \leq d_n, \]

for all \( n, m \geq 1, \ x \in [0, 1] \).

Therefore \( \{f_n\}_{n=1}^\infty \) is a Cauchy sequence, hence there exists a function \( f \) such that \( f_n(x) \to f(x) \) as \( n \to \infty \) and

\[
|f_n(x) - f(x)| \leq d_n, \quad \text{for all } n \geq 1, \ x \in [0, 1]. \tag{3.5}\]

We claim that \( f \in BV_{(p_n, \gamma_p)} \) but \( f \notin BV_{(q_n, \gamma_q)} \).

To see this, let \( \{I_j\}_{j=1}^s \) be a collection of nonoverlapping subintervals of \( [0, 1] \) such that \( \inf |I_j| \geq \frac{1}{2^n} \) and set

\[
\sigma_k = 1 \leq j \leq s : \left( \sum_{i=1}^{2k+1} \frac{1}{\lambda_i} \right)^{-1} \leq |I_j| < \left( \sum_{i=1}^{2k} \frac{1}{\lambda_i} \right)^{-1}, \quad k = 1, 2, \ldots . \]

Note that for \( j \in \sigma_k \), if \( a_j = \inf I_j \) and \( b_j = \sup I_j \), using (3.5) we get

\[
|f(I_j)| \leq |f(a_j) - f_k(a_j)| + |f_k(a_j) - f_k(b_j)| + |f_k(b_j) - f(b_j)| \leq d_k + |f_k(a_j) - f_k(b_j)| + d_k \leq 4d_k.
\]

Therefore, if we define \( t_n \) to be the smallest positive integer with \( 2^n \leq \sum_{j=1}^{2^{t_n}} \frac{1}{\lambda_j} \), it follows that

\[
\sum_{j=1}^s \frac{|f(I_j)|}{\lambda_j} = \sum_{k=0}^{t_n-1} \sum_{j=1}^{2^n} \frac{|f(I_j)|}{\lambda_j} \leq \sum_{k=0}^{t_n-1} (4d_k)^{p_n} \sum_{j \in \sigma_k} \frac{1}{\lambda_j}
\]

\[
= 4p_n \sum_{k=0}^{t_n-1} \left( \sum_{j=1}^{2k} \frac{1}{\lambda_j} \right)^{p_n} \sum_{j \in \sigma_k} \frac{1}{\lambda_j}
\]

\[
\leq 2p_n \sum_{k=0}^{t_n-1} \left( \sum_{j=1}^{2k+1} \frac{1}{\lambda_j} \right)^{p_n} \sum_{j \in \sigma_k} \frac{1}{\lambda_j}
\]

\[
\leq 2p_n \sum_{k=0}^{t_n-1} \sum_{j \in \sigma_k} |I_j| \leq 8p_n
\]

where in the third inequality we have used the fact that:

\[
\left( \sum_{j=1}^s \frac{1}{\lambda_j} \right)^{-1} \leq 2 \left( \sum_{j=1}^{2^{t_n+1}} \frac{1}{\lambda_j} \right)^{-1}
\]

Consequently we have

\[
\left( \sum_{j=1}^s \frac{|f(I_j)|}{\lambda_j} \right)^{\frac{1}{p_n}} \leq 8 \quad \text{for all } n \geq 1
\]

which means that \( f \in BV_{(p_n, \gamma_p)} \).

Finally we show that \( f \notin BV_{(q_n, \gamma_q)} \). Too see this, note that from the definition of \( f \) we have

\[
|f(j/2^n) - f((j-1)/2^n)| = d_n \quad \text{for } n \geq 1
\]

Hence it follows that:

\[
\left( \sum_{j=1}^{2^n} \frac{|f(I_j)|}{\lambda_j} \right)^{\frac{1}{q_n}} = \left( d_n \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right)^{\frac{1}{q_n}} \leq \left( \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right)^{\frac{1}{q_n}} - \frac{1}{p_n}
\]

where \( I_j := [(j-1)/2^n, j/2^n] \).

Note that we have earlier assumed that \( \lim \sup_n \left( \sum_{j=1}^{2^n} \frac{1}{\lambda_j} \right)^{\frac{1}{q_n}} = \infty \). As a result, we see that \( f \notin BV_{(q_n, \gamma_q)} \), as desired.
The mutual relationship between the generalized Wiener classes \( \Lambda BV(p_n, q_n) \) is rather chaotic even in the special case where \( p_n = q_n = 1 \) for all \( n \). It is worth mentioning that in order to determine when \( \Lambda BV(p) \subseteq \Gamma BV(q_n, \tau_q) \) ([4, Theorem 1.4]), it has been assumed that \( p \leq q \), or in Theorem 3 we have assumed that \( \Lambda = \Gamma \). So, it would be highly desirable to find a condition that implies the general inclusion \( \Lambda BV(p_n, q_n) \subseteq \Gamma BV(q_n, \tau_q) \) without any additional restrictions on the \( p_n, q_n, \Lambda \) and \( \Gamma \). The following theorem provides such a condition which is new even for special cases; see the corollaries to this result. (Note, of course, that we deal with the case of interest where \( \Gamma \) is unbounded.)

The inclusion \( \Lambda BV(p_n, q_n) \subseteq \Gamma BV(q_n, \tau_q) \) holds whenever:

\[
\sup_{1 \leq n < \infty} \sum_{k=1}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_m}{p_n}} < \infty
\]

**Proof.** Assume that \( f \in \Lambda BV(p_n, q_n) \). For an arbitrary but fixed \( n \), let \( \{I_j\}_{j=1}^{s} \) be a nonoverlapping collection of subintervals of \([0, 1]\) with \( \inf |I_j| \geq \frac{1}{2s} \), and put \( q = q_n/p_n \), \( x_j = |f(I_j)|^{p_n} \), \( y_j = 1/\lambda_j \), \( z_j = 1/\gamma_j \). Using inequality 2.3, we may also assume that the \( x_j \) are arranged in descending order. Now, by Abel’s transformation and applying (2.1) together with Lemma 2.1 we obtain:

\[
\sum_{k=1}^{s} \frac{|f(I_k)|^{q_n}}{\gamma_k} = \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) \sum_{j=1}^{k} |f(I_j)|^{q_n} + \frac{1}{\gamma_s} \sum_{j=1}^{s} |f(I_j)|^{q_n} \\
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) \left( \sum_{j=1}^{k} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right) \frac{q_n}{p_n} \\
+ \frac{1}{\gamma_s} \left( \sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right) \frac{q_n}{p_n} \\
\leq \sum_{k=1}^{s-1} \Delta \left( \frac{1}{\gamma_k} \right) V \left( f; p_n \uparrow p \right)^{q_n} \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} \\
+ \frac{1}{\gamma_s} V \left( f; p_n \uparrow p \right)^{q_n} \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} \\
\leq \sum_{k=s}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) V \left( f; p_n \uparrow p \right)^{q_n} \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} \\
+ \sum_{k=s}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) V \left( f; p_n \uparrow p \right)^{q_n} \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} \\
= V \left( f; p_n \uparrow p \right)^{q_n} \sum_{k=1}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} < \infty
\]

where we have used the fact that:

\[
\frac{1}{\gamma_s} \max_{1 \leq m \leq s} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}} \\
\leq \sum_{k=s}^{\infty} \Delta \left( \frac{1}{\gamma_k} \right) \max_{1 \leq m \leq k} m \left( \sum_{j=1}^{m} \frac{1}{\lambda_j} \right)^{-\frac{q_n}{p_n}}
\]

Taking suprema over all collections \( \{I_j\}_{j=1}^{s} \) as above, and over all \( n \) yields \( V_1 \left( f; q_n \uparrow q \right) < \infty \). That is, \( f \in \Gamma BV(q_n, \tau_q) \).

The inclusion \( \Lambda BV(p) \subseteq \Gamma BV(q) \) holds whenever:

\[
\sum_{n=1}^{\infty} \Delta \left( \frac{1}{\gamma_n} \right) \max_{1 \leq k \leq n} k \left( \sum_{j=1}^{k} \frac{1}{\lambda_j} \right)^{-\frac{2}{p}} < \infty
\]

The inclusion \( \Lambda BV \subseteq \Gamma BV \) holds whenever:

\[
\sum_{n=1}^{\infty} \Delta \left( \frac{1}{\gamma_n} \right) n \left( \sum_{j=1}^{n} \frac{1}{\lambda_j} \right)^{-1} < \infty
\]
4 Relationships between the classes $\Lambda V[h_1]^{(p)}$

A sequence $h$ of positive reals is said to be a modulus of variation if it is nondecreasing and concave, i.e., $h(tn + (1 - t)m) \geq th(n) + (1 - t)h(m)$ whenever $h$ is defined at $n$, $m$ and $tn + (1 - t)m$. Let $1 \leq p < \infty$ and $\Lambda$ be a $\Lambda$-sequence.

**Definition 4.1** For a bounded real function $f$ on $[a, b]$, the sequence

$$
\nu_{p, \Lambda}(f; n) := \sup_{j=1}^n \frac{|f(I_j)|^p}{\lambda_j}
$$

is the modulus of variation associated to $f$, where the supremum is taken over all collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[a, b]$. The class of all functions $f$ for which $\nu_{p, \Lambda}(f; n) = O(h(n))$ as $n \to \infty$, is denoted by $\Lambda V[h_1]^{(p)}$.

This class was first introduced in a more general context and was studied in connection with the Riemann–Stieltjes integration theory [7]. Note that many of the classes previously considered may be obtained by making special choices of $\Lambda$, $p$ and $h$. This provides us with a general setting to unify a number of inclusion theorems (see Theorem 4 and its corollaries). For instance if $p = 1$, then taking $\Lambda = \{1\}$ we get the Chanturiya class $V[p]$, and taking $h = \{1\}$ we obtain the Waterman class $ABV$.

Let $1 \leq p \leq q < \infty$. Suppose that either the sequence

$$
\left\{ \left( \sum_{j=1}^n \frac{1}{\gamma_j} \right)^{\frac{1}{q}} \left/ \left( \sum_{j=1}^n \frac{1}{\lambda_j} \right)^{\frac{1}{p}} \right. \right\}_n
$$

or

$$
\left\{ \left( \sum_{j=1}^n \frac{1}{\gamma_j} \right)^{\frac{1}{q}} \left/ \left( \sum_{j=1}^n \frac{1}{\lambda_j} \right)^{\frac{1}{p}} \right. \right\}_n
$$

is nondecreasing. Then the inclusion $\Lambda V[h_1]^{(p)} \subseteq \Gamma V[h_2]^{(q)}$ holds whenever:

$$
\sup_{1 \leq n < \infty} \left( \sum_{j=1}^n \frac{1}{\gamma_j} / h_2(n) \right)^{\frac{1}{q}} \left( h_1(n) / \sum_{j=1}^n \frac{1}{\lambda_j} \right)^{\frac{1}{p}} < \infty
$$

**Proof.** Let $f \in \Lambda V[h_1]^{(p)}$ and consider a fixed $n$. Let $\{I_j\}_{j=1}^n$ be a nonoverlapping collection of subintervals of $[0, 1]$. Set $x_j := |f(I_j)|^p$, $y_j := 1/\lambda_j$ and $z_j := 1/\gamma_j$. In view of the equimonotonic sequences inequality (2.3) we can, and do, assume that the $x_j$ are arranged in descending order. Now, applying (2.1) with $q/p \geq 1$ in place of $q$ we obtain:

$$
\left( \sum_{j=1}^n \frac{|f(I_j)|^q}{\gamma_j} \right)^{\frac{1}{q}}
\leq \sum_{j=1}^n \frac{|f(I_j)|^p}{\lambda_j} \max_{1 \leq k \leq n} \left( \sum_{j=1}^k \frac{1}{\gamma_j} \right)^{\frac{1}{q}} \left( \sum_{j=1}^k \frac{1}{\lambda_j} \right)^{-\frac{1}{p}}
$$

Therefore, we get

$$
\left( \sum_{j=1}^n \frac{|f(I_j)|^q}{\gamma_j} \right)^{\frac{1}{q}}
\leq \left( \sum_{j=1}^n \frac{|f(I_j)|^p}{\lambda_j} \right)^{\frac{1}{q}} \max_{1 \leq k \leq n} \left( \sum_{j=1}^k \frac{1}{\gamma_j} \right)^{\frac{1}{q}} \left( \sum_{j=1}^k \frac{1}{\lambda_j} \right)^{-\frac{1}{p}}
\leq C h_1(n)^{\frac{1}{p}} h_2(n)^{\frac{1}{q}} = C h_2(n)^{\frac{1}{q}}
$$

for some positive constant $C$, depending only on $f$. As a result, taking supremum over all collections $\{I_j\}_{j=1}^n$ as above, it follows that

$$
\nu_{q, \Gamma}(f; n) \leq C h_2(n),
$$

which means that $f \in \Gamma V[h_2]^{(q)}$.

Making suitable choices of $\Lambda$, $\Gamma$, $p$, $q$, $h_1$ and $h_2$ to invoke the preceding theorem, we obtain the following corollaries.

(1) The following inclusion holds:

$$
\Lambda BV \subseteq V \left[ n / \sum_{j=1}^n 1/\lambda_j \right]
$$

(4) Let $1 \leq p \leq q < \infty$. Then the inclusion $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q)}$ holds whenever

$$
\sup_{1 \leq n < \infty} \left( \sum_{j=1}^n \frac{1}{\gamma_j} \right)^{\frac{1}{q}} \left( \sum_{j=1}^n \frac{1}{\lambda_j} \right)^{-\frac{1}{p}} < \infty
$$

(2) The inclusion $V[h_1] \subseteq V[h_2]$ holds whenever

$$
\sup_{1 \leq n < \infty} h_1(n) / h_2(n) < \infty
$$
We end this paper with a characterization of when $\Lambda V[h]^{(p)}$ coincides with $B[0,1]$, the space of all bounded functions on $[0,1]$. Then we have:

The equality $\Lambda V[h]^{(p)} = B[0,1]$ holds if and only if

$$\sup_{1 \leq n < \infty} h(n)^{-1} \sum_{j=1}^{n} \frac{1}{\lambda_j} < \infty \quad (4.7)$$

**Proof.** Assume $(4.7)$ and let $f \in B[0,1]$. Then there exists some positive constant $C$ such that $\sum_{j=1}^{n} \frac{1}{\lambda_j} \leq C h(n)$ for all $n$. So if $\{I_j\}_{j=1}^{n}$ is a collection of nonoverlapping subintervals of $[0,1]$, we have:

$$\sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j} \leq 2^p \|f\|_p^p \sum_{j=1}^{n} \frac{1}{\lambda_j} \leq 2^p C \|f\|_p^p h(n)$$

where $\|f\|_p = \sup \{|f(x)| : x \in [0,1]\}$. Thus, $f \in \Lambda V[h]^{(p)}$.

Conversely, let $\Lambda V[h]^{(p)} = B[0,1]$. To show that $(4.7)$ holds, we construct an $f \in B[0,1]$ as follows:

$$f(x) := \begin{cases} 1 & \text{if } x \in \left(\frac{1}{2^k}, \frac{1}{2^k} + \frac{1}{2} \right); k = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the collection

$$I_k = \left[\frac{1}{2^k}, \frac{1}{2^k} + \frac{2}{3}. \frac{1}{2^k} \right]; \quad k = 1, 2, \ldots, n$$

Then we have:

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} = \sum_{k=1}^{n} \frac{|f(I_k)|^p}{\lambda_k} \leq \nu_{p,\Lambda}(f; n) \sup_{1 \leq r < \infty} \frac{\nu_{p,\Lambda}(f; r)}{h(r)}$$

where the last inequality is a result of our assumption that $f \in \Lambda V[h]^{(p)}$. The proof is complete.

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**References**


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