New Results on Ideals in MV-algebras

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Abstract

In the present paper, by considering the notion of ideals in MV-algebras, we study some kinds of ideals in MV-algebras and obtain some results on them. For example, we present definition of ultra ideal in MV-algebras, and we get some results on it. In fact, by definition of ultra ideals, we present new conditions to have prime ideals, positive implicative ideals and maximal ideals in MV-algebras. Also, we state some properties on contracted or extended ideals as useful examples of ideals in MV-algebras. Finally, we try to prove the Chinese reminder theorem in MV-algebras.

Keywords: MV-algebra; Ideal; Ultra ideal; Chinese reminder theorem; Pseudo-hoops.

1 Introduction

MV-algebras were defined by C. C. Chang [3, 4] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C*-algebras. They are also naturally related to Ulam’s searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. In particular, emphasis has been put the ideal theory of MV-algebras [8, 11]. Hoo, Iseki and Tanaka introduced the notions of implicative and quasi-implicative ideals of MV-algebras [12, 13]. Many important properties can be derived from the fact, established by Chang that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. Recently, some researchers worked on MV-algebras and ideals in them (see [2, 10, 17, 18, 19]). For continuing of study of ideals in MV-algebras, we present definition of ultra ideal in MV-algebras and verify the relationship between it and some other ideals. Also, we introduce contraction and extension of an ideal in MV-algebras and we get related results.

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2 Preliminaries

In this section, we review some definitions and related lemmas and theorems in MV-algebras that we use in the next sections.

Definition 2.1 [5] An MV-algebra is a structure $M = (M, \oplus, 0)$ of type $(2, 1, 0)$ such that:

(MV1) $(M, \oplus, 0)$ is an Abelian monoid,
(MV2) $(a')' = a,$
(MV3) $0' \oplus a = 0'$,
(MV4) $(a' \oplus b)' \oplus (b' \oplus a)' \oplus a,$
(MV5) $a \oplus b = (a' \oplus b')'$,
(MV6) $a \oplus 1 = 1,$
(MV7) $(a \oplus b) \oplus (b \oplus a) = a \oplus b,$
(MV8) $a \oplus a' = 1,$
for every $a, b \in M$.

It is clear that $(M, \oplus, 1)$ is an Abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on $M$ by $a \vee b = (a' \oplus b')'$ and $a \wedge b = a \ominus (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a bounded distributive lattice. An MV-algebra $M$ is a Boolean algebra if and only if the operation “$\ominus$” is idempotent, i.e., $x \ominus x = x$, for every $x \in M$. In MV-algebra $M$, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \ominus b' = 0$, (iii) $b = a \ominus (b \ominus a)$, (iv) there exists $c \in M$ such that $a \ominus c = b$, for every $a, b, c \in M$. For any two elements $a, b$ of $M$, $a \leq b$ if and only if $a, b$ satisfy the above equivalent conditions (i)–(iv). An ideal of MV-algebra $M$ is a subset $I$ of $M$, satisfying the following condition: (I1) $0 \in I$, (I2) $x \leq y$ and $y \in I$ imply that $x \in I$, (I3) $x \ominus y \in I$, for every $x, y \in I$. Let $I$ be an ideal of $M$ and $I \neq M$ (we say $I$ is a proper ideal of $M$). Then (i) $I$ is a prime ideal if and only if $x \ominus y \in I$ or $y \ominus x \in I$, for every $x, y \in M$. A proper ideal $I$ of $M$ is a maximal ideal of $M$ if and only if no proper ideal of $M$ strictly contains $I$. In MV-algebra $M$, the distance function $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv) $d(x, y) = d(x', y')$, (v) $d(x \ominus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let $I$ be an ideal of MV-algebra $M$. Then we denote $x \sim y$ ($x \equiv_I y$) if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on $M$. Denote the equivalence class containing $x$ by $\bar{x}$ and $\frac{d}{dM} = \{\bar{x} : x \in M\}$. Then $(\frac{M}{M}, \oplus, 0)$ is an MV-algebra, where $\frac{a}{M} = x$ and $\frac{d}{M} = \frac{x \oplus y}{M}$, for all $x, y \in M$. Let $M$ and $K$ be two MV-algebras. A mapping $f : M \to K$ is called an MV-homomorphism if (H1) $f(0) = 0$, (H2) $f(x \oplus y) = f(x) \oplus f(y)$ and (H3) $f(x') = (f(x))'$, for every $x, y \in M$. If $f$ is one to one (onto), then $f$ is called an MV-monomorphism (epimorphism). If $f$ is onto and one to one, then $f$ is called an MV-isomorphism.(see [5])

Definition 2.2 [6, 9] (i) An $l$-group is an algebra $(G, +, -0, \vee, \wedge)$, where the following properties hold:

(a) $(G, +, -0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.

A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian $l$-groups with strong unit will be simply called $lu$-groups.

The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by $\text{MV}$. The category whose objects are pairs $(G, u)$, where $G$ is an Abelian $l$-group and $u$ is a strong unit of $G$ and whose homomorphisms are $l$-group homomorphisms is denoted by $\text{UG}$. The functor that establishes the categorial equivalence between $\text{MV}$ and $\text{UG}$ is $\Gamma : \text{UG} \to \text{MV}$, where $\Gamma(G, u) = [0, u]_G$, for every $lu$-group $(G, u)$ and $\Gamma(h) = h|_{[0, u]}$, for every $lu$-group homomorphism $h$.

Lemma 2.1 [5] Let $M$ be an MV-algebra. Then $x \leq y$ implies that $x \ominus z \leq y \ominus z$ and $x + z \leq y + z$, for every $x, y, z \in M$.

Definition 2.3 [15] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:

(BCK1) $(x * y) * (x * z) * (z * y) = 0$,
(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = x$,
(BCK5) if $x * y = y * x = 0$, then $x = y$, for all
x, y, z ∈ X.

The relation x ≤ y which is defined by x * y = 0 is a partial order on X with 0 as least element.

In BCK-algebra X, for any x, y, z ∈ X, we have (BCK6) (x * y) * z = (x * z) * y.

Let (X, +, 0) be a BCK-algebra. Subset ∅ ≠ I ⊆ X is called an ideal of X, if 0 ∈ I and for any x, y ∈ X, x + y ∈ I and y ∈ I, imply that x ∈ I. A nonempty subset I of X is said to be a positive implicative ideal if 0 ∈ I and (x * y) * z ∈ I, y * z ∈ I imply that x * z ∈ I, for any x, y, z ∈ X. Furthermore, any positive implicative ideal must be an ideal. See [15]

Theorem 2.1 [5] If (M, ⊕', 0, 1) is an MV-algebra, then (M, ⊕, 0) is a BCK-algebra.

Corollary 2.1 [5] (i) Every prime ideal I of an MV-algebra M is contained in a unique maximal ideal of M.

(ii) Every proper ideal of an MV-algebra M is an intersection of prime ideals of M.

Lemma 2.2 [5] Let M be an MV-algebra and ∅ ≠ W ⊆ M. If the generated ideal by W is denoted by < W >, then < W > := {x ∈ M : x ≤ w₁ + · · · + wₙ, for some w₁, · · · , wₙ ∈ W}.

Proposition 2.1 [5] Let M, N be MV-algebras and J be a maximal ideal of N. Then for any homomorphism h : M → N, the inverse image h⁻¹(J) is a maximal ideal of M.

Lemma 2.3 [5] Let M, N be two MV-algebras and f : M → N be an MV-homomorphism. Then the following properties hold:

(i) Ker(f) is an ideal of M,

(ii) if f is an MV-epimorphism, then M/ Ker(f) ∼= N,

(iii) f(x) ≤ f(y) iff x ⊗ y ∈ Ker(f),

(iv) f is injective iff Ker(f) = {0}.

Definition 2.4 [6] A product MV-algebra (or PMV-algebra, for short) is a structure A = (A, ⊕, ·', 0), where (A, ⊕, ·', 0) is an MV-algebra and ·' is a binary associative operation on A such that the following property is satisfied: if x + y is defined, then x·z + y·z and x·z + z·y are defined and (x + y)·z = x·z + y·z, z.(x + y) = z·x + z·y, for every x, y, z ∈ A, where ·' is the partial addition on A. A unity for the product is an element e ∈ A such that e·x = x·e = x, for every x ∈ A. If A has a unity for product, then e = 1. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation.

3 Some results on ideals

In this section, we verify some results on ideals.

Proposition 3.1 Let M be an MV-algebra and I ⊆ M. Then

(i) I is an ideal of M if and only if the following holds:

(i) 0 ∈ I,

(ii) x ⊗ y ∈ I,

(iii) if x ⊗ y, y ∈ I, then x ∈ I, for any x, y ∈ M.

(2) I is an ideal of M if and only if the following holds:

(i) 0 ∈ I,

(ii) x ⊗ y ∈ I,

(iii) if z ⊗ y, y ⊗ x ∈ I, then z ⊗ x ∈ I, for any x, y, z ∈ M.

Proof. (1) (⇒) Let I be an ideal of M. Then (i) and (ii) are clear. Now, let x ⊗ y, y ∈ I. Then by (ii) and (MV7), (y ⊗ x) ⊗ x = (x ⊗ y) ⊗ y ∈ I. Since x ≤ (y ⊗ x) ⊗ x ∈ I, we have x ∈ I.

(⇐) Let (i), (ii) and (iii) be true. If x ≤ y and y ∈ I, then x ⊗ y = x ⊗ y′ = 0 ∈ I and so by (iii), x ∈ I. Hence, I is an ideal of M.

(2) (⇒) Let I be an ideal of M. Then (i) and (ii) are clear. Now, let z ⊗ y, y ⊗ x ∈ I, for any x, y, z ∈ M. Then by Theorem 2.1 and (BCK1), ((z ⊗ x) ⊗ (z ⊗ y)) ⊗ (y ⊗ x) = 0 and so by (1), (z ⊗ x) ∈ I.

(⇐) Let (i), (ii) and (iii) be true. If x ≤ y and y ∈ I, then x ⊗ y = x ⊗ y′ = 0 ∈ I. Since y ⊗ 0 = y ∈ I, by (iii), x = x ⊗ 0 ∈ I. Hence, I is an ideal of M.

Theorem 3.1 Let J be an ideal of MV-algebra M and a ∈ M. Then

< J ∪ {a} > := {x ∈ M : ∃n ∈ N, (x' ⊕ na') ∈ J}.

Moreover, < J ∪ {a} > is the least ideal of M containing J ∪ {a}.

Proof. Let T = {x ∈ M : ∃n ∈ N, (x' ⊕ na') ∈ J}. If x ∈ < J ∪ {a} >, then by Lemma 2.2,
there exist $b_1, \ldots, b_m \in J \cup \{a\}$ such that $x \leq b_1 \oplus b_2 \oplus \cdots \oplus b_m$, and so $x \ominus (b_1 \oplus b_2 \oplus \cdots \oplus b_m) = 0$. It means that $((x \ominus b_1) \ominus (b_2) \ominus \cdots) \ominus b_m = 0 \in J$. We consider two cases. Let $b_i \neq a$, for any $1 \leq i \leq m$. Then by Theorem 2.1 and (BCK6), $((x \ominus a) \ominus b_1) \ominus (b_2) \ominus \cdots) \ominus b_m = 0 \in J$. Since $b_1, \ldots, b_m \in J$, we have $x \ominus a \in J$ and so $x \in T$. If there exists $b_i = a$, for some $1 \leq i \leq m$, then by renumbering, there exist $n, k \in N$ and $n, k < m$ such that $((x' \ominus na) \ominus b_1) \ominus \cdots) \ominus b_k = 0 \in J$. It results that $(x' \ominus na) \in J$ and so $x \in T$. Now, let $x \in T$. Then there exists $n \in N$ such that $(x' \ominus na) \in J$. Let $u = (x' \ominus na) \ominus x \in J \cup \{a\}$. Then there is $u \in J \cup \{a\}$ such that $(x' \ominus na) \ominus u = u \ominus u = 0$. Hence, $x \in J \cup \{a\}$.

Finally, we will show that $J \cup \{a\} >$ is the least ideal of $M$ containing $J \cup \{a\}$. Let $C$ be an ideal of $M$ containing $J \cup \{a\}$. We must show that $J \cup \{a\} > C$. Since $a \in C$, we have $(x' \ominus na) \ominus na \in C$. Now, by (MV4), we have $x \leq (x \ominus na) \ominus x = (x' \ominus na) \ominus na \in C$. It results that $x \in C$. Therefore, $J \cup \{a\} > C$.

**Proposition 3.2** Let $a, b \in M$ and $J$ be an ideal of $M$. Then $J \cup \{a\} > \cap \leq J \cup \{b\} >$.

**Proof.** Let $x \in J \cup \{a\} > \cap \leq J \cup \{b\} >$. Then by Theorem 3.1, there exist $m, n \in N$ such that $(x' \ominus na) \in J$ and $(x' \ominus mb) \in J$. Let $u = (x' \ominus na) \ominus x \in J \cup \{a\}$. By Theorem 2.1 and (BCK6), we have

$$((x \ominus u) \ominus v) \ominus na = (x \ominus u) \ominus (v \ominus na) = (x \ominus na) \ominus v = (x' \ominus na) \ominus u \ominus v = (u \ominus u) \ominus v = 0.$$  

Similarly, we have $((x \ominus u) \ominus v) \ominus mb = (x' \ominus mb) \ominus v = (x \ominus v) \ominus u = 0$. Let $t = (x \ominus u) \ominus v$. We have $a \leq a \oplus b$. Then by Lemma 2.1, $t \ominus (a \oplus b) \leq t \ominus a$ and $(t \ominus (a \oplus b)) \ominus (a \oplus b) \leq (t \ominus a) \ominus (a \oplus b) = (t \ominus (a \oplus b)) \ominus a \leq (t \ominus a) \ominus a$. Hence, $(t \ominus (a \oplus b)) \ominus (a \oplus b) \leq (t \ominus a) \ominus a$. Similarly, it results that $(t' \ominus n(a \oplus b)) \ominus (a \oplus b) \leq (t' \ominus na) \ominus 0$ and so $((x \ominus u) \ominus v) \ominus na \ominus 0 = 0$. It is easy to show that $(x' \ominus na) \ominus u \ominus v = 0$. Since $u, v \in J$,

**Notation:** In general, the converse of Proposition 3.2, is not true.

**Example 3.1** Let $M = \{0, 1, 2, 3\}$ and operation “$\ominus$” is defined on $M$ as follows:

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If $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$, then $(M, \ominus, 0, 3)$ is an MV-algebra and $I = \{0, 1\}$ is an ideal of $M$. It is easy to show that $I \cup \{1, 2\} = I \cup \{\{1, 2, 3\} = \{x : x \in N, (x' + n)^{-}\} = \{0, 1, 2, 3\}, I \cup \{1\} = \{0, 1\}$ and $I \cup \{2\} = \{0, 1, 2, 3\}$. It results that $I \cup \{1, 2\} > 0 < I \cup \{1\} > \cap \leq I \cup \{2\}$.

**4 Ultra ideals**

In this section, we present definition of ultra ideals in MV-algebras. Then we verify some properties about them, and we obtain the relationship between ultra ideals and some other ideals.

**Definition 4.1** Let $M$ be an MV-algebra and $I$ be a non trivial ideal of $M$. Then $I$ is called an ultra ideal of $M$ if for every $x \in M$, $x \in I$ if and only if $x' \notin I$.

**Example 4.1** Let $M = \{0, 1, 2, 3, 4\}$ and the operation “$\ominus$” on $M$ is defined as follows:

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If $0' = 4$, $1' = 4$, $2' = 3$, $3' = 2$ and $4' = 0$, then $(M, \ominus, 0, 4)$ is an MV-algebra and $I = \{0, 1, 2\}$, $J = \{0, 1, 3\}$ and $K = \{0, 1\}$ are ideals of $M$. It is easy to show that $I, J$ are ultra ideals of $M$. Since $2' = 3 \notin K$ and $2 \notin K$, $K$ is not an ultra ideal of $M$. 
Theorem 4.1 Let $I$ be an ultra ideal of MV-algebra $M$, $J$ be a proper ideal of $M$ and $I \subseteq J$. Then $J$ is an ultra ideal of $M$, too.

Proof. Let $x \in J$. If $x' \in J$, then by ($I_3$), $1 = x \oplus x' \in J$, which is a contradiction. Now, let $x' \notin J$. If $x \notin J$ and so $x' \in J \subseteq J$, which is a contradiction.

By Theorem 2.1, in MV-algebra $(M, \mathbb{v}, 0, 1)$, if $I$ is an ideal of BCK-algebra $(M, \mathbb{v}, 0)$ and it satisfies in ($I_3$), then $I$ is an ideal of MV-algebra $(M, \mathbb{v}, 0, 1)$, too. Hence, in this case, definition of positive implicative ideals in BCK-algebras can be translated to MV-algebras. Then we can present the definition of positive implicative ideals in MV-algebras as follows:

Let $M$ be an MV-algebra and $0 \neq I \subseteq M$. Then $I$ is called a positive implicative ideal of $M$ if the following hold: (i) $0 \in I$, (ii) $x \oplus y \in I$, (iii) if $(x \oplus y) \ominus z \in I$ and $y \ominus z \in I$, then $x \ominus z \in I$, for any $x, y, z \in M$. Also, in this field, all of proved theorems of ideals in BCK-algebras are true in MV-algebras.

Example 4.2 (i) Let $M = \{0, 1, 2\}$ and operation $\oplus$ be defined by

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If $0' = 2$, $1' = 1$ and $2' = 0$, then $(M, \mathbb{v}, 0, 2)$ is an MV-algebra. It is easy to show that $I = \{0, 1\}$ is a positive implicative ideal of $M$.

(ii) In Example 4.1, $K$ is a positive implicative ideal of $M$.

(iii) Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and let $0$ be the matrix with all elements 0. It is easy to see that $M_2(\mathbb{R})$ is an $l$-group. If $v = \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} \right)$, then $(M_2(\mathbb{R}), v)$ is an $l$-group and so $M = \Gamma(M_2(\mathbb{R}), v)$ is an MV-algebra. It is easy to see that $I(M) = \{\{0\}, M\}$, where $I(M)$ is the set of ideals of $M$. It is easy to see that $\{0\}$ is not a positive implicative ideal of $M$.

In following, we verify the relationship between ultra ideals and positive implicative (prime) ideals.

Theorem 4.2 Let $I$ be an ultra ideal of $M$.

(i) $I$ is a positive implicative ideal of $M$, (ii) $I$ is a prime ideal of $M$.

Proof. (i) Let $(z \odot y) \ominus x \in I$, $y \ominus x \in I$, where $x, y, z \in M$. We must show that $z \ominus x \in I$. Let $z \ominus x \notin I$. Then $(z \ominus x)' \in I$. Since $x \ominus (z \ominus x)' = x \ominus (z \ominus x) = x \ominus (z \ominus x') = 0 \in I$, we get $x \notin I$. Now, since $y \ominus x$, $x \in I$, we have $y \in I$. On the other hand, by Theorem 2.1 and (BCK6), since $(z \ominus x) \odot y = (z \ominus y) \ominus x \notin I$, we have $z \ominus x \in I$, which is a contradiction. Therefore, $I$ is a positive implicative ideal of $M$.

(ii) If $I$ is not a prime ideal of $M$, then there exist $x, y \in M$ such that $x \ominus y \notin I$ and $y \ominus x \notin I$. Since $I$ is an ultra ideal of $M$, we have $(x \odot y)' \in I$ and $(y \odot x)' \in I$. Then $1 = (x' \odot y) \ominus (y' \odot x) = (x \ominus y)' \ominus (y \ominus x)' \in I$, which is a contradiction. Therefore, $I$ is a prime ideal of $M$.

Example 4.3 (i) In Example 4.1, $K$ is a positive implicative ideal, but it is not an ultra ideal.

(ii) In example 4.2 (i), $\{0\}$ is a prime ideal of $M$, but it is not an ultra ideal of $M$.

(iii) In example 4.2 (iii), $\{0\}$ is neither a positive implicative ideal of $M$ nor an ultra ideal of $M$. Also, $\{0\}$ is not a prime ideal of $M$.

Definition 4.2 Let $M$ be an MV-algebra. $B \subseteq M$ is said to have the finite union property if $a_1 \oplus a_2 \oplus \cdots \oplus a_n \neq 1$, for any $a_1, \cdots, a_n \in B$ and $a_i \neq 1$, where $1 \leq i \leq n$.

Example 4.4 In Example 4.1, $B = \{0, 1, 2\}$ has finite union property, but $C = \{2, 3\}$ has not finite union property (note that $2 \oplus 3 = 4$).

Theorem 4.3 Let $M$ be an MV-algebra, $B \subseteq A$ and $1 \notin B$. Then $\langle B \rangle$ is a proper ideal of $M$ if and only if $B$ has the finite union property.

Proof. ($\Rightarrow$) Let $\langle B \rangle$ be a proper ideal of $M$ and $B$ has not the finite union property. Then there exist $a_1, \cdots, a_n \in B$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 1$. By Lemma 2.2, $1 \in \langle B \rangle$ and so $\langle B \rangle = M$, which is a contradiction.

($\Leftarrow$) Let $B$ has the finite union property and $\langle B \rangle = M$. Then $1 \in \langle B \rangle$ and so by Lemma 2.2, there exist $a_1, \cdots, a_n \in B$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n \geq 1$, which is a contradiction.
Note. It is easy to see that every non trivial ideal of an MV-algebra has the finite union property. The proof is similar to the proof of Theorem 4.3 (⇒).

**Lemma 4.1** Let $M$ be an MV-algebra, $x \in M$ and $I$ be an ideal of $M$ such that $I$ have the finite union property. If $x \notin I$ and $x' \notin I$, then $I \cup \{x\}$ has the finite union property.

**Proof.** Let $B = I \cup \{x\}$. We will show that $b_1 + b_2 + \cdots + b_n \neq 1$, for any $b_1, \ldots, b_n \in B$ and $b_i \neq 1$. If $b_1, \ldots, b_n \in I$, then the proof is clear.

If W. O. L. G, $b_1 = x$ and $b_1 + b_2 + \cdots + b_n = 1$, for some $b_1, \ldots, b_n \in B$, then $(x' \otimes (b_2 + \cdots + b_n'))' = x' \otimes (b_2 + \cdots + b_n) = 1$ and so $x' \notin I$. Since $b_1 + \cdots + b_n \in I$, by Proposition 3.1(1), we have $x' \in I$, which is a contradiction. Therefore, $b_1 + b_2 + \cdots + b_n \neq 1$, for any $b_1, \ldots, b_n \in B$ and so $I \cup \{x\}$ has the finite union property.

**Theorem 4.4** Let $M$ be an MV-algebra and $I \subseteq M$. Then $I$ is an ultra ideal of $M$ if and only if $I$ is a non trivial maximal ideal of $M$.

**Proof.** ($\Rightarrow$) Let $I$ be an ultra ideal of $M$, and $J$ is not a maximal ideal of $M$. Then there exists a proper ideal $J$ of $M$ such that $I \subseteq J$ and so there exists $x \in J$ such that $x \notin I$. It results that $x' \in I$ and so $x' \in J$. Since $1 \otimes x = 1 \otimes x' = (0 \otimes x')' = x' \in J$ and $x \in J$, we get $1 \in J$, which is a contradiction.

($\Leftarrow$) Let $I$ be a maximal ideal of $M$. If $x \in I$ and $x' \in I$, for some $x \in M$, then $1 \in I$, which is a contradiction. Hence, $x \in I$ implies that $x' \notin I$. Now, let there exists $x \in A$ such that $x' \notin I$ and $x \notin I$. Consider $B = I \cup \{x\}$. Then by Lemma 4.1, $B$ has the finite union property. Hence, by Theorem 4.3, $\prec B \succ$ is a proper ideal of $M$, which is a contradiction. Because, $I \subseteq \prec B \preceq M$ and $I$ is a maximal ideal of $M$. Hence, $x' \notin I$ implies that $x \in I$. Therefore, $I$ is an ultra ideal of $M$.

**Lemma 4.2** Let $M$ be an MV-algebra and $I \subseteq M$. If $I$ has the finite union property, then there exists an ultra ideal $B$ of $M$ such that $I \subseteq B$.

**Proof.** Let $E = \{B : I \subseteq B, \text{where } B \text{ is a proper ideal of } M\}$. Since $I$ has the finite union property, by Theorem 4.3, $\prec I \succ$ is a proper ideal of $M$. Since $I \subseteq \prec I \succ$, we have $\prec I \succ \subseteq E$ and so $E \neq \emptyset$. Let $F = \{B_t\}_{t \in N}$ be a chain in $E$ and $B_1 = \bigcup_{t \in N} B_t$. Since $B_1$ is an upper bound of $F$ in $E$ and $B_1$ is an ideal of $M$, $B_1 \subseteq E$. Hence, by Zorn’s lemma, $E$ has a maximal element $B$ and so by Theorem 4.4, $B$ is an ultra ideal of $M$ such that $I \subseteq B$.

**Theorem 4.5** Any proper ideal in MV-algebra $M$, contained at least one ultra ideal.

**Proof.** Let $I$ be a proper ideal of $M$. Since $I \subseteq \prec I \succ$, by Theorem 4.3, $I$ has the finite union property and so by Lemma 4.2, there exists an ultra ideal $B$ of $M$ such that $I \subseteq B$.

## 5 Contraction and Extension of ideals in MV-algebras

In this section, we verify some properties on contracted or extended ideals as useful examples of ideals in MV-algebras. Also, we try to prove the Chinese reminder theorem in MV-algebras.

**Remark:** Let $M, N$ be MV-algebras, $f : M \rightarrow N$ be an MV-homomorphism, $I \subseteq M$ and $J$ be an ideal of $N$. Then we set $f^{-1}\langle J \rangle = J^c$ and $\langle f(I) \rangle = I^e$. It is clear that $J^c$ (contraction of $J$) is an ideal of $M$ and $I^e$ (extension of $I$) is an ideal of $N$.

**Theorem 5.1** Let $M, N$ be MV-algebras, $f : M \rightarrow N$ be an MV-homomorphism, $I$ be an ideal of $M$ and $J$ be an ideal of $N$. Then

(i) $I \subseteq I^e$,

(ii) $J^c \subseteq J$,

(iii) $J^c = J^{cc}$,

(iv) $I^e = I^{ce}$,

(v) If $K = \{I : I$ is an ideal of $M$ and $I^{cc} = I\}$, $E = \{J : J$ is an ideal of $N$ and $J^{cc} = J\}$, $K' = \{J^c : J$ is an ideal of $N\}$ and $E' = \{I^e : I$ is an ideal of $M\}$, then $K = K'$, $E = E'$ and there exists an isomorphism $\Phi : K \rightarrow E$.

**Proof.**

(i) The proof is clear.

(ii) Let $y \in J^{ce} = \prec f(J^c) \succ$. Then there exist $t_1, t_2, \ldots, t_k \in J^c$ such that $y \leq f(t_1) + \cdots + f(t_k)$ and so $y \leq f(f^{-1}(a_1)) + \cdots + f(f^{-1}(a_k))$, where $f(t_i) = a_i \in J$, for any $1 \leq i \leq k$. It results that $y \leq a_1 + \cdots + a_k$ and so $y \in \prec J \succ = J$. Hence,
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Example 5.1 (i) In example 3.1, let f : M → M be zero homomorphism. Consider I = \{0\} that is an ideal of M. We have I^c = < f(I) > = \{0\} and I^c = M. Then I \neq I^c.
(ii) In Example 3.1, let f : M → M be defined by f(0) = f(1) = 0 and f(2) = f(3) = 3. It is easy to see that f is an MV-homomorphism. Consider J = \{0\} that is an ideal of M. We have J^c = \{0, 1\} and J^c = < \{0, 1\} > = \{x ∈ M : x ≤ w_1 + ⋯ + w_n, for some w_1, ⋯, w_n ∈ \{0, 1\}\} = \{0, 1\}. Hence J^c \neq J.

Definition 5.1 [10] Let I be an ideal of an MV-algebra M. Then we set \text{rad}(I) = \bigcap_{i≤m} m, where m is any maximal ideal of M. Moreover, if there is not any maximal ideal of M containing I, then we let \text{rad}(I) = M.

Notation: By Corollary 2.1, any proper ideal of a PMV-algebra is contained in a maximal ideal (note that every PMV-algebra is an MV-algebra).

Theorem 5.2 Let M, N be MV-algebras, I_1, I_2, I be ideals of M, J_1, J_2, J be ideals of N and f : M → N be an MV-homomorphism. Then
(i) (I_1 ∩ I_2)^c \subseteq I_1^c ∩ I_2^c,
(ii) (J_1 ∩ J_2)^c = J_1^c ∩ J_2^c,
(iii) (I_1 ⊕ I_2)^c \subseteq (f(I_1) ⊕ f(I_2))^c, where I_1 ⊕ I_2 = \{a + b : a ∈ I_1, b ∈ I_2\},
(iv) \text{rad}(I)^c \subseteq \text{rad}(I^c),
(v) \text{rad}(J^c) \subseteq (\text{rad}(J))^c.

Proof. (i) Let y ∈ (I_1 ∩ I_2)^c. Then by Lemma 2.2, there exist a_1, ⋯, a_n ∈ I_1 ∩ I_2 such that y ≤ f(a_1) + ⋯ + f(a_k). Since a_i ∈ I_1 and a_i ∈ I_2, we have f(a_i) ∈ f(I_1) and f(a_i) ∈ f(I_2), for any 1 ≤ i ≤ n. It results that y ∈ I_1^c ∩ I_2^c.
(ii) The proof is routine.
(iii) Let y ∈ (I_1 ∩ I_2)^c. Then by Lemma 2.2, there exist a_1 + b_1 ∈ I_1 ∩ I_2, for any 1 ≤ i ≤ n such that y ≤ f(a_1 + b_1) + ⋯ + f(a_n + b_n) = f(a_1) + f(b_1) + ⋯ + f(a_n) + f(b_n). It results that y ∈ (f(I_1) ∩ f(I_2))^c.
(iv) Let y ∈ (\text{rad}(J))^c = < f(\bigcap_{i≤K} K) >, where K is every maximal ideal of M. Then there exist a_1, ⋯, a_k ∈ \bigcap_{i≤K} K such that y ≤ f(a_1) + ⋯ + f(a_k). We must show that y ∈ \bigcap_{i≤K} K, where L is any maximal ideal of N containing < f(K) >. We have a_i ∈ K, for any maximal ideal of M containing I. Then f(a_i) ∈ f(K) < < f(K) >. Let < f(K) > \neq N. Then by above Notation, f(a_i) ∈ n, where L is a maximal ideal of N containing < f(K) > (if < f(K) > = N, then there is no maximal ideal of N containing < f(K) > and so by definition of 5.1, we consider L = N). On the other hand, I ⊆ K implies that < f(I) > ⊆ < f(K) > ⊆ L. It results that f(a_i) ∈ \bigcap_{i≤K} L = \text{rad}(I).
(v) Let x ∈ (\text{rad}(J))^c = f^{-1}(\bigcap_{i≤L} L), where L is any maximal ideal of N. Then f(x) ∈ \bigcap_{i≤L} L \subseteq L and so x ∈ f^{-1}(L) = L^c. It results that x ∈ \bigcap_{i≤L} L^c = \bigcup_{j≤L^c} L^c and so by Proposition 2.1, x ∈ \text{rad}(J^c). Hence, (\text{rad}(J))^c \subseteq \text{rad}(J^c).

Lemma 5.1 Let A be a PMV-algebra. Then \bigcap_{i≤I} A = A ⊕ A ⊕ ⋯ ⊕ A is a PMV-algebra.

Proof. We define \{a_i\}_{i=1}^n + \{b_i\}_{i=1}^n = \{a_i + b_i\}_{i=1}^n, \{a_i\}_{i=1}^n \cdot \{b_i\}_{i=1}^n = \{a_i b_i\}_{i=1}^n and \{(a_i)_{i=1}^n\} = \{a_i\}_{i=1}^n, for every \{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n ∈ \bigcap_{i≤I} A. It is easy to show that (\bigcap_{i≤I} A, ⊕, ⋯, \{0\}) is a PMV-algebra.

Theorem 5.3 Let M be an MV-algebra and I_1, ⋯, I_n be ideals of M. Then there exists an MV-homomorphism Φ : M → M that is an MV-monomorphism if and only if
(i)\bigcap_{i=1}^n I_i = \{0\},
(ii) if Φ is onto, then \bigcap_{i=1}^n I_i = M, for any 1 ≤ i, j ≤ n, where I_i \bigoplus I_j = \{a + b : 0 \leq a \in I_i, b \in I_j\}. 

(iii) if \( y = (\frac{x_a}{M_i}, \cdots, \frac{x_n}{M_i}) \in \sum_{i=1}^n M_i \) implies that \( x_i \cap \bigwedge_{i=1}^n x_i \in I_i \), then \( \phi \) is onto.

**Proof.** By Lemma 5.1, \( \sum_{i=1}^n M_i \) is an \( MV \)-algebra (note that every \( PMV \)-algebra is an \( MV \)-algebra). We define \( \Phi(a) = (\frac{a}{I_1}, \cdots, \frac{a}{I_n}) \), for any \( a \in M \). It is clear that \( \Phi(0) = 0 \). It is easy to show that \( \Phi(a + b) = \Phi(a) + \Phi(b) \), for any \( a, b \in M \). We have \( \Phi(a') = (\frac{a}{I_1}, \cdots, \frac{a}{M_i}) = (\phi(a))' \). Hence, \( \phi \) is an \( MV \)-homomorphism.

\( i \) Let \( \phi \) be an \( MV \)-monomorphism. Then by Lemma 2.3(iv), \( Ker(\phi) = \{0\} \). If \( a \in \sum_{i=1}^n I_i \), then \( a \in I_i \) and so \( d(a,0) = a \in I_i \), for any \( 1 \leq i \leq n \). It means that \( \frac{a}{I_i} = \frac{a}{I_i} \) and so \( \Phi(a) = (\frac{a}{I_1}, \cdots, \frac{a}{I_n}) = (\frac{0}{I_1}, \cdots, \frac{0}{I_n}) = 0 \). Hence, \( a \in Ker(\phi) = \{0\} \). It results that \( \sum_{i=1}^n I_i = \{0\} \). Similarly, if \( \sum_{i=1}^n I_i = \{0\} \), then \( Ker(\phi) = \{0\} \) and so \( \phi \) is an \( MV \)-monomorphism.

\( ii \) Let \( \phi \) be an \( MV \)-epimorphism. We show that \( \cap I_i \cap I_j \geq M \). Since \( 0,1 \in M \), we have \( (\frac{1}{I_1}, \frac{0}{I_2}, \cdots, \frac{0}{I_n}) \in \sum_{i=1}^n M_i \). Since \( \phi \) is onto, there exists \( x \in M \) such that \( \phi(x) = (\frac{1}{I_1}, \cdots, \frac{0}{I_n}) \). It results that \( x' = d(1, x) \in I_1 \), \( x = d(0, x) \in I_2 \) and so \( 1 = x' \oplus x \in I_1 \oplus I_2 \). It means that \( \cap I_1 \cap I_2 \geq M \). Similarly, we can show that \( \cap I_i \cap I_j \geq M \), for any \( 1 \leq i, j \leq n \).

\( iii \) Let \( y = (\frac{x_a}{M_i}, \cdots, \frac{x_n}{M_i}) \in \sum_{i=1}^n M_i \). Then we consider \( x = \bigwedge_{i=1}^n x_i \). Since \( x \leq x_i \in I_i \), we have \( x \in I_i \). Since \( d(x, x_i) = (x \ominus x_i) \oplus (x_i \ominus x) = 0 \oplus (x_i \ominus x) \in I_i \), we have \( \frac{x}{I_i} = \frac{x_i}{I_i} \) for any \( 1 \leq i \leq n \). It means that \( \phi(x) = (\frac{x}{I_1}, \cdots, \frac{x}{I_n}) = (\frac{x_1}{I_1}, \cdots, \frac{x_n}{M_i}) = y \). Therefore, \( \phi \) is an \( MV \)-epimorphism.

6 Conclusion

We obtained some new results in ideals theory and opened new fields to anyone that is interested to studying and development of ideals in \( MV \)-algebras.

References


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