Duality of $g$-Bessel sequences and some results about RIP $g$-frames

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Abstract

In this paper, we develop the duality concept for $g$-Bessel sequences and Bessel fusion sequences in Hilbert spaces. We obtain some results about dual, pseudo-dual and approximate dual of frames and fusion frames. We also expand every $g$-Bessel sequence to a frame by summing some elements. We define the restricted isometry property for $g$-frames and generalize some results from (B. G. Bodmann et al, Fusion frames and the restricted isometry property, Num. Func. Anal. Optim. 33 (2012) 770-790) to $g$-frame situation. Finally we study the stability of $g$-frames under erasure of operators.

Keywords: $G$-frames; Fusion frames; Dual frames; Pseudo-dual frames; Approximate dual frames; Bessel sequences.

1 Introduction

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $\{W_i\}_{i \in I}$ be a sequence of closed subspaces of $\mathcal{K}$, where $I$ is a subset of $\mathbb{Z}$. For any frame $\{f_i\}_{i \in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i \in I}$ for which

$$f = \sum_{i \in I} < f, g_i > f_i \quad \forall f \in \mathcal{H}.$$  

If $\{f_i\}_{i \in I}$ is a Bessel sequence with bound $B < 1$, how can we find two sequences $\{g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ such that $\{f_i + g_i\}_{i \in I}$ and $\{p_i\}_{i \in I}$ are dual frames, i.e., such that

$$f = \sum_{i \in I} < f, p_i > (f_i + g_i)$$

$$= \sum_{i \in I} < f, f_i + g_i > p_i,$$

for all $f \in \mathcal{H}$. In this paper we obtain some the more general results of the type (1). Let $\mathcal{L}(\mathcal{H}, W_i)$ be the collection of all bounded linear operators from $\mathcal{H}$ into $W_i$. Recall that a family of operators $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : i \in I\}$ is said to be a generalized frame, or simply a $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq D \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1.1)$$

The constants $C$ and $D$ are called $g$-frame bounds and $\sup_{i \in I} \Lambda_i$ is called the multiplicity of the $g$-frame. We call $\Lambda$ a tight $g$-frame if $C = D$ and it is a Parseval $g$-frame if $C = D = 1$. $\Lambda$ is called a $\varepsilon$-$g$-frame for $\mathcal{H}$ if $C = \frac{1}{1+\varepsilon}$ and $D = 1+\varepsilon$ for some $\varepsilon > 0$. If the right-hand side of (1.1) holds, then $\Lambda$ is said a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. The representation space associated with a $g$-Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined.
by
\[
\left\{ \{g_i\}_{i \in I} \mid g_i \in W_i, \sum_{i \in I} \|g_i\|^2 < \infty \right\}.
\]
The synthesis operator of $\Lambda$ is defined by
\[
T_{\Lambda} : \left( \sum_{i \in I} \oplus W_i \right)_{\ell^2} \to \mathcal{H}
\]
\[
T_{\Lambda} \{g_i\}_{i \in I} = \sum_{i \in I} \Lambda_i^* g_i.
\]
The adjoint operator of $T_{\Lambda}$, which is called the analysis operator also obtain as follows
\[
T_{\Lambda}^* : \mathcal{H} \to \left( \sum_{i \in I} \oplus W_i \right)_{\ell^2}
\]
\[
T_{\Lambda}^* f = \{\Lambda_i f\}_{i \in I}.
\]
By composing $T_{\Lambda}$ with its adjoint $T_{\Lambda}^*$, we obtain the fusion frame operator
\[
S_{\Lambda} : \mathcal{H} \to \mathcal{H}
\]
\[
S_{\Lambda} f = T_{\Lambda} T_{\Lambda}^* f = \sum_{i \in I} \Lambda_i^* \Lambda_i f,
\]
which is a bounded, self-adjoint, positive and invertible operator and $C_{\mathcal{H}} \leq S_{\Lambda} \leq D_{\mathcal{H}}$. The canonical dual $g$-frame for $\{\Lambda_i\}_{i \in I}$ is defined by $\{\Lambda_i\}_{i \in I}$ with $\tilde{\Lambda}_i = \Lambda_i S_{\Lambda}^{-1}$, which is also a $g$-frame for $\mathcal{H}$ with $g$-frame bounds $\frac{1}{T}$ and $\frac{1}{S}$, respectively. Also we have
\[
f = \sum_{i \in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i \in I} \tilde{\Lambda}_i^* \Lambda_i f \quad \forall f \in \mathcal{H}.
\]

For more details about the theory and applications of frames we refer the readers to [1, 8, 9, 10, 11] and for fusion frames to [2, 4, 5, 7], about $g$-frames to [3, 12, 13].

The paper is organized as follows: Section 2, contains an extension of $g$-Bessel sequences to dual $g$-frames. In this Section, we consider the dual, pseudo-dual and approximate dual frames, fusion frames and we obtain several characterizations of all this dual frames. In Section 3, we generalize the restricted isometry property to the $g$-frame situation. In Section 4, we study the conditions which under removing some element from a $g$-frame, again we obtain another $g$-frame.

## 2 Dual, approximate dual and pseudo-dual of $g$-frames

Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be $g$-Bessel sequences for $\mathcal{H}$ with synthesis operators $T_{\Lambda}$ and $T_{\Gamma}$ respectively. Then we say that $\Lambda$ and $\Gamma$ are dual $g$-frames for $\mathcal{H}$ if $T_{\Lambda} T_{\Gamma}^* = I_{\mathcal{H}}$ or $T_{\Gamma} T_{\Lambda}^* = I_{\mathcal{H}}$.

In the following we show that any pair of $g$-Bessel sequences can be extended to pair of dual $g$-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [9] to the situation of $g$-frames.

**Theorem 2.1** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ and $\Gamma = \{\Gamma_i\}_{i \in I}$ be two $g$-Bessel sequences for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. Then there exist $g$-Bessel sequences $\{\Xi_j\}_{j \in J}$ and $\{\Omega_j\}_{j \in J}$ for $\mathcal{H}$ with respect to $\{V_j\}_{j \in J}$, such that $\{\Lambda_i\}_{i \in I} \cup \{\Xi_j\}_{j \in J}$ and $\{\Gamma_i\}_{i \in I} \cup \{\Omega_j\}_{j \in J}$ form a pair of dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I} \cup \{V_j\}_{j \in J}$.

**Proof.** Assume that $\{\Phi_j\}_{j \in J}$ and $\{\Psi_j\}_{j \in J}$ are any pair of dual $g$-frames for $\mathcal{H}$ with respect to $\{V_j\}_{j \in J}$ and let $\Theta = I_{\mathcal{H}} - T_{\Gamma} T_{\Lambda}^*$. Then for any $f \in \mathcal{H}$ we have
\[
f = \Theta f + T_{\Gamma} T_{\Lambda}^* f
\]
\[
= \sum_{j \in J} \Psi_j^* \Phi_j \Theta f + \sum_{i \in I} \Gamma_i^* \Lambda_i f.
\]
If we set $\Xi_j = \Phi_j \Theta$ and $\Omega_j = \Psi_j$ for all $j \in J$. Then $\{\Lambda_i\}_{i \in I} \cup \{\Xi_j\}_{j \in J}$ and $\{\Gamma_i\}_{i \in I} \cup \{\Omega_j\}_{j \in J}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I} \cup \{V_j\}_{j \in J}$.

**Theorem 2.2** Let $\mathcal{F}$ be a Bessel sequence for $\mathcal{H}$ with Bessel bound $B < 1$ and let $\mathcal{E}$ be Parseval frame for $\mathcal{H}$. Then there exists a Bessel sequence $\mathcal{G}$ for $\mathcal{H}$ such that $\mathcal{F} + \mathcal{E}$ and $\mathcal{G} + \mathcal{E}$ are dual frames.

Let $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{E} = \{e_i\}_{i \in I}$. Since $B < 1$, $I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*$ is an invertible operator in $\mathcal{L}(\mathcal{H})$. If we define
\[
\Theta = -(I_{\mathcal{H}} + T_{\mathcal{F}} T_{\mathcal{E}}^*)^{-1} T_{\mathcal{F}} T_{\mathcal{E}}^*
\]
and $g_i = \Theta^* e_i$ for all $i \in I$. Then $\mathcal{G} = \{g_i\}_{i \in I}$ is a
Bessel sequence for $\mathcal{H}$ and for all $f \in \mathcal{H}$ we have

$$f = (I_{\mathcal{H}} + T_f T_f^* f + T_f^* T_f f) \Theta f + T_f T_f^* f + T_f^* T_f f$$

$$= \sum_{i \in I} \Theta f, e_i > e_i + \sum_{i \in I} f, e_i > e_i$$

$$+ \sum_{i \in I} \Theta f, e_i > f_i + \sum_{i \in I} f, e_i > f_i$$

$$= \sum_{i \in I} f, g_i > e_i > (f_i + e_i),$$

which this finishes the proof. The following corollaries are generalizations of Theorem 2.2 to the $g$-frames situation. We leave the proofs to interested readers.

**Corollary 2.1** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be a $g$-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with $g$-Bessel bound $B < 1$. Then there exists a $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, such that $\{\Xi_i + \Lambda_i\}_{i \in I}$ and $\{\Xi_i + \Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, where $\{\Xi_i\}_{i \in I}$ is a Parseval $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

**Corollary 2.2** For every $g$-Bessel sequence $\Lambda = \{\Lambda_i\}_{i \in I}$ with Bessel bound $B < 1$ and each Parseval $g$-frame $\Xi = \{\Xi_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, there exists a $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ such that $\{\Xi_i + \Lambda_i\}_{i \in I}$ and $\{\Xi_i + \Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

**Corollary 2.3** For every $g$-Bessel sequence $\{\Lambda_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ there exist $g$-Bessel sequence $\{\Gamma_i\}_{i \in I}$ and a tight $g$-frame $\{\Xi_i\}_{i \in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ such that $\{\Xi_i + \Lambda_i\}_{i \in I}$ and $\{\Xi_i + \Gamma_i\}_{i \in I}$ are dual $g$-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$.

Let $\mathcal{W} = \{W_i\}_{i \in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $\mathcal{A} = \{\alpha_i\}_{i \in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$. A sequence $\mathcal{W}_a = \{(W_i, \alpha_i)\}_{i \in I}$ is a fusion frame, if there exist real numbers $0 < C \leq D < \infty$ such that for all $f \in \mathcal{H}$:

$$C \|f\|^2 \leq \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \leq D \|f\|^2, \quad (2.2)$$

where $\pi_{W_i}$ is the orthogonal projection from $\mathcal{H}$ onto $W_i$. The constant $C, D$ are called the fusion frame bounds. If the right-hand inequality of $(2.2)$ holds, then we say that $\mathcal{W}_a$ is a Bessel fusion sequence with Bessel fusion bound $D$. Moreover if $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ is a frame for $W_i$ for all $i \in I$. Then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is called a fusion frame system for $\mathcal{H}$. The constants $A, B$ are called the local frame bounds if they are the common frame bounds for the local frame $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ for all $i \in I$. A collection of dual frames $\mathcal{G}_i = \{g_{ij}\}_{j \in J_i}$, $i \in I$ associated with the local frames is called local dual frames. By Theorem 3.2 from [7], if $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for $\mathcal{H}$ with frame bounds $C, D$ and local frame bounds $A, B$, then $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $AC$ and $BD$. Also if $\mathcal{F} = \{\alpha_i f_{ij}\}_{i \in I, j \in J_i}$ is a frame for $\mathcal{H}$ with frame bounds $C$ and $D$, then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i \in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $\frac{C}{\alpha_i}$ and $\frac{D}{\alpha_i}$.

**Definition 2.1** Let $\mathcal{W}_a = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be Bessel fusion sequences for $\mathcal{H}$ with synthesis operators $T_{\mathcal{W}_a}$ and $T_{\mathcal{Z}_\beta}$ respectively. Then

(i) $\mathcal{W}_a, \mathcal{Z}_\beta$ are dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_a} T_{\mathcal{Z}_\beta}^* = I_{\mathcal{H}}$ or $T_{\mathcal{Z}_\beta} T_{\mathcal{W}_a}^* = I_{\mathcal{H}}$.

(ii) $\mathcal{W}_a, \mathcal{Z}_\beta$ are approximate dual fusion frames for $\mathcal{H}$ if $\|I_{\mathcal{H}} - T_{\mathcal{W}_a} T_{\mathcal{Z}_\beta}^*\| < 1$ or $\|I_{\mathcal{H}} - T_{\mathcal{Z}_\beta} T_{\mathcal{W}_a}^*\| < 1$.

(iii) $\mathcal{W}_a, \mathcal{Z}_\beta$ are called pseudo-dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_a} T_{\mathcal{Z}_\beta}^* \neq I_{\mathcal{H}}$ or $T_{\mathcal{Z}_\beta} T_{\mathcal{W}_a}^* \neq I_{\mathcal{H}}$.

**Theorem 2.3** For each $i \in I$ let $\alpha_i > 0$ and $J_i = J_{i1} \cup J_{i2}$ be a partition of $J_i$ and let $\mathcal{W} = \{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ and $\mathcal{Z} = \{(Z_i, \beta_i, \{g_{ij}\}_{j \in J_i})\}_{i \in I}$ be two fusion frame systems for $\mathcal{H}$. Define

$$u_{ij} = \begin{cases} \frac{1}{\sqrt{2}} f_{ij}, & j \in J_{i1} \\ \frac{1}{\sqrt{2}} \pi_{W_i} \tilde{g}_{ij}, & j \in J_{i2} \end{cases}$$

and 

$$v_{ij} = \begin{cases} \frac{1}{\sqrt{2}} \pi_{Z_i} \tilde{f}_{ij}, & j \in J_{i1} \\ \frac{1}{\sqrt{2}} g_{ij}, & j \in J_{i2} \end{cases}$$

for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

1. $\mathcal{W}_a = \{(W_i, \alpha_i)\}_{i \in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.
(2) $\{a_iu_{ij}\}_{i \in I, j \in J_i}$ and $\{b_iu_{ij}\}_{i \in I, j \in J_i}$ are (dual, pseudo-dual, approximate dual) frames for $H$.

**Proof.** This claim follows immediately from the fact that for $f \in H$ we have

$$\sum_{i \in I} \sum_{j \in J_i} <f, \beta_i v_{ij}> > a_i u_{ij}$$

$$= \sum_{i \in I} \sum_{j \in J_{i1}} <f, v_{ij}> > u_{ij}$$

$$+ \sum_{i \in I} \sum_{j \in J_{i2}} <f, v_{ij}> > u_{ij}$$

$$= \sum_{i \in I} \sum_{j \in J_{i1}} <f, \frac{1}{\sqrt{2}} \pi Z_i \tilde{f}_{ij}> > \frac{1}{\sqrt{2}} f_{ij}$$

$$+ \sum_{i \in I} \sum_{j \in J_{i2}} <f, \frac{1}{\sqrt{2}} g_{ij}> > \frac{1}{\sqrt{2}} \pi W_i \tilde{g}_{ij}$$

$$= \sum_{i \in I} \sum_{j \in J_{i1}} <f, \frac{1}{\sqrt{2}} \pi Z_i(f), \tilde{f}_{ij}> > f_{ij}$$

$$+ \sum_{i \in I} \sum_{j \in J_{i2}} <f, \frac{1}{\sqrt{2}} \pi W_i(f), \tilde{g}_{ij}> > \tilde{g}_{ij}$$

$$= \sum_{i \in I} \sum_{j \in J_{i2}} <f, \pi Z_i(f), \tilde{f}_{ij}> > \pi Z_i(f)$$

Theorem 2.4 Let $\{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$ be a fusion frame system and let $Z_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ be a fusion Bessel sequence for $H$. Put $g_{ij} = \pi Z_i(\tilde{f}_{ij})$ for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

(1) $W_\alpha = \{(W_i, \alpha_i)\}_{i \in I}$ and $Z_\beta = \{(Z_i, \beta_i)\}_{i \in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.

(2) $F = \{a_if_{ij}\}_{i \in I, j \in J_i}$ and $G = \{b_ig_{ij}\}_{i \in I, j \in J_i}$ are (dual, pseudo-dual, approximate dual) frames for $H$.

**Proof.** First we prove that $G$ is a Bessel sequence for $H$. Let $D$ be the Bessel fusion bound of $Z_\beta$ and $A, B$ be the local frame bounds of $\{(W_i, \alpha_i, \{f_{ij}\}_{j \in J_i})\}_{i \in I}$, then for all $f \in H$ we have

$$\sum_{i \in I} \sum_{j \in J_i} <f, \beta_i g_{ij}> > 2$$

$$= \sum_{i \in I} \sum_{j \in J_i} \beta_i^2 <f, \pi Z_i(\tilde{f}_{ij})> > 2$$

$$= \sum_{i \in I} \sum_{j \in J_i} \beta_i^2 <\pi Z_i(f), \tilde{f}_{ij}> > 2$$

$$\leq \frac{1}{2} \sum_{i \in I} \beta_i^2 \|\pi W_i \pi Z_i(f)\|^2$$

$$\leq \frac{1}{2} \sum_{i \in I} \beta_i^2 \|\pi Z_i(f)\|^2 \leq \frac{D}{2} \|f\|^2.$$
From this the result follows at once.

**Theorem 2.6** Let \( \mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I} \) be a fusion frame and let \( \mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I} \) be a Bessel fusion sequence for \( \mathcal{H} \). Suppose that \( T: \mathcal{H} \rightarrow \mathcal{H} \) is a bounded invertible operator such that \( T W_i \subseteq Z_i \) for all \( i \in I \). Then \( \mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i \in I} \) and \( T \mathcal{W}_\alpha = \{(TW_i, \alpha_i)\}_{i \in I} \) are pseudo-dual fusion frames for \( \mathcal{H} \). Moreover if \( T \mathcal{W}_\alpha \) is a Parseval fusion frame then \( \mathcal{Z}_\alpha \) and \( T \mathcal{W}_\alpha \) are dual fusion frames.

**Proof.** Since \( TW_i \subseteq Z_i \) hence \( \pi_{TW_i} \pi_{Z_i} = \pi_{Z_i} \pi_{TW_i} = \pi_{TW_i} \) for all \( i \in I \). It follows that \( T \pi_{TW_i} T^*_{Z_i} = T \pi_{Z_i} T^*_{TW_i} = S_{TW_i} \) which finishes the proof.

**Definition 2.2** Let \( \{W_i\}_{i \in I} \) and \( \{\tilde{W}_i\}_{i \in I} \) be closed subspaces in \( \mathcal{H} \) and \( \varepsilon > 0 \). If for every \( f \in \mathcal{H} \) we have

\[
\sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\tilde{W}_i}(f)\|^2 \leq \varepsilon \|f\|^2.
\]

Then we say that \( \{(\tilde{W}_i, \alpha_i)\}_{i \in I} \) is a \( \varepsilon \)-perturbation of \( \{(W_i, \alpha_i)\}_{i \in I} \).

**Theorem 2.7** Let \( \mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i \in I} \), \( \mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i \in I} \) be Bessel fusion sequences with Bessel fusion bounds \( D_1, D_2 \) respectively for \( \mathcal{H} \). Let \( \tilde{\mathcal{W}}_\alpha = \{(\tilde{W}_i, \alpha_i)\}_{i \in I} \) be a \( \varepsilon \)-perturbation of \( \mathcal{W}_\alpha \) and \( \varepsilon D_2 < 1 \). If \( \mathcal{W}_\alpha \) and \( \mathcal{Z}_\beta \) are dual fusion frames, then \( \mathcal{W}_\alpha \) and \( \mathcal{Z}_\beta \) are also approximate dual fusion frames for \( \mathcal{H} \).

**Proof.** By Proposition 2.4 from [4] \( \tilde{\mathcal{W}}_\alpha \) is a Bessel fusion sequence for \( \mathcal{H} \). Now for all \( f \in \mathcal{H} \) we have

\[
\|f - T_{\tilde{\mathcal{W}}_\alpha} T^*_{\tilde{\mathcal{Z}}_\alpha}(f)\|^2 \\
= \|T \pi_{\mathcal{W}_\alpha} T^*_{\mathcal{Z}_\alpha}(f) - T \pi_{\tilde{\mathcal{Z}}_\alpha} T^*_{\mathcal{W}_\alpha}(f)\|^2 \\
\leq \sup_{\|g\|=1} \left| T \pi_{\mathcal{W}_\alpha} T^*_{\mathcal{Z}_\alpha}(f) - T \pi_{\tilde{\mathcal{Z}}_\alpha} T^*_{\mathcal{W}_\alpha}(f), g > \right|^2 \\
\leq \sup_{\|g\|=1} \left( \sum_{i \in I} \alpha_i \beta_i \|\pi_{W_i}(f) - \pi_{\tilde{W}_i}(f)\| \|\pi_{Z_i}(g)\| \right)^2 \\
\leq \sup_{\|g\|=1} \left( \sum_{i \in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\tilde{W}_i}(f)\|^2 \right) \\
\times \sum_{i \in I} \beta_i^2 \|\pi_{Z_i}(g)\|^2 \leq \varepsilon D_2 \|f\|^2.
\]

This yields

\[
A \dim \mathcal{H} \leq \sum_{i \in I} \|\Lambda_i\|_{HS}^2 \leq B \dim \mathcal{H}.
\]

From this the claim follows immediately.

**Theorem 3.1** Let \( \Lambda = \{\Lambda_i\}_{i=1}^M \) be a g-frame for \( \mathcal{H}_N \) with respect to \( \{W_i\}_{i=1}^M \). Then

(i) The optimal g-frame bounds of \( \Lambda \) are the smallest and biggest eigenvalues of g-frame operator \( S_\Lambda \).

(ii) If \( \{\lambda_i\}_{i=1}^N \) is a representation of eigenvalues of \( S_\Lambda \). Then

\[
\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\Lambda_i\|_{HS}^2.
\]
and

$$\lambda_j = \sum_{i=1}^{M} \| \Lambda_i e_j \|^2,$$

where \( \{ e_j \}_{j=1}^{N} \) is the orthonormal basis consisting of eigenvectors of \( S_A \).

**Proof.** To prove (i), since \( S_A \) is a self-adjoint, \( \mathcal{H}_N \) has an orthonormal basis include eigenvectors of \( S_A \). Let \( \{ e_j \}_{j=1}^{N} \) be an orthogonal basis of \( \mathcal{H}_N \) include of eigenvectors of \( S_A \). Let \( \{ \lambda_j \}_{j=1}^{N} \) be eigenvalues of \( \{ e_j \}_{j=1}^{N} \). Then for any \( f \in \mathcal{H}_N \) we have

$$\sum_{i=1}^{M} \| \Lambda_i f \|^2 = < S_A f, f > = \sum_{j=1}^{N} < f, e_j > S_A e_j, f > = \sum_{j=1}^{N} < f, e_j > < S_A e_j, f > = \sum_{j=1}^{N} < f, e_j > < \lambda_j e_j, f > = \sum_{j=1}^{N} \lambda_j | < f, e_j > |^2.$$

Now from

$$\lambda_{\text{min}} \leq \lambda_i \leq \lambda_{\text{max}}, \quad (1 \leq i \leq N)$$

we obtain

$$\lambda_{\text{min}} \| f \|^2 \leq \sum_{i=1}^{M} \| \Lambda_i f \|^2 \leq \lambda_{\text{max}} \| f \|^2.$$

To prove (ii) we have:

$$\sum_{j=1}^{N} \lambda_j = \sum_{j=1}^{N} < S_A e_j, e_j > = \sum_{j=1}^{N} \sum_{i=1}^{M} \| \Lambda_i e_j \|^2 = \sum_{i=1}^{M} \sum_{j=1}^{N} \| \Lambda_i e_j \|^2 = \sum_{i=1}^{M} \| \Lambda_i \|^2_{HS}.$$

We also have

$$\lambda_j = < \lambda_j e_j, e_j > = < S_A e_j, e_j > = \sum_{i=1}^{M} \| \Lambda_i e_j \|^2.$$

**Corollary 3.1** Let \( \{ \Lambda_i \}_{i=1}^{M} \) be an A-tight g-frame for \( \mathcal{H}_N \) with respect to \( \{ W_i \}_{i=1}^{M} \) and \( \| \Lambda_i \|_{HS} = 1 \) for all \( 1 \leq i \leq M. \) Then \( A = \frac{M}{N} \).

**Proof.** This is a direct result from Proposition 3.1.

**Definition 3.1** Let \( \Lambda_i \in \mathcal{L} (\mathcal{H}, W_i) \) for all \( i \in I. \) Then

(i) \( \{ \Lambda_i \}_{i \in I} \) is called an orthonormal g-system for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \), if \( \Lambda_i \Lambda_i^* g_j = \delta_{ij} g_j \) for all \( i, j \in I, g_j \in W_j. \)

(ii) If \( \mathcal{H} = \{ \Lambda_i^* (W_i) \}_{i \in I} \), then we say that \( \{ \Lambda_i \}_{i \in I} \) is g-complete.

(iii) We say that \( \{ \Lambda_i \}_{i \in I} \) is a g-orthonormal basis for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \), if it is a g-orthonormal g-complete system for \( \mathcal{H} \) with respect to \( \{ W_j \}_{j \in J} \).

(iv) \( \{ \Lambda_i \}_{i \in I} \) is called a g-Riesz basis for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \), if \( \{ \Lambda_i \}_{i \in I} \) is g-complete and there exist real numbers \( 0 < A \leq B < \infty \) such that:

$$A \sum_{j \in J} \| g_j \|^2 \leq \sum_{j \in J} \| \Lambda_i^* g_j \|^2 \leq B \sum_{j \in J} \| g_j \|^2,$$

for all finite subset \( J \subset I \) and \( g_j \in W_j \). Moreover, \( \{ \Lambda_i \}_{i \in I} \) is called an \( \varepsilon \)-g-Riesz basis for \( \mathcal{H} \), if \( A = \frac{1}{1+\varepsilon} \) and \( B = 1 + \varepsilon \) for some \( \varepsilon > 0 \). Also \( \{ \Lambda_i \}_{i \in I} \) is an \( \varepsilon \)-g-Riesz sequence if \( \{ \Lambda_i \}_{i \in I} \) is an \( \varepsilon \)-g-Riesz basis for \( \{ \Lambda_i^* (W_i) \}_{i \in I} \).

The next proposition is similar to a result of Bodmann, Cahill and Casazza [6] to the situation of g-frames.

**Proposition 3.2** Let \( \{ \Lambda_i \}_{i \in I} \) be an \( \varepsilon \)-g-Riesz sequence for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \) and let \( \{ I_j \}_{j=1}^{L} \) be a partition of \( I. \) Then

$$\frac{1}{1+\varepsilon} \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k g_{jk} \|^2 \leq \sum_{j=1}^{L} \| \sum_{k \in I_j} g_{jk} \|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \| \sum_{k \in I_j} \Lambda_k^* g_{jk} \|^2,$$

where \( g_{jk} \) is the projection of \( g_j \) onto \( S_{I_j} \).
for every $1 \leq j \leq L$ and any sequence $\{g_{jk}\}_{k\in I_j} \in (\sum_{k\in I_j} + W_k)_{l^2}$. Also
\[
\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq \left\| \sum_{j=1}^{L} \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq (1+\varepsilon)^2 \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2.
\]

**Proof.** Let $1 \leq j \leq L$ and $\{g_{jk}\}_{k \in I_j} \in (\sum_{k \in I_j} + W_k)_{l^2}$
\[
\frac{1}{1+\varepsilon} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq \frac{1}{1+\varepsilon} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} g_{jk} \right\|^2 \leq \frac{1}{1+\varepsilon} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2.
\]
This yields
\[
\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq \frac{1}{1+\varepsilon} \sum_{j=1}^{L} \left\| \sum_{k \in I_j} g_{jk} \right\|^2 \leq (1+\varepsilon) \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2 \leq (1+\varepsilon)^2 \sum_{j=1}^{L} \left\| \sum_{k \in I_j} \Lambda_k^* g_{jk} \right\|^2.
\]

It is known that if $\{\Lambda_i\}_{i \in I}$ is a $\varepsilon$-$g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, then $\{\Lambda_i\}_{i \in I}$ is a $g$-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$ with bounded bounds $A$ and $B$. The next lemma is analogous to Lemma 3.3 in [6] to the situation of $g$-frames.

**Lemma 3.1.** Let $\Lambda = \{\Lambda_i\}_{i \in I}$ be an $\varepsilon$-$g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. Then for all $n \in \mathbb{N}$
\[
\frac{1}{(1+\varepsilon)^n} \left\| \sum_{i \in I} \Lambda_i^* s_i \right\|^2 \leq (1+\varepsilon)^n \left\| \sum_{i \in I} \Lambda_i^* s_i \right\|^2 \leq (1+\varepsilon)^n \left\| \sum_{i \in I} \Lambda_i^* s_i \right\|^2.
\]

**Proof.** Since $\{\Lambda_i\}_{i \in I}$ is an $\varepsilon$-$g$-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$, so this family is a $g$-frame for $\mathcal{H}$ with bounds $1/(1+\varepsilon)^n$ and $1/(1+\varepsilon)^n$. Hence $1/(1+\varepsilon)^n \leq \left\| S_{\Lambda} \right\| \leq (1+\varepsilon)^n$ and $1/(1+\varepsilon)^n \leq \left\| S_{\Lambda}^{-1} \right\| \leq (1+\varepsilon)^n$. On the other hand, for any $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have $\left\| S_{\Lambda}^{-1} \right\| \left\| f \right\| \leq \left\| S_{\Lambda} \right\| \left\| f \right\|$. From this we have $\left\| S_{\Lambda}^{-1} \right\| \left\| f \right\| \leq \left\| S_{\Lambda} \right\| \left\| f \right\|$. Consequently
\[
\frac{1}{(1+\varepsilon)^n} \left\| S_{\Lambda}^{-1} \right\| \left\| f \right\| \leq (1+\varepsilon)^n \left\| f \right\| \leq (1+\varepsilon)^n \left\| f \right\|.
\]
This shows that $\frac{1}{(1+\varepsilon)^n} \left\| f \right\| \leq (1+\varepsilon)^n \left\| f \right\|$ and so $\frac{1}{(1+\varepsilon)^n} \left\| f \right\| \leq (1+\varepsilon)^n I_{\mathcal{H}}$. Therefore
\[
\left\| <f, g> \right\| \leq 2\varepsilon + \varepsilon^2.
\]

**Proposition 3.3.** Let $\{\Lambda_i\}_{i \in I}$ be an $\varepsilon$-$g$-Riesz sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i \in I}$. Then
\[
\left\| <f, g> \right\| \leq 2\varepsilon + \varepsilon^2,
\]
for all partition $\{I_1, I_2\}$ of $I$ and $f \in \mathcal{H}$, $g \in \mathcal{H}$. Then we have
\[
\left\| <\varphi, \psi> \right\| \leq \frac{2}{1+\varepsilon} \sum_{i \in F_1 \cup F_2} \left\| g_i \right\|^2 - 1 \leq \frac{2}{1+\varepsilon} \sum_{i \in F_1 \cup F_2} \left\| g_i \right\|^2 - 1 \leq \frac{2}{1+\varepsilon} \sum_{i \in F_1 \cup F_2} \left\| g_i \right\|^2 - 1 \leq \frac{2}{1+\varepsilon} \left\| \varphi \right\|^2 + \left\| \psi \right\|^2 - 1 \leq 2\varepsilon + \varepsilon^2.
\]
This yields
\[
\left\| <\varphi, \psi> \right\| \leq 2\varepsilon + \varepsilon^2,
\]
which implies that $\left\| <f, g> \right\| \leq 2\varepsilon + \varepsilon^2$. 

**Definition 3.2.** For every $1 \leq i \leq M$, let $\Lambda_i \in \mathcal{L}(\mathcal{H}_i, W_i)$. Then we say that the family $\{\Lambda_i\}_{i \in I}$ has the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$, if for every $I \subseteq \{1, 2, ..., M\}$ with $|I| \leq s$, the family $\{\Lambda_i\}_{i \in I}$ is an $\varepsilon$-$g$-Riesz sequence for $\mathcal{H}$, with respect to $\{W_i\}_{i \in I}$. 

The next theorem is a generalization of Theorem 4.2 in [6] to the g-frames situation.

**Theorem 3.2** Let \( \{\Lambda_i\}_{i=1}^M \) be a tight g-frame for \( \mathcal{H}_N \) with respect to \( \{W_i\}_{i=1}^M \) with the restricted isometry property with constant \( 0 < \varepsilon < 1 \) for sets of size \( s \leq N \). Suppose that \( \{I_j\}_{j=1}^L \) is an arbitrary partition of \( \{1, 2, \ldots, M\} \) with \( |I_j| \leq s \). Define \( V_j = \{\Lambda_i^*(W_i)\}_{i \in I_j} \) for all \( 1 \leq j \leq L \), then \( \{V_j\}_{j=1}^L \) is a fusion frame for \( \mathcal{H}_N \) with fusion frame bounds \( \sum_{i=1}^M \|\Lambda_i\|_{HS}^2, (1+\varepsilon)\sum_{i=1}^M \|\Lambda_i\|_{HS}^2 \) and

\[
\frac{1}{1+\varepsilon} \sum_{i \in I_j} \|\Lambda_i f\|^2 \leq \|\pi_V f\|^2 \leq (1+\varepsilon) \sum_{i \in I_j} \|\Lambda_i f\|^2.
\]

**Proof.** By the hypothesis \( \{\Lambda_i\}_{i \in I_j} \) is a g-frame for \( V_j \) with respect to \( \{W_i\}_{i \in I_j} \) for all \( 1 \leq j \leq L \) with g-frame bounds \( \frac{1}{1+\varepsilon}, 1 \) respectively. Let \( S_j \) be g-frame operator of \( \{\Lambda_i\}_{i \in I_j} \) and \( \{\lambda_i\}_{i \in N}^N \) be the orthonormal basis of eigenvectors of \( S_j \) with eigenvalues \( \lambda_i \), then \( \lambda_i = 0 \) for all \( |I_j| \leq N \) and \( (\lambda_1 \lambda_2 \ldots \lambda_{|I_j|}) \leq 1 + \varepsilon \). Since \( \{\lambda_i\}_{i \in I_j} \) is an orthonormal basis for \( V_j \), hence

\[
\pi_V f = \sum_{i=1}^N <f, e_i>e_i,
\]

for any \( f \in \mathcal{H}_N \). Now we have

\[
S_j f = S_j(\sum_{i=1}^N <f, e_i>e_i) = \sum_{i=1}^N <f, e_i>e_i = \sum_{i=1}^|I_j| <f, e_i> \lambda_i e_i
\]

which implies that

\[
|<S_j f, f>| = \sum_{i=1}^|I_j| \lambda_i|<f, e_i>|^2
\]

Thus we have

\[
\frac{1}{1+\varepsilon} \sum_{i \in I_j} \|\Lambda_i f\|^2 = \frac{1}{1+\varepsilon} <S_j f, f> = \sum_{i \in I_j} \lambda_i |<f, e_i>|^2 \leq \|\pi_V f\|^2
\]

\[
\leq \sum_{i \in I_j} \lambda_i(1+\varepsilon)|<f, e_i>|^2 = (1+\varepsilon) <S_j f, f> = (1+\varepsilon) \sum_{i \in I_j} \|\Lambda_i f\|^2.
\]

It follows that

\[
\frac{1}{1+\varepsilon} \sum_{j=1}^L \sum_{i \in I_j} \|\Lambda_i f\|^2 \leq \sum_{j=1}^L \|\pi_V f\|^2
\]

\[
\leq (1+\varepsilon) \sum_{j=1}^L \sum_{i \in I_j} \|\Lambda_i f\|^2.
\]

Now by Proposition 3.1 we have

\[
\sum_{i=1}^M \|\Lambda_i\|_{HS}^2 \|f\|^2 \leq \sum_{j=1}^L \|\pi_V f\|^2
\]

\[
\leq (1+\varepsilon) \sum_{i=1}^M \|\Lambda_i\|_{HS}^2 \|f\|^2.
\]

**Corollary 3.2** Under the assumptions of Theorem 3.2 if

\[
\{1, 2, \ldots, L\} \subseteq \{1, 2, \ldots, M\}
\]

and there exists a family \( \{J_j\}_{j=1}^L \) such that \( \sum_{j=1}^L |J_j| \leq s \) and \( J_j \subset I_j \) for all \( 1 \leq j \leq L \). Then

\[
\frac{1}{(1+\varepsilon)^2} \sum_{j=1}^L \sum_{i \in J_j} \lambda_i^2 g_i^2 \leq \sum_{j=1}^L \sum_{i \in J_j} \lambda_i^2 g_i^2
\]

\[
\leq (1+\varepsilon)^2 \sum_{j=1}^L \sum_{i \in J_j} \lambda_i^2 g_i^2.
\]

**Proof.** This follows from the Proposition 3.2. The following theorem will give another method for obtaining a fusion frame from an unit norm tight frame for \( \mathcal{H}_N \) without having the restricted isometry property. Another form of this result can be found in [6] Theorem 4.2.

**Theorem 3.3** Let \( \{f_i\}_{i=1}^M \) be an unit norm tight frame of vectors for \( \mathcal{H}_N \) and let \( \{I_j\}_{j=1}^L \) be a partition of \( \{1, 2, \ldots, M\} \). Define \( W_j = \{f_i\}_{i \in I_j} \), then the family \( \{W_j\}_{j=1}^L \) is a fusion frame for \( \mathcal{H}_N \) with fusion frame bounds \( B_M A \) and \( B_M A \) where

\[
A = \min_{j=1}^L \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, B = \max_{j=1}^L \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}
\]

and \( \{\lambda_{jk}\}_{k=1}^{\dim W_j} \) is the family of eigenvalues of frame operator associated to \( \{f_i\}_{i \in I_j} \).

**Proof.** Let \( S_j \) be the frame operator associated to \( \{f_i\}_{i \in I_j} \) and let \( \{e_{jk}\}_{k=1}^N \) be the orthonormal
basis for \( \mathcal{H}_N \) of eigenvectors of \( S_j \) with eigenvalues \( \{\lambda_{jk}\}_{k=1}^N \). Then \( \lambda_{jk} = 0 \) for any \( \dim W_j < k \leq N \) and \( \{e_{jk}\}_{k=1}^{\dim W_j} \) is an orthonormal basis for \( W_j \). Thus

\[
< S_j f, f > = \sum_{k=1}^{\dim W_j} \lambda_{jk} |f, e_k|^2.
\]

Now for any \( f \in \mathcal{H}_N \) we have

\[
\min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i \in I_j} < f, f_i >^2
\]

\[
= \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\max_{1 \leq k \leq \dim W_j} \lambda_{jk}} |< f, e_{jk} >|^2
\]

\[
\leq \|\pi_{W_j}\|^2
\]

\[
\leq \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\min_{1 \leq k \leq \dim W_j} \lambda_{jk}} |< f, e_{jk} >|^2
\]

\[
= \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |< S_j f, f_i >|^2
\]

This yields

\[
\sum_{j=1}^{L} \sum_{i \in I_j} \min_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |< f, f_i >|^2
\]

\[
\leq \sum_{j=1}^{L} \|\pi_{W_j} f\|^2
\]

\[
\leq \sum_{j=1}^{L} \sum_{i \in I_j} \max_{1 \leq k \leq \dim W_j} \frac{1}{\lambda_{jk}} |< f, f_i >|^2.
\]

Put

\[
A = \min_{j=1}^{L} \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^{L} \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}.
\]

Then

\[
\frac{AM}{N} \|f\|^2 \leq \sum_{j=1}^{L} \|\pi_{W_j} f\|^2 \leq \frac{BM}{N} \|f\|^2.
\]

The next corollary generalizes Theorem 3.3 to the g-frames situation which the proof leave to interested readers.

**Corollary 3.3** Let \( \{\Lambda_i\}_{i=1}^M \) be a tight g-frame for \( \mathcal{H}_N \) with respect to \( \{W_i\}_{i=1}^M \) and let \( \{I_j\}_{j=1}^L \) be a partition of \( \{1, 2, \ldots, M\} \). Define

\[
V_j = \{\Lambda_i^* (W_i)\}_{i \in I_j}.
\]

Then the family \( \{V_j\}_{j=1}^L \) is a fusion frame for \( \mathcal{H}_N \) with fusion frame bounds

\[
\frac{A \sum_{i=1}^{L} \|\Lambda_i\|^2_{HS}}{N} \quad \text{and} \quad \frac{B \sum_{i=1}^{L} \|\Lambda_i\|^2_{HS}}{N},
\]

where

\[
A = \min_{j=1}^{L} \min_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^{L} \max_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}.
\]

and \( \lambda_{jk} \) is the family of eigenvalues of g-frame operator associated to \( \{\Lambda_i\}_{i \in I_j} \).

### 4 Stability of g-frames

Our purpose of this section is to study the conditions which under removing some element from a g-frame, again we obtain another g-frame. The next theorem gives an erasure result of g-frames so that Theorem 4.3 obtained in [5] is a special case of it.

**Theorem 4.1** Let \( \Lambda = \{\Lambda_i\}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I} \) with g-frame bounds \( A \) and \( B \) and let \( J \subset I \). Then \( \{\Lambda_i\}_{i \in I-J} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{W_i\}_{i \in I-J} \) with bounds

\[
\frac{A^2}{B} \|(I_\mathcal{H} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i)^{-1}\|^{-2} \quad \text{and} \quad B,
\]

if and only if \( I_\mathcal{H} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i \) is a bounded invertible operator on \( \mathcal{H} \).

**Proof.** For any \( f \in \mathcal{H} \) we have

\[
f = \sum_{i \in I} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f = \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f + \sum_{i \in I-J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i f.
\]

Thus

\[
I_\mathcal{H} - \sum_{i \in J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i = \sum_{i \in I-J} S_{\Lambda_i}^{-1} \Lambda_i^* \Lambda_i.
\]
Moreover we have
\[
\| (I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f \| = \left\| \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f \right\|
\]
\[
= \sup_{\|g\|=1} | < \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f, g > |
\]
\[
= \sup_{\|g\|=1} | \sum_{i \in I - J} < \Lambda_i f, \Lambda_i S_{\Lambda}^{-1} g > |
\]
\[
\leq \sup_{\|g\|=1} \sum_{i \in I - J} \| \Lambda_i f \| \| \Lambda_i S_{\Lambda}^{-1} g \|
\]
\[
\leq \sup_{\|g\|=1} ( \sum_{i \in I - J} \| \Lambda_i f \|^2 )^{\frac{1}{2}} ( \sum_{i \in I - J} \| \Lambda_i S_{\Lambda}^{-1} g \|^2 )^{\frac{1}{2}}
\]
\[
\leq \frac{\sqrt{B}}{A} ( \sum_{i \in I - J} \| \Lambda_i f \|^2 )^{\frac{1}{2}}.
\]

Now if \( I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i \) is invertible on \( \mathcal{H} \). Then
\[
\frac{A^2}{B} \| (I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i)^{-1} \|^{-2} \| f \|^2 
\]
\[
\leq \frac{A^2}{B} \| (I_H - \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f \|^2
\]
\[
\leq \sum_{i \in I - J} \| \Lambda_i f \|^2.
\]

On the other hand, since \( \Lambda \) is a g-frame hence \( \{ \Lambda_i \}_{i \in I - J} \) is a g-Bessel sequence. It follows that \( \{ \Lambda_i \}_{i \in I - J} \) is a g-frame. Conversely, suppose that \( \{ \Lambda_i \}_{i \in I - J} \) is a g-frame for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I - J} \), with g-frame bounds \( A \) and \( B \). We first show that \( I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i \) is injective. Let
\[
(I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f = 0 \Rightarrow
\]
\[
S_{\Lambda}^{-1} \left( \sum_{i \in I - J} \Lambda_i^* \Lambda_i f \right) = \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f = 0
\]

hence \( \sum_{i \in I - J} \Lambda_i^* \Lambda_i f = 0 \). It follows that
\[
A \| f \|^2 \leq \sum_{i \in I - J} \| \Lambda_i f \|^2
\]
\[
= \sum_{i \in I - J} < \Lambda_i f, \Lambda_i f >
\]
\[
= < \sum_{i \in I - J} \Lambda_i^* \Lambda_i f, f > = 0
\]

which implies that \( f = 0 \). Also, if \( (I_H - \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i) f = 0 \) then \( \sum_{i \in I - J} \Lambda_i^* \Lambda_i S_{\Lambda}^{-1} f = 0 \) and therefore \( S_{\Lambda}^{-1} f = 0 \), it follows that \( f = 0 \). This finishes the proof.

**Corollary 4.1** Let \( \{ \Lambda_i \}_{i \in I} \) be a g-frame for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \) and let \( J \subset I \). If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = 0 \), then \( \{ \Lambda_i \}_{i \in I - J} \) is not a g-frame for \( \mathcal{H} \).

**Proof.** If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = 0 \), then \( \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0 = 0 \). It follows that
\[
\sum_{i \in I - J} \| \Lambda_i f_0 \|^2 = \sum_{i \in I - J} < \Lambda_i f_0, \Lambda_i f_0 >
\]
\[
= < \sum_{i \in I - J} S_{\Lambda}^{-1} \Lambda_i^* \Lambda_i f_0, f_0 > = 0
\]

Therefore \( \{ \Lambda_i \}_{i \in I - J} \) is not a g-frame.

**Corollary 4.2** Let \( \{ \Lambda_i \}_{i \in I} \) be a A-tight g-frame for \( \mathcal{H} \) with respect to \( \{ W_i \}_{i \in I} \) and let \( J \subset I \). If there exists \( 0 \neq f_0 \in \mathcal{H} \) such that \( \sum_{i \in J} \Lambda_i^* \Lambda_i f_0 = A f_0 \), then \( \{ \Lambda_i \}_{i \in I - J} \) is not a g-frame for \( \mathcal{H} \).

**5 Conclusion**

In this paper, we proved that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame and computed its optimal bounds. We also showed that a Bessel sequence is an inner summand of a frame and changed every Bessel sequence to a dual frame by summing it with any Parseval frame. Moreover, we proved that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim in [9] to the situation of g-frames. We defined the restricted isometry property for g-frames and generalized some results from [6] to g-frames.

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