Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 8, No. 4, 2016 Article ID IJIM-00774, 8 pages
Research Article

Science and Research Branch (IAU)

Approximation solution of two-dimensional linear stochastic Volterra-Fredholm integral equation via two-dimensional Block-pulse functions

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Received Date: 2015-10-25 Revised Date: 2016-02-11 Accepted Date: 2016-11-02

Abstract

In this paper, a numerical efficient method based on two-dimensional block-pulse functions (BPFs) is proposed to approximate a solution of the two-dimensional linear stochastic Volterra-Fredholm integral equation. Finally the accuracy of this method will be shown by an example.

Keywords : Block-pulse function; Two-dimensional equation; Stochastic integral equation; Volterra-fredholm integral; Operational matrix; Brownian motion process; Ito integral.

1 Introduction

The The nonlinear and linear Volterra-Fredholm ordinary integral equations arise from various physical and biological models. The essential features of these models are of wide applicable. These models provide an important tool for modeling a numerous problems in engineering and science [6, 7]. Modelling of certain physical phenomena and engineering problems [8, 9, 10, 11, 12] leads to two-dimensional nonlinear and linear Volterra-Fredholm ordinary integral equations of the second kind. Some numerical schemes have been inspected for resolvent of two-dimensional ordinary integral equations by several probers. Computational complexity of mathematical operations is the most important obstacle for solving ordinary integral equations in higher dimensionas.

These include the Nystrom method, collocation method, Gauss product quadrature rule method, Galerkin method, using triangular fuctions, Legender polynomial method, differential transform method, meshless method, Bernstein polynomials method and Haar wavelet method [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. It's easy to show that the Volterra integral form of the general hyperbolic differential equation [9] is given by two-dimentional integral equation

$$g(x,y) = f(x,y) + \int_0^y \int_0^x K(x,y,s,t)g(s,t)dsdt.$$

If we import statistical noise in the general hyperbolic differential equation [9], we can obtain two-dimensional linear stochastic Volterra integral equation of the second kind, i.e.

$$g(x,y) = f(x,y) + \int_0^y \int_0^x K_1(x,y,s,t)g(s,t)dsdt$$

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$$+\int_0^y\int_0^x K_2(x,y,s,t)g(s,t)dB(s)dB(t), \quad (1.1)$$

$$(x,y)\in[0,T_1)\times[0,T_2) \quad , \quad s\leqslant x<t\leqslant y.$$

Alike two-dimensional linear Volterra-Fredholm ordinary integral equations we can obtain two-dimensional linear stochastic Volterra-Fredholm integral equation of the second kind from Eq. 1.1 as

$$g(x,y)=f(x,y)$$
$$+\int_0^1\int_0^1 K_1(x,y,s,t)g(s,t)dsdt$$
$$+\int_0^y\int_0^x K_2(x,y,s,t)g(s,t)dsdt$$
$$+\int_0^y\int_0^x K_3(x,y,s,t)g(s,t)dB(s)dB(t), \quad (1.2)$$
$$(x,y)\in[0,T_1)\times[0,T_2) \quad , \quad s\leqslant x<t\leqslant y.$$

where the kernels $K_1(x,y,s,t)$, $K_2(x,y,s,t)$, $K_3(x,y,s,t)$ and the function $f(x,y)$ in 1.2 are known functions whereas $g(x,y)$ is unknown function and is called the solution of two-dimensional stochastic integral equation. Also $B(s)$ is a Brownian motion process and $\int_0^y\int_0^x K_3(x,y,s,t)g(s,t)dB(s)dB(t)$ is the double Wiener-Itô integral. The condition $s\leqslant x<t\leqslant y$ is necessary for adaptability to the filtration $\{F_t;0\leq t\leq 1\}$ where $F_t=\sigma\{B(s);0\leq s\leq t\}$. Some detailed treatments of numerical method for solving the one-dimensional case of Eqs. 1.1 and 1.2 can be found in [2, 3, 4]. According to [1] in this paper we apply two-dimensional block-pulse functions (BPFs), constructed on $[0,T_1)\times[0,T_2)$ to solve Eq. 1.2. Our method consists of reducing 1.2 to a set of algebraic equations by expanding unknown function as two-dimensional BPFs with unknown coefficients. For validation the stochastic double Wiener-Ito integral we need the following lemma and definition [5]:

Lemma 1.1 *Put $\phi(t,s)=K(x,y,s,t)g(s,t)$. Let ϕ be a function in $L^2([0,1]^2)$. Then there exists a sequence ϕ_n of off-diagonal step functions such that*

$$\lim_{n\to\infty}\int_a^b\int_a^b \mid \phi(t,s)-\phi_n(t,s)\mid^2 dtds=0.$$

Definition 1.1 *Let $\phi\in L^2([0,1]^2)$. Then the double Wiener-Ito integral of ϕ in $L^2(\Omega)$ is defined as*

$$\int_a^b\int_a^b \phi(t,s)dB(t)dB(s)$$
$$=\lim_{n\to\infty}\int_a^b\int_a^b \phi_n(t,s)dB(t)dB(s).$$

This paper is organized as follows: In Section 2 we introduce BPFs and their properties. In Section 3 we solve the two-dimensional linear stochastic Volterra-Fredholm integral equation 1.2 by finding the ordinary and stochastic operational matrixes. In Section 4 we apply the proposed method in an example, showing the accuracy and efficiency with %95 confidence interval for it.

2 Two dimentional BPFs

The block-pulse functions are a set of orthogonal functions with piecewise constant values and are usually applied as a useful tool in the analysis, synthesis, identification and other problems of contorol and systems science. This set of functions was first introduced to electrical engineers by Harmuth in 1969, and have been extensively applied due to their simple and easy operations for one-dimensional problems [26, 27, 28, 29]. The complete details for one-dimensional BPFs is given in [26, 27]. These discussions can also be extended to the two-dimensional BPFs [27] that are presented in this section.

2.1 Definition and properties

An (n_1n_2)-set of two-dimensional BPFs $\psi_{a_1,a_2}(x,y)$ $(a_1=1,2,...,n_1)$; $(a_2=1,2,...,n_2)$ is defined in the region of $x\in[0,T_1)$ and $y\in[0,T_2)$ as:

$$\psi_{a_1,a_2}(x,y)$$
$$=\begin{cases} 1, & for \begin{array}{l}(a_1-1)k_1\leq x<a_1k_1\\ (a_2-1)k_2\leq y<a_2k_2\end{array}\\ 0, & \quad otherwise,\end{cases}$$

where (n_1,n_2) are arbitary positive integers and we have

$$k_1=\frac{T_1}{n_1} \quad , \quad k_2=\frac{T_2}{n_2}.$$

Similar to the one-dimensional case, we have the elementary properties for two-dimensional BPFs that are as follows:

1. Disjointness. The BPFs are disjoined with each other:

$$\psi_{a_1,a_2}(x,y)\psi_{b_1,b_2}(x,y)$$
$$= \begin{cases} \psi_{a_1,a_2}(x,y), & if\ a_1 = b_1\ ,\ a_2 = b_2 \\ 0, & otherwise. \end{cases}$$

2. Orthogonality. The BPFs are orthogonal with each other:

$$\int_0^{T_1}\int_0^{T_2} \psi_{a_1,a_2}(x,y)\psi_{b_1,b_2}(x,y)dydx$$
$$= \begin{cases} k_1k_2, & for\ \ a_1 = b_1\ ,\ a_2 = b_2 \\ 0, & otherwise, \end{cases}$$

in the region of $x \in [0,T_1)$ and $y \in [0,T_2)$, where $a_1, b_1 = 1,2,...,n_1$ and $a_2, b_2 = 1,2,...,n_2$.

3. Completeness. The BPF set is complete when n_1 and n_2 approaches to the infinity. This means that for every $h \in \mathcal{L}^2([0,T_1) \times [0,T_2))$ Parseval's identity holds:

$$\int_0^{T_1}\int_0^{T_2} h^2(x,y)dxdy$$
$$= \sum_{a_1=1}^{\infty}\sum_{a_2=1}^{\infty} h^2_{a_1,a_2}\|\psi_{a_1,a_2}(x,y)\|^2,$$

where

$$h_{a_1,a_2}$$
$$= \frac{1}{k_1k_2}\int_0^{T_1}\int_0^{T_2} h(x,y)\psi_{a_1,a_2}(x,y)dydx.$$

The set of two-dimensional BPFs can be written as a vector $\psi(x,y)$ of dimension n_1n_2:

$$\Psi(x,y) = [\psi_{1,1}(x,y), ..., \psi_{n_1,n_2}(x,y)]^T \qquad (2.3)$$

where $(x,y) \in [0,T_1) \times [0,T_2)$.
From the above representation and disjointness property, it follows:

$$\Psi(x,y)\Psi^T(x,y) =$$

$$\begin{pmatrix} \psi_{1,1}(x,y) & 0 & ... & 0 \\ 0 & \psi_{1,2}(x,y) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & \psi_{n_1,n_2}(x,y) \end{pmatrix}, \qquad (2.4)$$

$$\Psi^T(x,y)\Psi(x,y) = 1,$$

and

$$\Psi(x,y)\Psi^T(x,y)V = \tilde{V}\Psi(x,y), \qquad (2.5)$$

where V is an n_1n_2-vector and $\tilde{V} = diag(V)$. Moreover, it can be clearly concluded that for every $(n_1n_2) \times (n_1n_2)$ matrix A:

$$\Psi^T(x,y)A\Psi(x,y) = \hat{A}^T\Psi(x,y), \qquad (2.6)$$

where $\hat{A}$ is an n_1n_2-vector with elements equal to the diagonal entries of matrix A.

2.2 Two dimensional BPFs expansions

A function $h(x,y)$ defined over $[0,T_1)\times[0,T_2)$ can be expanded by the two-dimensional BPFs as

$$h(x,y)$$
$$\simeq \sum_{a_1=1}^{n_1}\sum_{a_2=1}^{n_2} h_{a_1,a_2}\psi_{a_1,a_2}(x,y) = H^T\Psi(x,y),$$

where F is an $(n_1n_2) \times 1$ vector given by

$$H = [h_{1,1}, ..., h_{1,n_2}, ..., h_{n_1,1}, ..., h_{n_1,n_2}]^T,$$

and $\Psi(x,y)$ is defined in 2.3.
The block-pulse coefficients, h_{a_1,a_2} , are obtained as

$$h_{a_1,a_2}$$
$$= \frac{1}{k_1k_2}\int_{(a_1-1)k_1}^{a_1k_1}\int_{(a_2-1)k_2}^{a_2k_2} h(x,y)dydx.$$

Similarly a function of four variables, $K(x,y,s,t)$, on $[0,T_1)\times[0,T_2)\times[0,T_3)\times[0,T_4)$ can be approximated with respect to BPFs such as:

$$K(x,y,s,t) \simeq \Psi(x,y)^T K\Psi(s,t),$$

where $\Psi(x,y)$ is two-dimensional BPF vector of dimension n_1n_2 and K is the $(n_1n_2) \times (n_3n_4)$, two-dimensional BPF coefficient matrix.

2.3 Operational matrix of integration

The integration of the vector $\Psi(x, y)$ defined in 2.3 can be approximately obtained as [1]

$$\int_0^{x_1}\int_0^{x_2}\Psi(y_1, y_2)dy_1dy_2 \simeq P\Psi(x_1, x_2)$$
$$= [O_{(n_1\times n_1)} \otimes O_{(n_2\times n_2)}]\Psi(x_1, x_2), \qquad (2.7)$$

where $x_1 \in [0, T_1), x_2 \in [0, T_2)$ and P is the $(n_1n_2)\times(n_1n_2)$ operational matrix of integration for two-dimensional BPFs where O is the operational matrix of one-dimensional BPFs defined over $[0, T)$ with $k = \frac{T}{n}$ and $T = T_1 = T_2$ as follows

$$O = \frac{k}{2}\begin{pmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

In 2.7, $\otimes$ denotes the Kronecker product defined as

$$D \otimes E = (d_{ij}E).$$

Also from (2.4) we have:

$$\int_0^1\int_0^1 \Psi(s, t)\Psi^T(s, t)dsdt$$
$$= \begin{pmatrix} k_1k_2 & 0 & \dots & 0 \\ 0 & k_1k_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_1k_2 \end{pmatrix} = D, \qquad (2.8)$$

where D is the $(n_1n_2) \times (n_1n_2)$ known matrix.

2.4 Stochastic operational matrix based on two-dimensional BPFs

Similarly we obtain the stochastic integration of the vector $\Psi(x, y)$ defined in 2.3 as [30, 1]

$$\int_0^{x_1}\int_0^{x_2}\psi(y_1, y_2)dB(y_1)dB(y_2)$$
$$\simeq P_s\psi(x_1, x_2)$$
$$= [O_{s,(n_1\times n_1)} \otimes O_{s,(n_2\times n_2)}]\psi(x_1, x_2), \qquad (2.9)$$

where $x_1 \in [0, T_1), x_2 \in [0, T_2)$ and P_s is the $(n_1n_2) \times (n_1n_2)$ stochastic operational matrix of integration for two-dimensional BPFs where O_s is $n_1 \times n_2$ stochastic operational matrix of one-dimensional BPFs defined over $[0, T)$ with $k = \frac{T}{n}$ and $T = T_1 = T_2$ as follows [2]

$$O_s = $$
$$\begin{pmatrix} B(\frac{k}{2}) & \dots & B(k) \\ \vdots & \ddots & B(2k) - B(k) \\ 0 & \dots & B(\frac{(2n-1)k}{2}) - B((n-1)k) \end{pmatrix}$$

In the next sections, it is assumed that $T_1 = T_2 = 1$, so two-dimensional BPFs are defined over $[0, 1) \times [0, 1)$ and $k_1 = \frac{1}{n_1}, k_2 = \frac{1}{n_2}$.

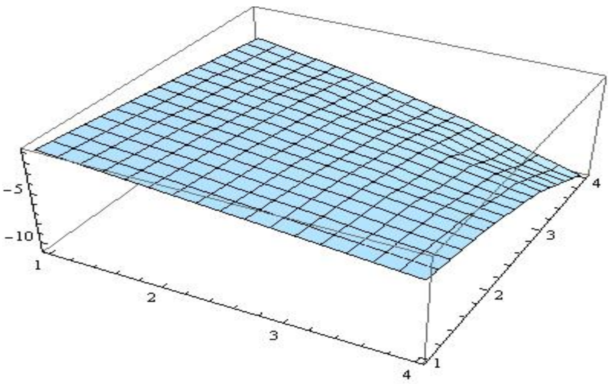

Figure 1: Approximate solution $n = 4$

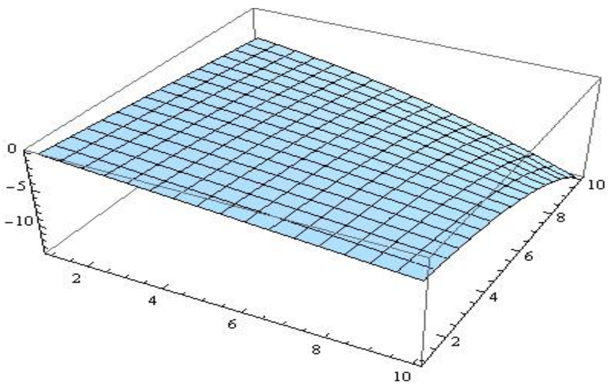

Figure 2: Approximate solution $n = 10$

3 Method of solution

In this section we solve two-dimensional linear stochastic Volterra-Fredholm integral equation 1.2 using two-dimensional BPFs.
Approximating functions $f(x, y)$, $K_1(x, y, s, t)$, $K_2(x, y, s, t)$, $K_3(x, y, s, t)$ and $g(x, y)$ with respect to two-dimensional BPFs by the way men-

Table 1: The solutions mean with %95 confidence interval for above example.

n	(x,y)	$\bar{g}(x,y)$	(L,U)
4	$(0,0.25)$	-1.58407	$(-1.61509,-1.55305)$
	$(0.25,0.5)$	-3.46561	$(-3.52655,-3.40467)$
	$(0.25,0.75)$	-6.88707	$(-6.97383,-6.80031)$
6	$(0,0.5)$	-2.16111	$(-2.19703,-2.12519)$
	$(0.33,0.67)$	-4.19015	$(-4.30596,-4.07435)$
	$(0.17,0.83)$	-4.08131	$(-4.15619,-4.00643)$
8	$(0,0.12)$	-0.82309	$(-0.84059,-0.80559)$
	$(0.25,0.5)$	-2.98738	$(-3.04395,-2.93080)$
	$(0.62,0.87)$	-8.29961	$(-8.41224,-8.18698)$
10	$(0,0.1)$	-1.02272	$(-1.02633,-1.01911)$
	$(0.3,0.6)$	-3.63773	$(-3.65491,-3.62055)$
	$(0.7,0.9)$	-9.62180	$(-9.64288,-9.60071)$

tioned in Section 2 gives

$$f(x,y) = F^T\Psi(x,y), \tag{3.10}$$

$$K_1(x,y,s,t) = \Psi^T(x,y)\Gamma_1\Psi(s,t), \tag{3.11}$$

$$K_2(x,y,s,t) = \Psi^T(x,y)\Gamma_2\Psi(s,t), \tag{3.12}$$

$$K_3(x,y,s,t) = \Psi^T(x,y)\Gamma_3\Psi(s,t), \tag{3.13}$$

and

$$g(x,y) = G_1^T\Psi(x,y), \tag{3.14}$$

where the vectors F and G_1 and matrices K_1, K_2 and K_3 are BPFs coefficients of $f(x,y)$, $g(x,y)$, $K_1(x,y,s,t)$, $K_2(x,y,s,t)$ and $K_3(x,y,s,t)$ respectively and $\Psi(x,y)$ is defined in 2.3. In 3.10, F is $(n_1n_2\times 1)$ known vector, also in 3.11, 3.12 and 3.13, Γ_1, Γ_2 and Γ_3 are $(n_1n_2)\times(n_1n_2)$ known matrices but in 3.14, G_1 is $(n_1n_2\times 1)$ unknown vector.

In Eq. 1.2 to approximate the first two-dimensional integral from 3.11 and 3.14 we get

$$\int_0^1\int_0^1 K_1(x,y,s,t)g(s,t)dsdt$$

$$= \int_0^1\int_0^1 \Psi^T(x,y)\Gamma_1\Psi(s,t)\Psi^T(s,t)G_1dsdt,$$

by using operational matrix D from Eq. 2.8 we have

$$= \Psi^T(x,y)\Gamma_1\left(\int_0^1\int_0^1 \Psi(s,t)\Psi^T(s,t)dsdt\right)$$

$$\times G_1 = \Psi^T(x,y)\Gamma_1 DG_1 = (\Gamma_1 DG_1)^T\Psi(x,y)$$
$$= G_2^T\Psi(x,y),$$

where G_2 is an (n_1n_2)-vector obtained from Γ_1DG_1. Therefore the approximation of the first two-dimensional integral with respect to two-dimensional BPFs may be compactly written as

$$\int_0^1\int_0^1 K_1(x,y,s,t)g(s,t)dsdt \simeq G_2^T\Psi(x,y). \tag{3.15}$$

To approximate the second two-dimensional integral in 1.2 from Eqs. 3.12 and 3.14 we get [1]

$$\int_0^y\int_0^x K_2(x,y,s,t)g(s,t)dsdt$$

$$\simeq \int_0^y\int_0^x \Psi^T(x,y)\Gamma_2\Psi(s,t)\Psi^T(s,t)G_1dsdt$$

$$= \Psi^T(x,y)\Gamma_2\left(\int_0^y\int_0^x \Psi(s,t)\Psi^T(s,t)G_1dsdt\right),$$

from Eq. 2.5 follows

$$= \Psi^T(x,y)\Gamma_2\left(\int_0^y\int_0^x \tilde{G}_1\Psi(s,t)dsdt\right)$$

$$= \Psi^T(x,y)\Gamma_2\tilde{G}_1\left(\int_0^y\int_0^x \Psi(s,t)dsdt\right),$$

Using operational matrix P in Eq. 2.7 we have

$$\int_0^y\int_0^x K_2(x,y,s,t)g(s,t)dsdt$$
$$\simeq \Psi^T(x,y)\Gamma_2\tilde{G}_1P\Psi(x,y),$$

in which $\Gamma_2\tilde{G}_1P$ is an $(n_1n_2)\times(n_1n_2)$ matrix. Eq. 2.6 follows:

$$\int_0^y\int_0^x K_2(x,y,s,t)g(s,t)dsdt \simeq G_3^T\Psi(x,y), \tag{3.16}$$

where G_3 is an (n_1n_2)-vector with components equal to the diagonal entries of matrix $\Gamma_2\tilde{G}_1P$. Similarly to approximate the stochastic integral case in 1.2 from Eqs. 3.13 and 3.14 we get

$$\int_0^y\int_0^x K_3(x,y,s,t)g(s,t)dB(s)dB(t)$$

$$\simeq \int_0^y\int_0^x \Psi^T(x,y)\Gamma_3\Psi(s,t)\Psi^T(s,t)$$

$$\times G_1dB(s)dB(t) = \Psi^T(x,y)\Gamma_3$$

$$\times\left(\int_0^y\int_0^x \Psi(s,t)\Psi^T(s,t)G_1dB(s)dB(t)\right),$$

from Eq. 2.5 we have

$$= \Psi^T(x,y)\Gamma_3\left(\int_0^y\int_0^x \tilde{G}_1\Psi(s,t)dB(s)dB(t)\right)$$

$$= \Psi^T(x,y)\Gamma_3\tilde{G}_1\left(\int_0^y\int_0^x \Psi(s,t)dB(s)dB(t)\right).$$

By using operational matrix P_s in Eq. 2.9 we have

$$\int_0^y\int_0^x K_3(x,y,s,t)g(s,t)dB(s)dB(t)$$

$$\simeq \Psi^T(x,y)\Gamma_3\tilde{G}_1P_s\Psi(x,y),$$

in which $\Gamma_3\tilde{G}_1P_s$ is an $(n_1n_2)\times(n_1n_2)$ matrix. Eq. 2.6 follows:

$$\int_0^y\int_0^x K_3(x,y,s,t)g(s,t)dB(s)dB(t)$$

$$\simeq G_4^T\Psi(x,y), \tag{3.17}$$

where G_4 is an (n_1n_2)-vector with components equal to the diagonal entries of matrix $\Gamma_3\tilde{G}_1P_s$. Applying Eqs. 3.10, 3.14, 3.15, 3.16 and 3.17 in Eq. 1.2 we get

$$G_1^T\Psi(x,y) \simeq F^T\Psi(x,y) + G_2^T\Psi(x,y)$$

$$+ G_3^T\Psi(x,y) + G_4^T\Psi(x,y). \tag{3.18}$$

Replacing $\simeq$ with $=$, Eq. 3.18 gives

$$G_1 - G_2 - G_3 - G_4 = F. \tag{3.19}$$

The equation 3.19 generate a system of (n_1n_2) linear equations with (n_1n_2) unknown variable which can be solved using Newton's iterative method.

4 Numerical Example

In this section, the numerical example is given to demonstrate the applicability and accuracy of our method. For convenience we put $n_1 = n_2 = n$ so $k_1 = k_2 = \frac{1}{n}$.

Example 4.1 *Consider the following linear two-dimensional stochastic Volterra integral equation of second kind:*

$$g(x,y) = f(x,y)$$

$$+\int_0^1\int_0^1 (x+y+t-s)u(s,t)dsdt$$

$$+\int_0^y\int_0^x (x+y+t-s)u(s,t)dsdt$$

$$+\int_0^y\int_0^x (x+y+t+s)u(s,t)dB(s)dB(t)$$

where

$$f(x,y) = x+y-\frac{1}{12}xy(x^3+4x^2y+4xy^2+y^3).$$

The solutions mean $(\bar{g}(x,y))$ with %95 confidence interval (L,U) at the points that as for condition $s \leqslant x < t \leqslant y$ are selected for the present method for 500 iterative of system 3.19 is shown in Table 1. The numerical example is carried out for selected grid points which are proposed by difference as $(2^{-k}, k = 4,6,8,10)$. In Figs. 1-2, three-dimensional graphs of the approximate solution for various values of arbitary positive integer n are shown.

5 Conclusion

The two-dimensional integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the numerical solutions. The numerical solution of two-dimensional stochastic integral equations because of the randomness is very difficult or sometimes impossible. In this paper, the method based on two-dimensional BPFs and its operational matrix has been successfully used for approximate a solution of two-dimensional linear stochastic Volterra-Fredholm integral equations. This approach transformed a two-dimensional linear stochastic Volterra-Fredholm integral equation to a matrix equation which corresponds to a system of linear equations with unknown coefficients. The illustrative example is included to demonstrate the validity and applicability of the technique. Mathematica has been used for computations.

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