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Research Article

Flexibility of Variations in Radial and Non-Radial Data Envelopment Analysis Models

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Abstract

One of the major problems in Data Envelopment Analysis (DEA) is to determine the projection of inefficient Decision Making Units (DMUs) into the efficient frontier. In conventional DEA models, inputs and outputs of inefficient DMUs alter arbitrarily for reaching to the efficient frontier. Nevertheless, sometimes the ability of DMUs is defined and restricted. Moreover, there are situations in the real world applications that limited resources exist. Therefore, in these cases inputs and outputs cannot vary irrationally. Actually, there are pre-specified alteration levels of inputs and outputs. For this purpose, the current study proposes DEA-based models, radial and non-radial models, to evaluate the relative efficiency of DMUs with restricted input and output variables. Furthermore, non-radial super-efficiency models are extended for ranking efficient DMUs. An example from the banking sector is used to illustrate the proposed approach.

Keywords : Data Envelopment Analysis (DEA); Efficiency; Input/Output; Variations.

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1 Introduction

Data envelopment analysis (DEA), popularized by Charnes et al. [5] and Banker et al. [2], is a non-parametric technique to evaluate the relative efficiency of DMUs with multiple inputs and multiple outputs. The set of observations in DEA define a production possibility set (PPS) and the boundary points of this set construct the efficient frontier. Decision making units (DMUs) that belong to the boundary are called efficient and the others are inefficient. The reference set for inefficient units consists of efficient units and determines a virtual unit on the efficient frontier. In conventional DEA models, inefficient DMUs reduce their inputs and increase their outputs (with considering desirable factors) arbitrarily to meet the efficient frontier. These variations can be made in different ways: radially and non-radially. In radial models, inefficient DMUs can be improved by fixing the outputs (inputs) and radially reducing the inputs (increasing the outputs) until the efficient frontier is met. However, non-radial models consider the input excesses and the output shortfalls simultaneously in arriving at a point on the efficient frontier which is most distant from inefficient DMU. In many real applications of DEA, because of some limitations in resources and DMU's ability, these changes cannot be made arbitrarily. For instance, in evaluating the efficiency of banks, a factor like the number of staffs is considered as an input and a factor like income is deemed as an output. As-

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sume in a survey of banks, 20 staffs exist in a bank while income is 4000 dollars. In addition, suppose this bank is specified as an inefficient bank after evaluating by means of conventional DEA models; that it should decrease staffs to 5 individuals and increase income to 8000 dollars for reaching to the efficient frontier. Nonetheless, the bank is not able to achieve the aforementioned situation. In these situations, there are predefined variation levels of inputs and outputs that are determined by decision makers. Unlike the classical DEA models, the target unit for inefficient DMU is not necessarily efficient in these cases.

In the current paper with considering these predefined variation levels on inputs and outputs, restricted DEA models are proposed to determine the relative efficiency of DMUs with restricted variables. To illustrate, radial and non-radial models are introduced to assess the performance of DMUs where input and output variations are restricted. Furthermore, approaches are suggested for ranking the efficient DMUs. As far as we see the DEA literature, there is no study about the subject except Kordrostami et al. [8] that have considered the variation levels in radial models where undesirable outputs exist while in this study, radial and non-radial models are proposed that incorporate restricted variations. Moreover, slacks-based super-efficiency models are extended for ranking efficient units. Indeed, in DEA contexts, radial and non-radial models exist for ranking the efficient DMUs. Readers can refer to [1, 6, 9, 11, 6] for more information. In this study, non-radial super-efficiency models are used and generalized because the slacks-based super-efficiency DEA models are always feasible, that is, Tone's model [11] and Du et al.'s model [7] are extended for occasions that these restrictions exist. Also, the efficiency scores of Iranian bank branches are calculated and ranked by using the suggested methods.

The paper is organized as follows: Section 2 reviews some basic concepts and models in DEA that are applied and extended in this study. Next, the suggested approaches are provided and illustrated in Section 3. A case study of commercial bank branches in Iran is given in Section 4. Finally, conclusions appear in Section 5.

2 Preliminaries

Consider n DMUs, $DMU_j (j = 1, 2, ..., n)$, that each DMU consumes m inputs x_{ij}, $i = 1, 2, ..., m$ and produce s outputs y_{rj}, $r = 1, 2, ..., s$. Charnes et al. [5] proposed the following model, called CCR (Charnes, Cooper, and Rhodes) model, for evaluating the efficiency of DMUs.

$$\begin{aligned} & Min \ \ \theta \\ & s.t. \ \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ip}, \quad i = 1, 2, ..., m, \\ & \quad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad r = 1, 2, ..., s, \\ & \quad \lambda_j \geq 0, \ j = 1, 2, ..., n. \end{aligned} \tag{2.1}$$

If the constraint $\sum_{j=1}^{n} \lambda_j = 1$ is added to model (2.1), we will have the BCC model, introduced by Banker et al. [2]. The aforementioned models, the CCR and BCC models, are radial models. In DEA contexts, there are, also, non-radial models like slacks-based measure (SBM) of efficiency, the additive model. Readers can refer to Tone [10] and [4] for more information.

Further, as mentioned in the previous section, there are models for ranking efficient DMUs in the DEA literature. Here, we review non-radial models that are extended in this study. Tone [11] proposed the following model for distinguishing efficient DMUs.

$$\begin{aligned} & Min \ \ (1 + \tfrac{1}{m} \sum_{i=1}^{m} \tfrac{t_{ip}^-}{x_{ip}})/(1 - \tfrac{1}{s} \sum_{r=1}^{s} \tfrac{t_{rp}^+}{y_{rp}}) \\ & s.t. \ \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} \leq x_{ip} + t_{ip}^-, \quad i = 1, 2, ..., m, \\ & \quad \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} \geq y_{rp} - t_{rp}^+, \quad r = 1, 2, ..., s, \\ & \quad \lambda_j \geq 0, \ t_{ip}^- \geq 0, \ t_{rp}^+ \geq 0, j = 1, 2, ..., n, j \neq p \\ & \quad i = 1, 2, ..., m, \ \ r = 1, 2, ..., s. \end{aligned} \tag{2.2}$$

Furthermore, Du et al. [7] introduced the additive super-efficiency model for ranking efficient DMUs as follows:

$$\begin{aligned} & Min \ \ \sum_{i=1}^{m} t_{ip}^- + \sum_{r=1}^{s} t_{rp}^+ \\ & s.t. \ \sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} \leq x_{ip} + t_{ip}^-, \ i = 1, 2, ..., m, \\ & \quad \sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} \geq y_{rp} - t_{rp}^+, \ r = 1, 2, ..., s, \\ & \quad \lambda_j \geq 0, \ t_{ip}^- \geq 0, \ t_{rp}^+ \geq 0, j = 1, 2, ..., n, j \neq p \\ & \quad i = 1, 2, ..., m, \ \ r = 1, 2, ..., s. \end{aligned} \tag{2.3}$$

In both models (2.2) and (2.3), $t_{ip}^{-}$and t_{rp}^{+} indicate amounts by which inputs increase and outputs decrease for DMU_p to reach the frontier constructed by the remaining DMUs.

3 Flexibility of variations

In this section some radial and non-radial models are proposed that regard restricted variations. Actually, in the real world, there are occasions that the input and output factors of DMUs cannot change arbitrarily. To illustrate, a DMU is not able to reach some situations. In this study, knowledge of managers and decision makers about resources, products, and DMU's ability has a considerable effect on determining the efficiency of firms. The structure of this system is displayed as follows:

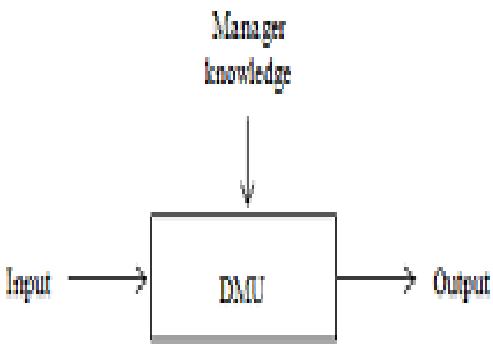

Figure 1: A System.

3.1 Restricted variations in radial Models

As previous section, suppose there are n DMUs, $DMU_j (j = 1, 2, ..., n)$, with m inputs x_{ij}, $i = 1, 2, ..., m$ and s outputs y_{rj}, $r = 1, 2, ..., s$. Inefficient units in DEA should increase their output levels and simultaneously decrease their input levels according to equations (3.4) to become efficient.

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{ip}, \ i = 1, 2, ..., m, \\ &\textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \ r = 1, 2, ..., s. \end{aligned} \tag{3.4}$$

The conventional DEA models assume that reducing inputs and increasing outputs can be made arbitrarily. In real applications, however, because of limited resources, infinite variations in inputs and outputs are impossible.

Suppose the i-th input of DMU_p is limited to decrease to $x_{ip} - \alpha_{ip} \geq 0$. Similarly, the r-th output of DMU_p is limited to increase to $y_{rp} + \beta_{rp} \geq 0$. In other words

$$\begin{aligned} &x_{ip} \to x_{ip} - \alpha_{ip}, \ i = 1, 2, ..., m, \\ &y_{rp} \to y_{rp} + \beta_{rp}, \ r = 1, 2, ..., s \end{aligned} \tag{3.5}$$

that $\alpha_p = (\alpha_{1p}, \alpha_{2p}, ..., \alpha_{mp})^t$ and $\beta_p = (\beta_{1p}, \beta_{2p}, ..., \beta_{sp})^t$. If $(\sum_{j=1}^{n} \lambda_j x_j, \sum_{j=1}^{n} \lambda_j y_j)$be the projection of DMU_p in PPS (that is T), clearly, we cannot expect this projection is located on the frontier.

Considering the restricted variations (i.e.(3.5)), the following constraints must be held:

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \ i = 1, 2, ..., m, \\ &\textstyle\sum_{j=1}^{n} \lambda_j y_{ij} \leq y_{rp} + B_{rp}, \ r = 1, 2, ..., s. \end{aligned} \tag{3.6}$$

Now consider the efficiency assessment of DMU_p in CRS environment as follows:

$$\begin{aligned} &\min \ \theta \\ &(\theta x_p, y_p) \in T. \end{aligned}$$

By the definition of T and taking the restrictions (3.6) into consideration, we have the following linear programming:

$$\begin{aligned} &Min \ \theta \\ &s.t. \ \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ip}, \quad i = 1, 2, ..., m, \\ &\quad \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp}, \quad r = 1, 2, ..., s, \\ &\quad \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1, 2, ..., m, \\ &\quad \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1, 2, ..., s, \\ &\quad \lambda_j \geq 0, \ j = 1, 2, ..., n. \end{aligned} \tag{3.7}$$

The first two constraints in (3.7) are the usual envelopment restrictions of the classical CCR-model. The last two constraints take the restricted variations imposed by decision makers into consideration. To avoid the weak efficient units in (3.7), the following revised model is proposed:

Table 1: Data for a Real Application.

Branch	Personnel	Cost	Debt	Resource	Income	Loan
1	19	2026	22585	187679	13304	102808
2	18	1953	21035	124349	2521	75509
3	11	1914	17861	72149	3153	57537
4	18	1753	39525	127370	5252	149860
5	17	1839	11796	89871	2673	51114
6	16	1989	9632	95288	4690	55757
7	14	1857	12830	150026	6783	106734
8	7	1511	14867	42654	2354	52485
9	12	1962	10383	97812	4782	67298
10	14	1430	15118	77031	1881	43487
11	17	1285	13955	89304	5766	84631
12	14	1409	11947	75923	2261	41442
13	9	1478	16423	47763	2028	43262
14	5	1500	3772	45732	756	14237
15	6	1153	31647	55222	863	41062
16	6	2429	4986	53323	2469	37418
17	8	2076	18700	69734	2433	57883
18	9	1838	7153	160138	8395	102353
19	9	1652	15773	49153	2364	47139
20	8	2100	7705	92365	5663	55543
21	32	167	446698	515578	40254	1277833
22	7	1944	3752	64236	1361	22347
23	9	1528	4875	89104	2681	45717
24	7	1728	30614	42012	2814	73925
25	7	2008	4584	69360	2240	27246
26	7	1670	4977	51438	2293	26531
27	6	1578	4495	39948	1151	20223
28	7	1514	9464	154284	1518	43928
29	7	1594	4953	61101	1855	25718
30	8	2079	5405	81544	1711	27985
31	9	1555	8109	79046	8085	58355
32	5	2051	5185	47876	648	14055
33	7	1543	9235	71606	4289	68341
34	8	2363	728	124146	8563	54541
35	6	1881	1577	77868	1965	33838
36	5	1537	3534	58696	2248	28746
37	13	2011	7806	148755	2697	38375
38	9	1609	6881	68792	4475	66061
39	5	1674	161	88358	756	14628
40	8	1608	3762	140413	3665	20398

$$
\begin{aligned}
& e_p^* = Min\ \ \theta - \varepsilon \left[\textstyle\sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+\right] \\
& s.t.\ \ \textstyle\sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, , 2, ..., m, \\
& \quad \textstyle\sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp}, \qquad r = 1, , 2, ..., s, \\
& \quad \textstyle\sum_{j=1}^n \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1, , 2, ..., m, \\
& \quad \textstyle\sum_{j=1}^n \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1, , 2, ..., s, \\
& \quad \lambda_j \geq 0, \ \ j = 1, 2, ..., n.
\end{aligned} \tag{3.8}
$$

in which ε is a very small positive constant (i.e. a non-Archimedean constant).

Definition 3.1 *DMU_p is said to be efficient in models (3.7) and (3.8) if and only if $e_p^* = 1$.*

Improvement in an inefficient unit is attained by the following formula:

$$
\begin{aligned}
& \hat{x}_{ip} \leftarrow \textstyle\sum_{j=1}^n \lambda_j x_{ij}, \ \ i = 1, 2, ..., m, \\
& \hat{y}_{rp} \leftarrow \textstyle\sum_{j=1}^n \lambda_j y_{rj}, \ \ r = 1, 2, ..., s.
\end{aligned} \tag{3.9}
$$

Table 2: The Levels of Variations for all Branches.

Branch	α_1	α_2	α_3	β_1	β_2	β_3
1	4	405.2	6775.5	56303.7	2660.8	5140.4
2	4	390.6	6310.5	3304.7	504.2	3775.45
3	2	382.8	5358.3	21644.7	630.6	2876.85
4	5	350.6	11857.5	38211	1050.4	7493
5	4	367.8	3538.8	26934.3	534.6	2555.7
6	5	397.8	2889.6	28586.4	938	2787.85
7	4	371.4	3849	45007.8	1356.6	5336.7
8	2	302.2	4460.1	12796.2	470.8	2624.25
9	3	392.4	3114.9	29343.6	956.4	3364.9
10	4	286	4535.4	23109.3	376.2	2174.35
11	5	257	4186.5	26791.2	1153.2	4231.55
12	3	281.8	3584.1	22776.9	452.2	2072.1
13	2	295.6	4926.9	14328.9	405.6	2163.1
14	1	300	1131.6	16566.6	151.2	711.85
15	1	230.6	9494.1	15996.9	172.6	2053.1
16	1	485.8	1495.8	20920.2	493.8	1870.9
17	2	415.2	5610	48041.4	486.6	2894.15
18	3	367.6	2145.9	14745.9	1679	5117.65
19	2	330.4	4731.9	27709.5	472.8	2356.95
20	2	420	2311.5	154673.4	1132.6	2777.15
21	7	33.4	134009.4	19270.5	8050.8	63891.65
22	1	388.8	1125.6	26731.2	272.2	1117.35
23	3	305.67	1462.5	12603.6	536.2	2285.85
24	2	345.6	9184.2	20808	562.8	3696.25
25	1	401.6	1375.2	15431.4	448	1362.3
26	2	334	1493.1	11984.4	458.6	1326.55
27	1	315.6	1348.5	46285.5	230.2	1011.15
28	1	302.8	2839.2	18330.3	303.6	2196.4
29	2	318.8	1485.9	24463.2	371	1285.9
30	2	415.8	1621.5	23713.8	342.2	1399.25
31	3	311	2432.7	23713.8	1617	2917.75
32	1	410.2	1555.5	14362.8	129.6	702.75
33	2	308.6	2770.5	21481.8	857.8	3417.05
34	2	472.6	218.4	37243.8	1712.6	2727.05
35	1	376.2	473.1	23360.4	393	1691.9
36	1	307.4	1060.2	17608.8	449.6	1423.8
37	4	402.2	2341.8	44626.5	539.4	1918.75
38	2	321.8	2064.3	20367.6	895	3303.05
39	1	334.8	48.3	26507.4	151.2	731.4
40	2	321.6	1128.67	42123.9	733	1019.9

An important point to be noted is that unlike the traditional DEA models, there is no guarantee that the peer unit $(\sum_{j=1}^{n}\lambda_j x_j, \sum_{j=1}^{n}\lambda_j y_j)$ is efficient.

The dual formulation of the LP model (3.7) is given by

$$\begin{aligned}
& Max\ \sum_{r=1}^{s}(u_r-\mu_r)y_{rp}+ \\
& \sum_{i=1}^{m}\rho_i(x_{ip}-\alpha_{ip})-\sum_{r=1}^{s}\mu_r\beta_{rp} \\
& s.t.\ \sum_{r=1}^{s}(u_r-\mu_r)y_{rj}- \\
& \sum_{i=1}^{m}(\nu_i-\rho_i)x_{ij}\leq 0,\ \ j=1,2,...,n, \\
& \quad \sum_{i=1}^{m}\nu_i x_{ip}=1, \\
& \quad u_r,\mu_r,\nu_i,\rho_i\geq 0,\ \ \forall i,\forall r.
\end{aligned} \tag{3.10}$$

Table 3: The Results for the Real Case Example.

Branch	Eff. CCR	Eff. RRVM	ζ_p^*	Ranking by ζ_p^* and ψ_p^*	τ_p^*	η_p^*	Ranking by η_p^* and $\delta_p^*(\rho)$
1	1	1	1	7	0	1	7
2	0.636025	0.855908	0.773792	11	10140.597	0.810738	10
3	0.422454	0.818182	0.653009	26	30895.25	0.653009	27
4	0.786154	0.921302	0.742109	13	45106.252	0.798673	13
5	0.522770	0.8	0.699359	18	33343.02	0.699701	20
6	0.537073	0.8	0.633792	34	35470.703	0.635167	34
7	0.894326	0.915496	0.75415	12	24923.982	0.80507	12
8	0.525947	0.8	0.627208	35	20654.177	0.678984	23
9	0.574844	0.8	0.633803	33	37175.2	0.633803	35
10	0.533131	0.8	0.645312	27	30375.754	0.645312	28
11	0.872278	0.941762	0.800346	10	22010.183	0.808505	11
12	0.550701	0.8	0.688427	21	28947.192	0.688427	21
13	0.384028	0.8	0.641628	29	22122.1	0.641628	30
14	0.467366	0.8	0.636722	30	18862.25	0.636722	31
15	0.449719	0.833333	0.740927	14	23273.741	0.760077	17
16	0.544604	0.874548	0.676026	24	25266.726	0.676026	25
17	0.493986	0.8	0.58737	39	57448.841	0.58737	39
18	1	1	1	5	0	1	6
19	0.397567	0.8	0.597256	37	35603.55	0.597256	37
20	0.696858	0.8	0.68915	20	60387.564	0.734586	19
21	1	1	1	1	0	1	1
22	0.495655	0.857143	0.642949	28	29636.15	0.642949	29
23	0.696698	0.820067	0.715532	16	16094.366	0.748255	18
24	0.551870	0.8	0.593213	38	34598.798	0.593213	38
25	0.535181	0.857143	0.678806	23	19019.5	0.678806	24
26	0.407578	0.8	0.635743	31	15598.65	0.635743	32
27	0.362532	0.833333	0.529263	40	49191.95	0.529263	40
28	1	1	1	6	0	1	5
29	0.473590	0.8	0.606592	36	27926.8	0.606592	36
30	0.534537	0.8	0.635448	32	27494.55	0.635448	33
31	1	1	1	8	0	1	8
32	0.461411	0.84069	0.681282	22	17161.257	0.681282	22
33	0.780388	0.829781	0.696452	19	27626.729	0.780674	16
34	1	1	1	3	0	1	3
35	0.783579	0.957404	0.86343	9	18228.542	0.86343	9
36	0.642561	0.8	0.670288	25	20850.404	0.670288	26
37	0.801389	0.848838	0.723024	15	36528.67	0.782222	15
38	0.723072	0.829503	0.7128	17	25365.177	0.792907	14
39	1	1	1	2	0	1	2
40	1	1	1	4	0	1	4

Theorem 3.1 *The radial restricted variation model (RRVM) represented in (3.7) is feasible and bounded.*

Proof. The feasibility of model (3.7) is obvious. Because $\theta = 1,\lambda_p = 1,\ \lambda_j = 0,\ j = 1, ..., n, j \neq p$ satisfies all constraints. Thus, it is a feasible solution. Furthermore, the optimal solution is not greater than one because the problem is minimized and a feasible solution with $\theta = 1$ exists. Moreover, $\theta > 0$. This is because the input and output vectors have at least a nonzero component. Assume $\theta = 0$, from the first constraint of model (3.7) it is obtained $\lambda = 0$ and from the second constraint of model (3.7) is achieved $y \leq 0$. But we have $y \geq 0$. Thus, $y = 0$, while it has been assumed input and output vectors are nonzero at least in one component. As a result, reduction ad absurdum is invalid, and $\theta > 0$. So $0 < \theta \leq 1$,

Table 4: The Results of Models (3.12), (3.15), and (3.16).

Branch	ψ_p^*	Ranking by ψ_p^*	ρ_p^*	$\delta_p^*(\rho)$	Ranking by $\delta_p^*(\rho)$
1	1.145064	7	888.56872	1.146	7
18	1.264889	5	1475.673	1.473	6
21	43.38264	1	363023.27	51.596	1
28	1.154989	6	1270.3899	1.537	5
31	1.041851	8	958.77753	1.059	8
34	1.785514	3	6745.8244	2.579	3
39	7.028259	2	32147.024	10.546	2
40	1.418697	4	29085.831	1.604	4

and it guarantees that model (3.7) (RRVM) is bounded, and this completes the proof.

Theorem 3.2 *Let $DMU_{\hat{p}}$ be the projection of DMU_p in model (3.7). Then $DMU_{\hat{p}}$ dominates DMU_p.*

Proof. Clearly, the first and second constraints of model (3.7) imply that

$$\hat{x}_{ip} = \sum_{j=1}^{n} \lambda_j x_{ij} =$$

$$\theta x_{ip} - s_i^- \leq x_{ip}, \quad i = 1, 2, ..., m,$$

$$\hat{y}_{rp} = \sum_{j=1}^{n} \lambda_j y_{rj} =$$

$$y_{rp} + s_r^+ \geq y_{rp}, \quad r = 1, 2, ..., s.$$

and strict inequality is held at least for one component, that is, $\theta < 1$ and $\theta x_{ip} < x_{ip}$; therefore, $\hat{x}_{ip} < x_{ip}$. This completes the proof.

3.2 Restricted Variations in Non-Radial Models

In this subsection, two non-radial restricted DEA approaches are provided. The first approach is an extension of SBM model proposed by Tone [10] as follows:

$$\begin{aligned} Min \quad & \zeta_p^* = (1 - \tfrac{1}{m} \textstyle\sum_{i=1}^{m} \frac{s_{ip}^-}{x_{ip}})/(1 + \tfrac{1}{s} \textstyle\sum_{r=1}^{s} \frac{s_{rp}^+}{y_{rp}}) \\ s.t. \quad & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} + s_{ip}^- = x_{ip}, \quad i = 1, 2, ..., m, \\ & \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} - s_{rp}^+ = y_{rp}, \quad r = 1, 2, ..., s, \\ & \textstyle\sum_{j=1}^{n} \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1, 2, ..., m, \\ & \textstyle\sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1, 2, ..., s, \\ & \lambda_j \geq 0, \ j = 1, 2, ..., n. \end{aligned} \tag{3.11}$$

s_{ip}^- and s_{rp}^+ called slacks show the excesses of inputs and shortfalls of outputs for DMU_p, respectively. The third and fourth constraints indicate the amount of variations in inputs and outputs, respectively.

Definition 3.2 *model (3.11) is efficient if and only if $\zeta_p^* = 1$. It means all inputs and outputs slacks are equal to zero.*

Furthermore, for ranking the efficient DMUs and discriminating the efficient DMUs, the following model is proposed. Model (3.12) is an extension of slacks-based super-efficiency model proposed by Tone [11].

$$\begin{aligned} Min \quad & \psi_p^* = \\ & (1 + \tfrac{1}{m} \textstyle\sum_{i=1}^{m} \frac{t_{ip}^-}{x_{ip}})/(1 - \tfrac{1}{s} \textstyle\sum_{r=1}^{s} \frac{t_{rp}^+}{y_{rp}}) \\ s.t. \quad & \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} \leq x_{ip} + t_{ip}^-, \quad i = 1, 2, ..., m, \\ & \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} \geq y_{rp} - t_{rp}^+, \quad r = 1, 2, ..., s, \\ & \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1, 2, ..., m, \\ & \textstyle\sum_{j=1, j\neq p}^{n} \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1, 2, ..., s, \\ & \lambda_j \geq 0, \ t_{ip}^- \geq 0, \ t_{rp}^+ \geq 0, j = 1, 2, ..., n, j \neq p \\ & i = 1, 2, ..., m, \ \ r = 1, 2, ..., s. \end{aligned} \tag{3.12}$$

where $\psi_p^* \geq 1$. Furthermore, models (3.11) and (3.12) can be transformed into the linear programming problems by using Charnes and Cooper transformation [3].

As another approach, Du et al.' s method [7] is also generalized for evaluating the efficiency of DMUs and ranking efficient DMUs when variation levels like the following exist:

$$
\begin{aligned}
&Max\ \ \tau_p^* = \textstyle\sum_{i=1}^m s_{ip}^- + \sum_{r=1}^s s_{rp}^+ \\
&s.t.\ \textstyle\sum_{j=1}^n \lambda_j x_{ij} + s_{ip}^- = x_{ip}, \quad i = 1,2,...,m, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j y_{rj} - s_{rp}^+ = y_{rp}, \quad r = 1,2,...,s, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1,2,...,m, \\
&\quad \textstyle\sum_{j=1}^n \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1,2,...,s, \\
&\quad \lambda_j \geq 0,\ j = 1,2,...,n.
\end{aligned} \tag{3.13}
$$

In the above model, DMU_p is efficient if and only if all slacks are zero. Furthermore, the following formula is used for estimating the efficiency score:

$$
\eta_p^* = (1 - \frac{1}{m}\sum_{i=1}^m \frac{s_{ip}^{-*}}{x_{ip}})/(1 + \frac{1}{s}\sum_{r=1}^s \frac{s_{rp}^{+*}}{y_{rp}}) \tag{3.14}
$$

in which $s_{ip}^{-*}$and s_{rp}^{+*} are obtained from model (3.13).

In this case, for distinguishing between efficient DMUs, the following model is presented:

$$
\begin{aligned}
&Min\ \ \rho_p^* = \textstyle\sum_{i=1}^m t_{ip}^- + \sum_{r=1}^s t_{rp}^+ \\
&s.t.\ \textstyle\sum_{j=1,j\neq p}^n \lambda_j x_{ij} \leq x_{ip} + t_{ip}^-, \quad i = 1,2,...,m, \\
&\quad \textstyle\sum_{j=1,j\neq p}^n \lambda_j y_{rj} \geq y_{rp} - t_{rp}^+, \quad r = 1,2,...,s, \\
&\quad \textstyle\sum_{j=1,j\neq p}^n \lambda_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1,2,...,m, \\
&\quad \textstyle\sum_{j=1,j\neq p}^n \lambda_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1,2,...,s, \\
&\quad \lambda_j \geq 0,\ t_{ip}^- \geq 0,\ t_{rp}^+ \geq 0, j = 1,2,...,n, j \neq p \\
&\quad i = 1,2,...,m, \quad r = 1,2,...,s.
\end{aligned} \tag{3.15}
$$

Then,

$$
\delta_p^*(\rho) = (\tfrac{1}{m}\textstyle\sum_{i=1}^m \frac{(x_{ip}+t_{ip}^{-*}(\rho))}{x_{ip}})/(\tfrac{1}{s}\sum_{r=1}^s \frac{(y_{rp}-t_{rp}^{+*}(\rho))}{y_{rp}}) \tag{3.16}
$$

is determined that $t_{ip}^{-*}(\rho)$ and $t_{rp}^{+*}(\rho)$ are attained from model (3.15). $\delta_p^*(\rho)$ is used as the super-efficiency score which $\delta_p^*(\rho) \geq 1$. t_{ip}^- and t_{rp}^+ in models (3.12) and (3.15) denote the increase of inputs and the decrease of outputs for the efficient DMU_p while the frontier has been made by the remaining DMUs.

Theorem 3.3 *Models (3.12) and (3.15) are feasible.*

Proof. As Tone [11] and Du et al. [7], we also assume
$\tilde{t}_{ip}^- = \max\{x_{ip}, \sum_{j=1,j\neq p}^n \tilde{\lambda}_j x_{ij}\} - x_{ip} \geq 0,\ i = 1,...,m,$
$\tilde{t}_{rp}^+ = y_{rp} - \min\{y_{rp}, \sum_{j=1,j\neq p}^n \tilde{\lambda}_j y_{rj}\} \geq 0,\ r = 1,...,s.$
Therefore,
$x_{ip} + \tilde{t}_{ip}^- = \max\{x_{ip}, \sum_{j=1,j\neq p}^n \tilde{\lambda}_j x_{ij}\} \geq \sum_{j=1,j\neq p}^n \tilde{\lambda}_j x_{ij}$ and
$y_{rp} - \tilde{t}_{rp}^+ = \min\{y_{rp}, \sum_{j=1,j\neq p}^n \tilde{\lambda}_j y_{rj}\} \leq \sum_{j=1,j\neq p}^n \tilde{\lambda}_j y_{rj}.$

Furthermore, $\tilde{\lambda}_j j = 1,...,n,\ j \neq p$ is considered as a non-negative set such that
$\sum_{j=1,j\neq p}^n \tilde{\lambda}_j x_{ij} \geq x_{ip} - \alpha_{ip}, \quad i = 1,2,...,m,$

$\sum_{j=1,j\neq p}^n \tilde{\lambda}_j y_{rj} \leq y_{rp} + \beta_{rp}, \quad r = 1,2,...,s.$

Thus,

$\tilde{t}_{ip}^-,\ i = 1,...,m,\ \tilde{t}_{rp}^+,\ r = 1,...,s,$ and $\tilde{\lambda}_j$ $j = 1,...,n,\ j \neq p$

is a feasible solution for models (3.12) and (3.15).

4 A Real Application

In this section we examine the validity of the restricted DEA models by using a real data set. We apply the approaches to a data set consisting 40 branches of a commercial bank in one region in Iran. We have used six variables from the data set as inputs and outputs. Each branch uses three inputs and three outputs. Inputs include number of staff, operational costs (excluding staff costs) and overdue debts; outputs are deposits (resources), amount of income and amount of loans. The chosen input and output measures that are used in the application are summarized in Table 1 (All monetary variables are stated in ten million of Iranian current Rials). Table 2 contains listing of the levels of variations in inputs and outputs of each branch j for $j = 1, ..., 40$ that are predicted by the board of management. The defined limited values are associated with management's points of view and unit location. In Table 2, columns 2, 3, and 4 show the variations levels in inputs(α_i) while the variations levels in outputs (β_r)are represented in columns 5, 6, and 7. Running the CCR- model (2.1), eight efficient units as 1, 18, 21, 28, 31, 34, 39, and 40 are obtained. This is confirmed by our proposed models. The results are listed in Table 3. As columns 2 and 3 of the table show, efficiency measures of inefficient units in model (3.7) (RRVM) are greater than that of the CCR-model. This means that the target unit obtained from model (3.7) is closer than the target obtained from the CCR model for the unit under evaluation. Furthermore, one can contrast the results of SBM and additive models with models (3.11) and (3.14), respectively. It is found that the efficiency measures of non-radial restricted variation models, models (3.11) and (3.14), will be greater than the SBM and additive models. This is the advantage of our models in the sense that we took the ability of the units into consideration. Columns 4, 6, and 7 in Table 3 show the results of models (3.11), (3.13) and (3.14), respectively. Also, the results of ranking branches by using the restricted variation SBM approach and the restricted variation additive approach can be seen in columns 5 and 8 of Table 3, respectively. In both approaches, branch 21 has been distinguished as the most efficient while branch 27 has been determined as the least efficient. Nevertheless, there are some differences between rankings of the two methods. Table 4 represents the results of models (3.12), (3.15) and (3.16). To illustrate, the results of ranking the efficient branches can be found in Table 4. As can be seen, except ranks of 18 and 28 branches, ranks of other branches are the same when model (3.12), models (3.15) and (3.16) are calculated.

5 Concluding Remarks

In the real world, there are application cases in which inefficient units cannot reduce their inputs and increase their outputs arbitrarily to become efficient. In these cases, the target units for these operational units do not necessarily belong to the efficient frontier. The current paper has proposed modified DEA models in such a restricted environment. Indeed, it has been imported these limitations in some DEA models and proposed new models, radial and non-radial models, in order to assess the relative efficiency of these application cases. In models proposed, inefficient units are not necessarily projected onto the efficient frontier, but the projections dominate inefficient units. Moreover, some non-radial ranking approaches have been extended for distinguishing the efficient DMUs where restricted variations exist. An application area investigated involved 40 branches of a commercial bank. It seems incorporating unbalanced data with missing values in the proposed models is an interesting subject for future research.

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