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Research Article

The spectral iterative method for Solving Fractional-Order Logistic Equation

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Abstract

In this paper, a new spectral-iterative method is employed to give approximate solutions of fractional logistic differential equation. This approach is based on combination of two different methods, i.e. the iterative method [35] and the spectral method. The method reduces the differential equation to systems of linear algebraic equations and then the resulting systems are solved by a numerical method. The solutions obtained are compared with Adomian decomposition method and iterative method used in [35] and Adams method [36].

Keywords : Adomian decomposition method (ADM); Iterative method (IM); Spectral method; Fractional logistic equation; Collocation method.

1 Introduction

To describe population growth in a limited environment, Verhulst [28] first presented the classical logistic equation and it has been very popular in population dynamics so far. We can apply the fractional derivative operator on the logistic equation to obtain the fractional order logistic model. Pierre Verhulst published this model in 1838 for the first time [14]. We can describe the continuous logistic model by first order ordinary differential equation. The discerete logistic model is a simple iterative equation which shows the chaotic property in certain regions [11, 29]. There are many variations of the population modeling. To describe the periodic doubling and chaotic characteristic in dynamical system we can use Verhulst model which is a classic example [11]. This model indicates that the population growth may be restricted by some factors like population density [12, 23].

Many studies are focussed on ordinary and partial fractional equations thanks to their recurrent appearance in different applications in fluid mechanics, viscoelasticity, biology, physics and engineering [13]. Most recently, a large amount of literatures are developed regarding the usage of fractional differential equations in non-linear dynamics. Consequently, the solutions of fractional differential equations of physical interest have been of great importance. We can not find exact solutions for most fractional differential equations, so approximate and numerical techniques are applied [15, 16, 19, 20, 21, 22]. Recently to solve the fractional differential equations several numerical and approximate methods, such as variational iteration method [17], iterative method [35], homotopy perturbation method [24],

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Adomian decomposition method [8], homotopy analysis method and collocation method [18, 26] have been employed.

We consider fractional logistic equation of the following form:

$$\begin{cases} D^{\alpha}y(x) = \mu y(x)(1-y(x)), \\ y(0) = y_0. \end{cases} \quad (1.1)$$

where $\mu > 0$, $x > 0$, $0 < \alpha \leq 1$.
The important application of the logistic equation is that it is a model of population growth. The population size at time x is denoted with $y(x)$ and the constant $\mu > 0$ defines the growth rate. Another application of Logistic equation is in medicine, where the logistic differential equation is used to model the growth of tumors. This application can be considered as an extension of the above mentioned use in the frame work of ecology. The existence and the uniqueness of the solution to the proposed problem (1.1) are introduced in details in [6].

In this paper, we describe preliminaries in Sec. 2, in Sec. *3.1* we describe the iterative method and in Sec. *3.2* we give a description of shifted fractional-order Legendre functions. In Sec. *3.3* we use collocation method to obtain the approximate solution for differential equation with initial conditions as a linear combination of Legendre functions. In Sec. *3.4*, we describe the new spectral-iterative method (NSIM) which is a combination of two different methods, one iterative and the other spectral. We study the numerical results in Sec. 4 and review the estimation of the errors in Sec. 5.

2 Preliminaries

Definition 2.1 *In order to proceed, we need the following definitions of fractional derivatives and integrals. First, we introduce the Riemann-Liouville definition of fractional integral operator* J_a^{α} .

Let $\alpha \in R^{+}$. The operator J_a^{α}, defined on the usual Lebesgue space $L_1[a,b]$ by

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad (2.2)$$

$$J_a^0 f(x) = f(x),$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^{α} can be found in [1]. For $f \in L_1[a,b], \alpha, \beta \geq 0$ and $\gamma > -1$, we mention only the following:

(1) $J_a^{\alpha} f(x)$ *exists for almost every* $x \in [a,b]$,

(2) $J_a^{\alpha} J_a^{\beta} f(x) = J_a^{\alpha+\beta} f(x)$,

(3) $J_a^{\alpha} J_a^{\beta} f(x) = J_a^{\beta} J_a^{\alpha} f(x)$,

(4) $J_a^{\alpha}(x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

Definition 2.2 *The fractional derivative of* $f(x)$ *in the Riemann-Liouville sense is defined as*

$$D_a^{\alpha} f(x) = D^m J_a^{m-\alpha} f(x)$$

$$= \frac{d^m}{dx^m} \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f(t)dt, \quad (2.3)$$

where $m \in N$ and satisfies the relations $m-1 < \alpha \leq m$, and $f \in L_1[a,b]$.

Properties of the operator D_a^{α} can be found in [1, 4]. For $m-1 < \alpha \leq m$, $x > a$ and $\gamma > -1$ we mention only the following:

(1) $D_a^{\alpha}(x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-a)^{\gamma-\alpha}$,

(2) $D_a^{\alpha} J_a^{\alpha} f(x) = f(x)$.

3 New spectral iterative method

	$\alpha = \frac{1}{2}, \mu = \frac{1}{5}, \beta = \frac{1}{4}$	
L	$\Vert RESy \Vert_\infty$	Cpu Times
5	$1.6E-07$	0.204
10	$1.0E-13$	0.219
15	$1.8E-20$	0.484
20	$1.6E-27$	0.782

Table 1.

	$\alpha=\frac{1}{4},\ \mu=\frac{1}{4},\ \beta=\frac{1}{10}$	
L	$\Vert RESy\Vert_\infty$	Cpu Times
5	$1.2E-08$	0.250
10	$9.0E-15$	0.232
15	$3.0E-21$	0.437
20	$8.0E-28$	1.484

Table 2.

	$\alpha=\frac{3}{10},\ \mu=\frac{1}{4},\ \beta=\frac{1}{10}$	
L	$\Vert RESy\Vert_\infty$	Cpu Times
5	$8.0E-09$	0.187
10	$3.0E-15$	0.250
15	$6.0E-22$	0.656
20	$7.0E-29$	1.531

Table 3.

	$\alpha=\frac{4}{5},\ \mu=\frac{1}{2},\ \beta=\frac{1}{5}$	
L	$\Vert RESy\Vert_\infty$	Cpu Times
5	$3.0E-06$	0.172
10	$2.0E-12$	0.266
15	$2.0E-19$	0.453
20	$6.0E-27$	0.844

Table 4.

	$\alpha=\frac{1}{2},\ \mu=\frac{1}{5},\ \beta=\frac{1}{4}$	
L	$\Vert RESy\Vert_\infty$	Cpu Times
5	$5.0E-08$	1.297
10		*failed*
15		*failed*
20		*failed*

Table 5.

3.1 Iterative method

Consider the following nonlinear differential equation:

$$L[y]+N[y]=f(x), \tag{3.4}$$

where L is a linear operator and N is a nonlinear operator from a Banach space E into E, f is a given function in E and we are looking for $y\in E$ satisfying (3.4).
Daftardar and Jafari [35], suggest that the solution of $y(x)$ be expanded by the infinite series solution

$$y(x)=\sum_{k=0}^{\infty}y_k(x), \tag{3.5}$$

and the nonlinear operator N in Eq. (3.4) is decomposed as follows:

$$N(y)=\sum_{i=0}^{\infty}A_i(y_0,y_1,\cdots,y_i), \tag{3.6}$$

where $A_0=0$ and A_i are obtained by

$$A_i=N\left(\sum_{k=0}^{i}y_k\right)-N\left(\sum_{k=0}^{i-1}y_k\right).$$

Substituting (3.5) and (3.6) into (3.4) gives the following recursive scheme:

$$\begin{cases} L[y_0]=f(x), \\ L[y_{i+1}]=-A_i, \qquad i=0,1,\cdots. \end{cases} \tag{3.7}$$

We define the $M+1-$th term approximation solution as

$$\phi_M(x)=\sum_{i=0}^{M}y_i(x), \tag{3.8}$$

where, if convergence happen,

$$y(x)=\lim_{M\to\infty}\phi_M(x).$$

3.2 Shifted fractional-order Legendre function

The Legendre polynomials, denoted by $l_n(x)$, are orthogonal with respect to the weight function $w(x)=1$ over $I=[-1,1]$, namely [9],

$$\int_{-1}^{1}l_n(x)l_m(x)dx=\frac{2}{2n+1}\delta_{nm},$$

where

$$\delta_{nm}=\begin{cases}1, & n=m,\\ 0, & O.W.\end{cases}$$

In order to use these polynomials on the interval $[0,1]$, we define the so-called shifted Legendre polynomials by introducing the change of variable $x=2t-1$. Let the shifted Legendre polynomials $l_n(x)$ be denoted by $L_n(t)$. The shifted Legendre polynomials are orthogonal with respect to the weight function $w(t)=1$ in the interval $[0,1]$ with the orthogonality property

$$\int_{0}^{1}L_n(t)L_m(t)dt=\frac{2}{2n+1}\delta_{nm}.$$

Then $L_i(t)$ can be obtained as follows:

$$L_{n+1}(t)=\frac{(2n+1)(2t-1)}{n+1}L_n(t)-\frac{n}{n+1}L_{n-1}(t), n=1,2,...$$

	$\alpha=\frac{1}{4}, \mu=\frac{1}{4}, \beta=\frac{1}{10}$	
L	$\|RESy\|_\infty$	Cpu Times
5	$4.0E-06$	3.891
10		*failed*
15		*failed*
20		*failed*

Table 6.

	$\alpha=\frac{3}{10}, \mu=\frac{1}{4}, \beta=\frac{1}{10}$	
L	$\|RESy\|_\infty$	Cpu Times
5	$3.0E-06$	3.906
10		*failed*
15		*failed*
20		*failed*

Table 7.

	$\alpha=\frac{4}{5}, \mu=\frac{1}{2}, \beta=\frac{1}{5}$	
L	$\|RESy\|_\infty$	Cpu Times
5	$2.0E-06$	4.297
10		*failed*
15		*failed*
20		*failed*

Table 8.

	$\alpha=\frac{1}{2}, \mu=\frac{1}{5}, \beta=\frac{1}{4}$	
L	$\|RESy\|_\infty$	Cpu Times
5	$3.0E-07$	1.375
10	$1.0E-11$	1.672
15	$1.2E-15$	2.328
20	$7.0E-20$	3.266

Table 9.

$$L_0(t) = 1, \quad L_1(t) = 2t-1. \tag{3.9}$$

Note that $L_n(0) = (-1)^n$ and $L_n(1) = 1$. The shifted fractional-order Legendre functions defined by introducing the change of variable $t = x^\alpha$ with $\alpha > 0$ on shifted Legendre polynomials, are denoted by $FL_i^\alpha(x)$[10].

Hence $FL_i^\alpha(x)$ satisfy the following recurrence relation

$$FL_{n+1}^\alpha(x) = \frac{(2n+1)(2x^\alpha-1)}{(n+1)} FL_n^\alpha(x) - \frac{n}{n+1} FL_{n-1}^\alpha(x), \quad n = 1,2,3,...,$$

$$FL_0^\alpha(x) = 1, \quad FL_1^\alpha(x) = 2x^\alpha - 1.$$

3.3 Collocation method

Consider the linear fractional differential equation:

$$\sum_{k=0}^{n} D^{\alpha_k} y(x) = g(x), \tag{3.10}$$

where $\alpha_k \in (k, k+1]$, with initial conditions

$$y^{(i)}(0) = \beta_i, \quad i = 0,1,\cdots,n. \tag{3.11}$$

The unknown function $y(t)$ in problem (3.10), can be approximated by a truncated series of Legendre functions,

$$y_m(t) = \sum_{j=0}^{m} c_j FL_j^\alpha(t), \tag{3.12}$$

where c_j are unknowns. Here, the main purpose is to find c_j. In order to achieve this end, putting (3.12) in (3.10) and (3.11) we obtain:

$$\sum_{j=0}^{m} c_j \sum_{k=0}^{n} D^{\alpha_k} FL_j^\alpha(t) = g(t), \tag{3.13}$$

$$\sum_{j=0}^{m} c_j FL_j^{\alpha^{(i)}}(0) = \beta_i, \; i = 0,1,\cdots,n. \tag{3.14}$$

Relation (3.14) forms a system with $n+1$ equations and $m+1$ unknowns, to construct the remaining $m-n$ equations, we substitute Legendre-Guass points $\left\{t_i\right\}_{i=1}^{m-n}$ in (3.13), to obtain $m-n$ equations. So, reduces the obtaining to the solution of the system $AC = b$, where A, C and b are $A = \left[\frac{A1}{A2}\right]$, $C = [c_0, c_1, \cdots, c_m]^T$, $b = \left[\frac{b1}{b2}\right]$ and matrices $A1_{(m-n)\times(m+1)}$ and $A2_{(n+1)\times(m+1)}$ are defined by

$$A1[i,j] = \sum_{k=0}^{n} D^{\alpha_k} FL_j^\alpha(t_i), \; i = 1,2,\cdots,m-n, \quad j = 0,1,\cdots,m,$$

$$A2[i,j] = FL_j^{\alpha^{(i)}}(0), \quad i = 0,1,\cdots,n, \quad j = 0,1,\cdots,m,$$

and vectors $b1_{(m-n)\times 1}$, $b2_{(m-n)\times 1}$ are defined by
$b1[i] = g(t_i), \quad i = 1,2,\cdots,m-n,$
$b2[i] = \beta_i, \quad i = 0,1,\cdots,n.$

	$\alpha=\frac{1}{4}, \mu=\frac{1}{4}, \beta=\frac{1}{10}$	
L	$\|RESy\|_\infty$	Cpu Times
5	$3.0E-06$	2.469
10	$1.2E-09$	3.422
15	$5.0E-13$	4.531
20	$1.6E-16$	5.984

Table 10.

	$\alpha=\frac{3}{10}, \mu=\frac{1}{4}, \beta=\frac{1}{10}$	
L	$\|RESy\|_\infty$	CPU Times
5	$3.0E-06$	2.515
10	$1.0E-09$	3.516
15	$2.5E-13$	4.828
20	$5.0E-17$	7.110

Table 11.

3.4 The methodology

Consider the fractional logistic equation

$$D^\alpha y(x)=\mu y(x)(1-y(x)), \tag{3.15}$$

where $\alpha\in(0,1]$ [27], with initial condition

$$y(0)=\beta. \tag{3.16}$$

The nonlinear equation (3.15), can be written by

$$\begin{cases} D^\alpha y(x)-\mu y(x)=-\mu y^2(x), \\ y(0)=\beta. \end{cases} \tag{3.17}$$

Substituting the $y(x)=\sum_{k=0}^{\infty} y_k(x)$ in the nonlinear fractional logistic equation, we have:

$$\sum_{k=0}^{\infty} D^\alpha y_k(x)-\sum_{k=0}^{\infty}\mu y_k(x)=-\mu\left(\sum_{k=0}^{\infty} y_k(x)\right)^2$$
$$=-\sum_{k=0}^{\infty} A_k,$$

where $A_0=0$ and

$$A_k=-\mu(\sum_{l=0}^{k-1} y_l)^2+\mu(\sum_{l=0}^{k} y_l)^2.$$

The solution of problem (3.17), is

$$y(x)=\sum_{k=0}^{\infty} y_k(x)$$

	$\alpha=\frac{4}{5}, \mu=\frac{1}{2}, \beta=\frac{1}{5}$	
L	$\|RESy\|_\infty$	CPU Times
5	$1.6E-05$	6.766
10	$2.0E-08$	12.250
15	$2.0E-11$	18.953
20	$9.0E-15$	23.984

Table 12.

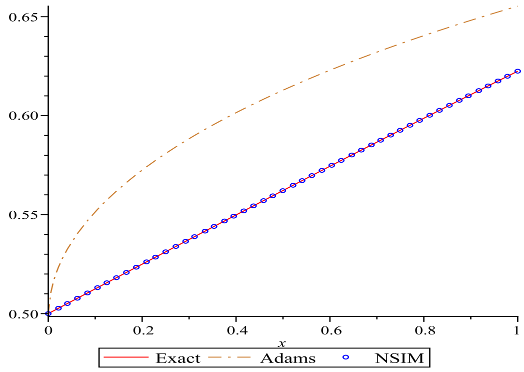

Figure 1: Comparing the exact solution and approximate solution by NSIM and Adams method.

where $y_k(x)$ satisfies in

$$D^\alpha y_k(x)-\mu y_k(x)=-A_k, \quad k=0,1,\cdots. \tag{3.18}$$

We solve the above linear equation using the spectral method. The function $y_k(x)$ can be approximated as

$$y_k(x)=\sum_{j=0}^{\infty} c_j^{(k)} FL_j^\alpha(x),$$

where the unknown coefficients $c_j^{(k)}$ are determined by using the collocation method. The residual function associated to the equation (3.18) is

$$RESy_k(x)=D^\alpha y_k(x)-\mu y_k(x)+A_k, \quad k=0,1,\cdots.$$

By imposing the initial condition (3.16), we have

$$\sum_{j=0}^{\infty} c_j^{(k)} FL_j^\alpha(0)=\begin{cases} \beta, & k=0, \\ 0, & k=1,2,\cdots. \end{cases}$$

For all k, the matrix form of the above system is:

$$MC^{(k)}=b^{(k)},$$

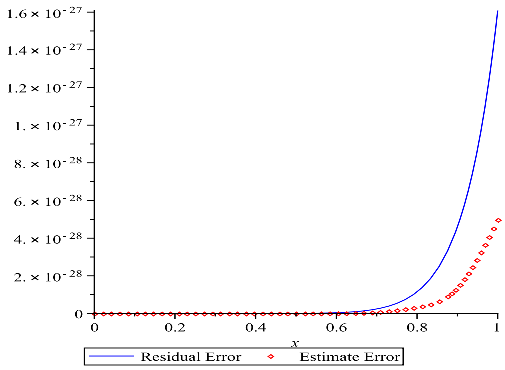

Figure 2: $\alpha = \frac{1}{2}$, $\mu = \frac{1}{5}$, $\beta = \frac{1}{4}$

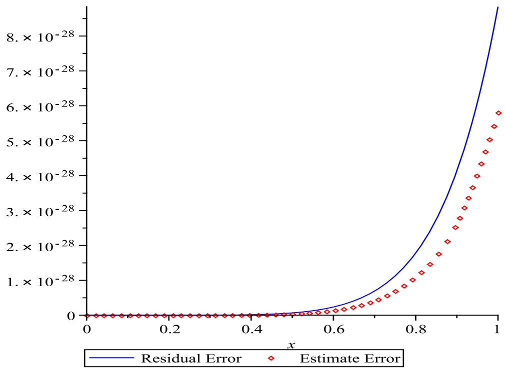

Figure 3: $\alpha = \frac{1}{4}$, $\mu = \frac{1}{4}$, $\beta = \frac{1}{10}$

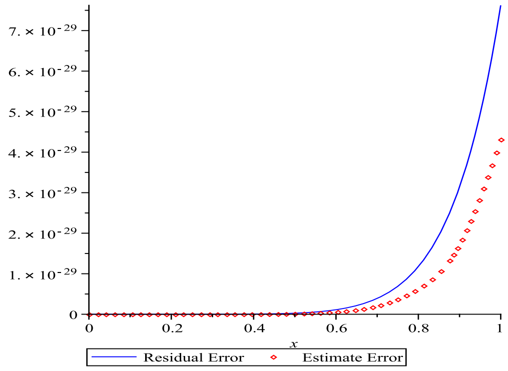

Figure 4: $\alpha = \frac{3}{10}$, $\mu = \frac{1}{4}$, $\beta = \frac{1}{10}$

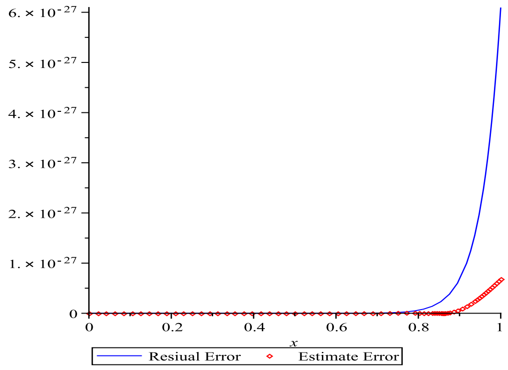

Figure 5: $\alpha = \frac{4}{5}$, $\mu = \frac{1}{2}$, $\beta = \frac{1}{5}$

$$M = [m_{ij}]_{(n+1)(n+1)},$$
$$C^{(k)} = [c_0^{(k)}, c_1^{(k)}, ..., c_n^{(k)}]^t,$$
$$b^{(k)} = [b_0^{(k)}, b_1^{(k)}, ..., b_n^{(k)}]^t.$$

Suppose that $\left\{x_i\right\}_{i=1}^{n}$ are zeros of Legendre polynomial of degree n, we have

$$m_{0j} = FL_j^{\alpha}(0),$$
$$j = 0, 1, 2, ..., n,$$
$$m_{ij} = D^{\alpha} FL_j^{\alpha}(x_i) - \mu FL_j^{\alpha}(x_i),$$
$$j = 0, 1, 2, ..., n, \quad i = 1, 2, ..., n.$$

For $i = 1, 2, 3, ..., n$, $k = 1, 2, 3, ...$, we have

$$b_0^{(0)} = \beta, \quad b_i^{(0)} = 0,$$

and

$$b_0^{(k)} = 0, \quad b_i^{(k)} = -A_k(x_i).$$

The approximate solution of (3.15) with $L+1$ terms is

$$y_{L,n} = \sum_{k=0}^{L} y_k(x), \tag{3.19}$$

where

$$y_k(x) = \sum_{j=0}^{n} c_j^{(k)} FL_j^{\alpha}(x). \tag{3.20}$$

4 Numerical study

Consider the following logistic initial value problem:

$$\begin{cases} D^{\alpha} y(x) = \mu y(x)(1 - y(x)), \\ y(0) = \beta. \end{cases} \tag{4.21}$$

We demonstrate the effectiveness of the proposed method (NSIM) by applying it on four values of α, β and μ for above problem. For each case, the maximum norm of the residual error of $y_{L,n}(x)$ is presented. Tables 1, 2, 3 and 4 shows the obtained numerical results of the (NSIM), tables 5, 6, 7 and 8 shows the obtained numerical results of the (IM) and tables 9, 10, 11 and 12 shows the obtained numerical results of the (ADM). The exact solution of (4.21) is $y(x) = \frac{e^{0.5x}}{1+e^{0.5x}}$ for $\alpha = 0.5$,

$\mu = 0.5$ and $\beta = 0.5$. The figure 1 shows the solutions obtained by (NSIM) and Adams method with $h = 0.001$ and exact solution.

All the computations associated with the method have been performed by a personal computer having the Intel Pentium 4, 2.8 GHz processor, 1GB RAM and using Maple 13 with 32 digits precision.

5 Estimation of the errors

The approximate solution of (4.21) is $y_{L,n}(x)$ and the exact solution is $y(x)$. Substituting $y_{L,n}(x)$ and $y(x)$ in (4.21), we obtain the following results.

$$\begin{cases} D^{\alpha}y(x) - \mu y(x)(1 - y(x)) = 0, \\ y(0) = \beta, \end{cases} \tag{5.22}$$

$$\begin{cases} D^{\alpha}y_{L,n}(x) - \mu y_{L,n}(x)(1 - y_{L,n}(x)) = R(x), \\ y_{L,n}(0) = \beta, \end{cases} \tag{5.23}$$

where $R(x)$ is the residual error. From (5.22) and (5.23) we obtain

$$\begin{cases} D^{\alpha}E_{L,n}(x) = \mu E_{L,n}(x)(1 + E_{L,n}(x) \\ \qquad\qquad -2y_{L,n}(x)) + R(x), \\ E_{L,n}(0) = 0, \end{cases} \tag{5.24}$$

where $E_{L,n} = y_{L,n}(x) - y(x)$ is error of solution. The solution of the (5.24) is an estimate of the error of $y_{L,n}(x)$. To hove convergence we should have $|R(x)| \simeq |E_{L,n}(x)|$ and $\lim_{L,n\to\infty} |E_{L,n}(x)| = 0$.

We calculate $E_{L,n}(x)$ by Adams method for $h = 0.1$ and compare with residual error of NSIM for $L = 20$ in figures 2, 3, 4 and 5.

6 Conclusion

In this paper we proposed a new method to solve logistic equations of fractional order. This method was based on combination of iterative and spectral methods, which reduced nonlinear differential equations to systems of linear algebraic equations. The obtained approximate solutions have shown the effectiveness of our new method.

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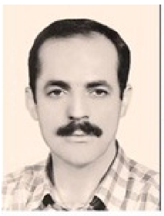

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