A Study on Intuitionistic Fuzzy and Normal Fuzzy M-Subgroup, M-Homomorphism and Isomorphism

M. Oqla Massadeh

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Abstract

In this paper, we introduce some properties of an intuitionistic normal fuzzy m-subgroup of m-group with m-homomorphism and isomorphism. We study the image, the pre-image and the inverse mapping of the intuitionistic normal fuzzy m-subgroups.

Keywords: Intuitionistic Fuzzy Sets; M-Groups; Intuitionistic Fuzzy M-Subgroups; Intuitionistic Normal Fuzzy M-Subgroups; M-Homomorphism.

1 Introduction

In 1971 Rosenfeld. A [8] introduced the concept of fuzzy subgroups. In 1981 Wu [10] studied the normal fuzzy subgroups. Gu. Wx et al [3] further studied in 1994 the fuzzy groups theory and gave some new concepts such as fuzzy m-subgroups, normal fuzzy m-subgroups. Several mathematicians have followed them in investigating the fuzzy m-subgroups in [5, 6, 9]. The intuitionistic fuzzy set idea was first published by Atanassov [1, 2] as a generalization of the fuzzy sets notion. The basic concepts of intuitionistic fuzzy subgroups are in [4, 7]. In this paper, we introduce some properties of an intuitionistic normal fuzzy m-subgroups of m-groups with m-homomorphism and isomorphism and we study the image, pre-image and other properties in this subject.

2 Preliminaries

Definition 2.1 [3] Let G be a group, M be a set, if

(i) \( m x \in G \quad \forall x \in G, \quad x \in M \).

(ii) \( m(xy) = (mx)y = x(my) \quad \forall x, y \in G, \quad x \in M \).

Then m is said to be a left operator of G, M is said to be a left operator set of G. G is said to be a group with operators. We use phrase ”G is an M-group” in stead of a group with operators. If a subgroup of M-group G is also M-group, then it is said to be an M subgroup of G.

Definition 2.2 [1] An intuitionistic fuzzy subse \( \mu \) in a set X is defined as an object of the form \( \mu = \{ < x, \delta_\mu(x), \lambda_\mu(x) > : x \in X \} \), where \( \delta_\mu : X \rightarrow [0, 1] \) and \( \lambda_\mu : X \rightarrow [0, 1] \) define the degree of membership and the degree of non- membership of the element \( x \in X \) respectively and for every \( x \in X \) satisfying \( 0 \leq \delta_\mu(x) + \lambda_\mu(x) \leq 1 \). All the intuitionistic fuzzy sets on X are written as \( IFS(X) \) for short.
Definition 2.3 [11] Let $X$, $Y$ be a non empty classical sets, $\Phi : X \to Y$ be a mapping and $\mu = \{ y \in Y, \delta_{\mu}(y), \lambda_{\mu}(y) \}$ be an intuitionistic fuzzy set on $Y$ ($\mu \in IFS(Y)$) $\Psi_{\Phi}^{-1} : IFS(Y) \to IFS(X)$ is the inverse mapping induced by $\Phi$, the pre-image $\Psi_{\Phi}^{-1}(\mu) = \{ x \in X ; \Psi_{\Phi}^{-1}(\delta_{\mu}(x)), \Psi_{\Phi}^{-1}(\lambda_{\mu}(x)) \}$. Where $\Psi_{\Phi}^{-1}(\delta_{\mu})$, $\Psi_{\Phi}^{-1}(\lambda_{\mu})$ obey the classical extension principle of Zadeh. L. A.

Definition 2.4 [11] Let $X$, $Y$ be a non empty classical sets, $\Phi : X \to Y$ be a mapping and $\mu = \{ y \in Y, \delta_{\mu}(y), \lambda_{\mu}(y) \}$ be an intuitionistic fuzzy set on $Y$ ($\mu \in IFS(Y)$) $\Psi_{\Phi} : IFS(Y) \to IFS(X)$ is the inverse mapping induced by $\Phi$, the image $\Psi_{\Phi}(\mu)$ of is an intuitionistic fuzzy set on $Y$, and define $\Psi_{\Phi}(\mu) = \{ y \in Y, \Psi_{\Phi}(\delta_{\mu}(y), \Psi_{\Phi}(\lambda_{\mu}(y)) \}$ Where

$$\Psi_{\Phi}(\delta_{\mu}(y)) = \begin{cases} \sup \{ \delta_{\mu}(x); \Phi(x) = y, x \in X \}; \\ \Phi^{-1}(y) \neq \phi, \\ 0; \Phi^{-1}(y) = \phi \end{cases}$$

$$\Psi_{\Phi}(\lambda_{\mu}(y)) = \begin{cases} \inf \{ \lambda_{\mu}(x); \Phi(x) = y, x \in X \}; \\ \Phi^{-1}(y) \neq \phi, \\ 0; \Phi^{-1}(y) = \phi \end{cases}$$

Definition 2.5 Let $G$ be an $M$-group and $\mu$ be an intuitionistic fuzzy group of $\delta_{\mu}(mx) \geq \delta_{\mu}(x)$ and $\lambda_{\mu}(mx) \leq \lambda_{\mu}(x)$ for all $x \in G$ and $m \in M$ then $\mu$ is said to be an intuitionistic fuzzy group with operator of $G$. We use the phrase $\mu$ is an intuitionistic fuzzy $M$-subgroup of $G$. All the intuitionistic fuzzy $M$-subgroups on $G$ are written as $IFMS(G)$ for short.

Example 2.1 Let $H$ be $M$-subgroup of an $M$-group $G$ and let $\mu$ be an intuitionistic fuzzy set in $G$ defined by.

$$\delta_{\mu}(x) = \begin{cases} 0.8 & : x \in H, \\ 0 & : \text{otherwise} \end{cases}$$

$$\lambda_{\mu}(x) = \begin{cases} 0.4 & : x \in H, \\ 0.6 & : \text{otherwise} \end{cases}$$

For all $x \in G$. Then it is easy to verify that $\mu$ is an intuitionistic fuzzy $M$-subgroup of $G$. All the intuitionistic fuzzy $M$-subgroups on $G$ are written as $IFMS(G)$ for short.

Proposition 2.1 If $\mu$ is an intuitionistic fuzzy $M$-subgroup of an $M$-group $G$, then for any $x, y \in G$ and $m \in M$

$$1-\delta_{\mu}(m(xy)) \geq \min \{ \delta_{\mu}(mx), \delta_{\mu}(my) \}$$

and

$$\lambda_{\mu}(m(xy)) \leq \max \{ \lambda_{\mu}(mx), \lambda_{\mu}(my) \}$$

and

$$\delta_{\mu}(m(xy)) \geq \delta_{\mu}(x) \land \delta_{\mu}(m(xy)) \leq \delta_{\mu}(x) \land \lambda_{\mu}(m(xy)) \geq \lambda_{\mu}(x).$$

Definition 2.6 Let $G$ be a $m$-group, $\mu$ be an intuitionistic fuzzy $m$-subgroup of $G$, then $\mu$ is called intuitionistic normal fuzzy $m$-subgroup if $\delta_{\mu}(m(xy^{-1})) \geq \delta_{\mu}(y)$ and $\lambda_{\mu}(m(xy^{-1})) \leq \lambda_{\mu}(m(xy))$ for all $x, y \in G$ and $m \in M$. All the intuitionistic fuzzy $M$-subgroups on $G$ are written as $INFMS(G)$ for short.

Definition 2.7 [5] Let $G_1$ onto $G_2$ be two $m$-groups, $\Psi$ be a homomorphism from $G_1$ onto $G_2$. If $\Phi(mx) = m \Phi(x)$ for all $x \in G_1$ and $m \in M$, then $\Psi$ is called m-homomorphism.

3 M-Homomorphism and isomorphism for intuitionistic fuzzy $m$-subgroups

Theorem 3.1 Let $G_1, G_2$ be $m$-groups, $\Phi : G_1 \to G_2$ be $m$-homomorphic mapping. If $\mu \in IFMS(G_1), \gamma \in IFMS(G_2)$. Then $\Psi_{\Phi}(\mu) \in IFMS(G_2)$ and $\Psi_{\Phi}^{-1}(\gamma) \in IFMS(G_1)$.

Theorem 3.2 Let $G_1, G_2$ be $m$-groups, $\Phi : G_1 \to G_2$ be $m$-homomorphic mapping. If $\mu$ be an intuitionistic fuzzy $m$-subgroup of $G_1$. Define for any $x \in G_1$, then $m^{-1} \in IFMS(G_1)$ and $\mu^{-1} \in IFMS(G_1).$

$$\delta_{\mu^{-1}}(x) = \delta_{\mu}(x^{-1})$$

and

$$\lambda_{\mu^{-1}}(x) = \lambda_{\mu}(x^{-1})$$

and $\Psi_{\Phi}(\mu^{-1}) = (\Psi_{\Phi}(\mu))^{-1}$.

Theorem 3.3 Let $G_1, G_2$ be $m$-groups, $\Phi : G_1 \to G_2$ be $m$-homomorphic surjective mapping. $\mu \in IFMS(G_1)$ then $\Psi_{\Phi}(\mu) \in IFMS(G_2)$.

Proof. By Theorem 3.1, clearly we have $\Psi_{\Phi}(\mu) \in IFMS(G_2)$. We need to prove the normality fuzzy for $\Psi_{\Phi}(\mu)$, for any $y_1, y_2 \in G_2, m \in M$ by the extension principle, $\Phi : G_1 \to G_2$ is $m$-homomorphic surjective mapping. This means that $\Phi(G_1) = G_2, \Phi^{-1}(my_1) \neq \phi$ and $\Phi^{-1}(my_2) \neq \phi, \Phi^{-1}(m(y_1y_21^{-1})) \neq \phi$ and we have

$$\Psi_{\Phi}(\delta_{\mu})(m(y_1y_21^{-1})) = \sup_{z \in \Phi^{-1}(m(y_1y_21^{-1}))} \delta_{\mu}(z)$$
\[ \Psi_\Phi(\delta_\nu)(my_2) = \sup_{z \in \Phi^{-1}(my_2)} \delta_\nu(z) \text{ For all } mx_2 \in \Phi^{-1}(my_2) \]

\[ \in \Phi^{-1}(my_2) \text{ and for all } mx_1 \in \Phi^{-1}(my_1), \text{ then } (mx_1)^{-1} \in \Phi^{-1}((my_1)^{-1}) \text{ since } \mu \in IFMS(G). \]

We get \( \delta_\nu \in (m(x_1x_2^{-1})) \geq \delta_\nu \cdot (mx_2) \), as \( \Phi \) is m-homomorphism then

\[ \Phi(m(x_1x_2^{-1})) = m(\Phi(x_1)\Phi(x_2)) = m(\Phi(x_1)\Phi(x_2)(\Phi(x_1))^{-1}) = m(y_1y_2y_1^{-1}). \]

Consequently \( m(x_1x_2 x_1^{-1}) \in \Phi^{-1}(m(y_1y_2y_1^{-1})) \), therefore

\[ \sup_{z \in \Phi(m(y_1y_2y_1^{-1}))} \delta_\nu(z) \geq \sup_{mx_1 \in \Phi^{-1}(my_1), mx_2 \in \Phi^{-1}(my_2)} \delta_\nu(mx_2) \]

This means that \( \Psi_\Phi(\delta_\nu)(m(y_1y_2y_1^{-1}) \geq \Psi_\Phi(\delta_\nu)(my_2) \) for all \( y_1, y_2 \in G_2, m \in M \). On the other hand, similarly \( y_1, y_2 \in G_2, m \in M \Phi^{-1}(my_1) \neq \phi \) and \( \Phi^{-1}(my_2) \neq \phi \),

\[ \Phi^{-1}(m(y_1y_2y_1^{-1})) \neq \phi \text{ and } mx_2 \in \Phi^{-1}(my_2), mx_1 \in \Phi^{-1}(my_1) \text{ then } (mx_1)^{-1} \in \Phi^{-1}(my_1) \text{ and } \lambda_\mu(m(x_1x_2^{-1})) \leq \lambda_\mu(mx_2), \text{ thus } \inf_{z \in \Phi^{-1}(m(y_1y_2y_1^{-1}))} \lambda_\mu(z) \leq \inf_{mx_2 \in \Phi^{-1}(my_2)} \lambda_\mu(mx_2) \]

This means that \( \Psi_\Phi(\lambda_\mu)(m(y_1y_2y_1^{-1})) \in \Psi_\Phi(\lambda_\mu)(my_2) \) for all \( y_1, y_2 \in G_2, m \in M \). Hence \( \Psi_\Phi(\mu)(my_2) \in INFMS(G_2) \).

**Theorem 3.4**: Let \( G_1, G_2 \) be \( m \)-groups, \( \Phi \): \( G_1 \rightarrow G_2 \) be \( m \)-homomorphism mapping. If \( \gamma \in INFMS(G_2) \), then \( \Psi_\Phi^{-1}(\gamma) \in INFMS(G_1) \).

**Proof**: By Theorem 3.1 \( \Psi_\Phi^{-1}(\gamma) \in INFMS(G_1) \), thus we need to prove the normality fuzzy. Since \( \gamma \in INFMS(G_2) \) for any \( x, y \in G_1, m \in M \) from the extension principle, we obtain

\[ \Psi_\Phi^{-1}(\delta_\nu)(m(xyx^{-1})) = (\delta_\nu)(\Phi(m(xyx^{-1})) = \delta_\nu(m(\Phi(x)\Phi(y)\Phi(x^{-1})) = \delta_\nu(m(\Phi(x)\Phi(y)\Phi(x^{-1})) \geq \delta_\gamma(m(\Phi(y))) = \Psi_\Phi(\delta_\gamma)(my) \]

Similarly we get

\[ \Psi_\Phi^{-1}(\lambda_\mu)(m(xyx^{-1})) = (\lambda_\mu)(\Phi(m(xyx^{-1})) = \lambda_\mu(m(\Phi(x)\Phi(y)\Phi(x^{-1})) = \lambda_\mu(m(\Phi(x)\Phi(y)\Phi(x^{-1})) \leq \lambda_\mu(m(\Phi(y))) = \Psi_\Phi(\lambda_\mu)(my) \]

Therefore \( \Psi_\Phi^{-1}(\gamma) \in INFMS(G_1) \).

**Theorem 3.5**: Let \( G_1, G_2 \) be \( m \)-groups, \( \Phi \): \( G_1 \rightarrow G_2 \) be \( m \)-homomorphism mapping. If \( \mu \in INFMS(G_2) \), then \( \mu^{-1} \in INFMS(G_1) \) and \( \Psi_\Phi(\mu^{-1}) = (\Psi_\Phi(\mu))^{-1} \).

**Proof**: Let \( \mu \) be intuitionistic fuzzy \( m \)-subgroup of \( G_1 \), then \( \mu^{-1} = \{ x \in G_1; \delta_{u^{-1}}(mx), \lambda_{u^{-1}}(mx) \in M \} \) where \( \delta_{u^{-1}}(mx) = \delta_u(mx) \) and \( \lambda_{u^{-1}}(mx) = \lambda_u(mx^{-1}) \) since \( \mu \in INFMS(G_1) \) and by Theorem 3.1. We know \( \mu^{-1} \in INFMS(G_1) \), for any \( x, y \in G_1, m \in M \) we have \( \delta_{u^{-1}}(mx) = \delta_u(mx^{-1}) \) and \( \lambda_{u^{-1}}(mx) = \lambda_u(mx^{-1}) \). Therefore \( \Phi(\mu^{-1}) = \Phi(\mu) \).

**Corollary 3.1**: Let \( G_1, G_2 \) be \( m \)-groups, \( \Phi \): \( G_1 \rightarrow G_2 \) be \( m \)-isomorphic mapping. If \( \gamma \in INFMS(G_1) \), then \( \Psi_\Phi^{-1}(\gamma) = \Psi_\Phi(\gamma) \).

**Theorem 3.6**: Let \( G_1, G_2 \) be \( m \)-groups, \( \Phi \): \( G_1 \rightarrow G_2 \) be an isomorphic mapping. If \( \mu \in INFMS(G_1) \), then \( \Psi_\Phi^{-1}(\mu) = \mu \).

**Proof**: Let \( x \in G_1, m \in M \) and \( \Phi(mx) = my \) as \( \Phi \) is an isomorphic mapping \( \Psi^{-1}(my) = \{ mx \} \). Applying the extension principle we obtain \( \Psi_\Phi^{-1}(\delta_\nu)(mx) = \Psi_\Phi(\delta_\nu)(\Phi(mx)) = \Psi_\Phi(\delta_\nu)(\Phi(mx)) = \sup_{mx \in \Phi^{-1}(my)} \lambda_\mu(mx) = \lambda_\mu(mx) \)

Hence \( \Psi_\Phi^{-1}(\mu) = \mu \).

**Corollary 3.2**: Let \( G_1, G_2 \) be \( m \)-groups. 1- If \( \Phi \): \( G_1 \rightarrow G_2 \) be an isomorphic mapping and \( \gamma \in INFMS(G_1) \) then \( \Psi_\Phi(\gamma) = \gamma \).

2- If \( \Phi \): \( G_1 \rightarrow G_2 \) be an automorphism mapping and \( \mu \in INFMS(G_1) \), then \( \Psi_\Phi(\mu) = \mu \) if \( \Psi_\Phi^{-1}(\mu) = \mu \).

**4 Conclusion**: Further work is in progress in order to develop the intuitionistic anti \( L \)-normal fuzzy \( m \)-subgroups and its applications and properties.

**References**


M. Oqla Massa’deh is Ph.D of applied mathematics. He is member of Department of Applied Science, Ajloun College, Al-Balqa’ Applied University, Jordan. His research interests include numerical solution of fuzzy functional equations and Fuzzy numerical Analysis.