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Numerical solution of general nonlinear Fredholm-Volterra integral equations using Chebyshev approximation

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Abstract

A numerical method for solving nonlinear Fredholm-Volterra integral equations of general type is presented. This method is based on replacement of unknown function by truncated series of well known Chebyshev expansion of functions. The quadrature formulas which we use to calculate integral terms have been estimated by Fast Fourier Transform (FFT). This is a grate advantage of this method which has lowest operation count in contrast to other early methods which use operational matrices (with huge number of operations) or involve intermediate numerical techniques for evaluating intermediate integrals which presented in integral equation or solve special case of nonlinear integral equations. Also rate of convergence are given. The numerical examples show the applicability and accuracy of the method.

Keywords : Nonlinear Fredholm-Volterra integral equation; Chebyshev polynomials; Error analysis; Fast Fourier Transform.

1 Introduction

In this paper we present a computational method for solving general nonlinear Fredholm-Volterra integral equations of the second kind:

$$x(s) = y(s) + \lambda_1 \int_0^s K_1(s,t) f(t, x(t))\, dt + \lambda_2 \int_0^1 K_2(s,t) g(t, x(t)) dt, \quad 0 \le s,t \le 1. \tag{1.1}$$

Several numerical methods for approximating the solution of linear and nonlinear integral equations are known [1]-[19]. Brunner in [7] applied a collocation-type method and Ordokhani in [17] applied rationalized Haar function to nonlinear Volterra-Fredholm integral equations. A variation of the Nystrom method was presented in [14]. A collocation type method was developed in [12]. Also more recent works have solved simple case of these equations with operational matrices with more huge computations and operation counts ([5],[11],[14],[15],[18],[19]). Borzabadi in [6] converted the nonlinear Fredholm integral equation to an optimal control problem and then used a linear programming to solve the problem. Orthogonal functions and polynomials receive attention in dealing with various problems such as integral equations. The main characteristic of using orthogonal basis is that it reduces these problems to solving a system of nonlinear algebraic equations by truncated approximating series

$$x(t) \simeq x_N(t) = \sum_{i=0}^{N-1} c_i T_i(t),$$

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where function $x(t) \in L^2([0,1])$ and $c_n = (f(t), T_n(t))$, in which $(.,.)$ denotes the inner product in $L^2([0,1])$ also $\mathbf{C}$ and $\mathbf{T}$ are matrices given by

$$\mathbf{C} = [c_0, c_1, \dots c_N]^T,$$
$$\mathbf{T}(t) = [T_0(t), T_1(t), \dots, T_N(t)]^T, \tag{1.2}$$

where $T_n(t)$, $0 \le n \le N$ are Chebyshev polynomials of the first kind and degree n which are orthogonal with respect to the weight function $\omega(t) = 1/\sqrt{1-t^2}$ on the interval [-1,1]. These polynomials satisfy the following recursive formula,

$$\begin{array}{l} T_0(t) = 1, \quad T_1(t) = t, \\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \\ m = 1, 2, \dots. \end{array}$$

2 Fast method of solution for general nonlinear integral equations

Consider the nonlinear integral equation (1.1). At first we approximate $x(t)$ as

$$x(t) \simeq \mathbf{C}^T\mathbf{T}(t), \tag{2.3}$$

then we substitute this approximation into eq. (1.1) to get

$$\begin{array}{l} \mathbf{C}^T\mathbf{T}(s) = y(s) + \lambda_1 \int_0^s K_1(s,t) \\ f(t, \mathbf{C}^T\mathbf{T}(t))dt + \lambda_2 \int_0^1 K_2(s,t) \\ g(t, \mathbf{C}^T\mathbf{T}(t))dt. \end{array} \tag{2.4}$$

In order to use Gaussian integration formula for eq. (2.4), we transfer the intervals $[0, s_i]$ and $[0, 1]$ into interval $[-1, 1]$ by transformations

$$\tau_1 = \frac{2}{s_i}t - 1, \qquad \tau_2 = 2t - 1.$$

For Chebyshev polynomials we consider the collocation points

$$s_i = \cos(\frac{i\pi}{N}), \qquad i = 0, 1, \dots, N, \tag{2.5}$$

let

$$\begin{array}{l} H_1(s,t) = K_1(s,t)f(t, \mathbf{C}^T\mathbf{T}(t)), \\ H_2(s,t) = K_2(s,t)g(t, \mathbf{C}^T\mathbf{T}(t)). \end{array}$$

Using collocation points (2.5) in transformed eq. (2.4), we get

$$\begin{array}{l} \mathbf{C}^T\mathbf{T}(s_i) = y(s_i) + \\ \lambda_1 \frac{s_i}{2} \int_{-1}^1 H_1(s_i, \frac{s_i(\tau_1+1)}{2})d\tau_1 + \\ \frac{\lambda_2}{2} \int_{-1}^1 H_2(s_i, \frac{(\tau_2+1)}{2})d\tau_2. \end{array} \tag{2.6}$$

Now we use Clenshaw-Curtis quadrature formula [10] to get

$$\begin{array}{l} \mathbf{C}^T\mathbf{T}(s_i) = y(s_i) + \\ \sum_{k=0}^{N}{}'' w_k[\lambda_1 \frac{s_i}{2} H_1(s_i, \frac{s_i(s_k+1)}{2}) + \\ \frac{\lambda_2}{2} H_2(s_i, \frac{(s_k+1)}{2})], \end{array} \tag{2.7}$$

for $i = 0, 1, \dots, N$, where

$$w_k = \frac{4}{N} \sum_{even\ n=0}^{N} {}'' \frac{1}{1-n^2} \cos(\frac{nk\pi}{N}), \tag{2.8}$$

and double prime means that the first and the last terms are halved. The system (2.7) consist of N+1 nonlinear equations which can be solved by usual iterative method such as Newton's method or simplex method. The Fast Fourier Transform (FFT) technique is used to evaluate the summation part in (2.7) in $O(N \log N)$ operations. In fact eq. (2.8) for weights w_k can also be viewed as the discrete cosine transformation of the vector v with entries:

$$v_n = \begin{cases} 2/(1-n^2), & n \ even \\ 0, & n \ odd. \end{cases}$$

The weights w_k therefore is computed directly in $O(N \log N)$ operations, this will be the faster computation when we integrate functions in (2.6) using the same value of N. Therefore one of the good advantages of this method to all early methods which use m-power of operational matrices with operation cost of at least $O(mN^3)$ or $O(m^2N^5)$ ([5],[9],[11],[13],[15],[18],[19]) (for the simple case $(x(t))^m$ as the nonlinear term of integral equations) is that the method is reasonable in cost and also very stable against rounding errors as we see in the next section.

3 Convergence and error analysis

In this section, we discuss the convergence of the Chebyshev polynomial method for the general nonlinear integral equation (1.1). The following proposition is fundamental to the convergence analysis:

Table 1: Mean absolute error for Example 4.1, order of operations and CPU times.

Method of [19]	N=16	e=0.0039	O(m^2N^2)	$\widetilde{CPU} = 4.5562$
method of [5]	N=16	e=1.52d-4	O(mN^3)	CPU=-
method of [13]	N=16	e=2.2011d-2	O(m^2N^2)	CPU=-
method of [12]	n=8,m=8	e=2.7255	O(m^2N^2)	CPU=-
present method:	N=6 N=16 N=32	e=1.21d-3 e=0.66d-8 e=0.28d-15	O(N $\ln N$)	CPU=4.01*10^-2 1.65*10^-1 4.13*10^-1

Table 2: Absolute error of Example 4.2 by introduced method (N= the number of basis functions) and their operation counts(OC).

t	N=5	metod of [17] (N=16)
0.0	0.306e-3	0.0e-3
0.1	0.305e-3	0.1e-3
0.2	0.304e-3	0.0e-3
0.3	0.311e-3	0.2e-3
0.4	0.336e-3	0.1e-1
0.5	0.391e-3	0.1e-3
0.6	0.485e-3	0.1e-3
0.7	0.620e-3	0.1e-3
0.8	0.785e-3	0.0e-3
0.9	0.953e-3	0.1e-3
1.0	0.107e-3	0.1e-3
OC	8.0472	65536

Table 3: Absolute error of Example [3] by introduced method (N= the number of basis functions).

t	N=3	N=5	N=7	metod of [6]
0.0	0.001e-3	0.229e-4	0.258e-5	0.2e-2
0.2	0.324e-3	0.305e-4	0.735e-5	0.1e-1
0.4	0.258e-3	0.167e-4	0.793e-5	0.2e-1
0.6	0.207e-3	0.075e-4	0.255e-5	0.1e-1
0.8	0.177e-3	0.214e-4	0.398e-5	0.0e-2
1.0	0.176e-3	0.062e-4	0.264e-5	0.1e-3

Table 4: Numerical results for Example 4.4.

	N	$\|e\|_\infty$	OC
method of [14]	20 50 100	2.051e-5 3.370e-6 9.182e-7	763 2121 4221
present method	20 50 100	2.725e-8 1.21e-11 0.66e-20	59.91 195.60 460.51

Proposition 3.1 *Let $x(t) \in H^k(-1,1)$ (Sobolev space) and $T_n(x(t)) = \sum_{i=0}^{n} c_i T_i(t)$ be the approximation polynomial of $x(t)$ in L_2 norm. Thus, the truncation error is:*

$$\|x(t) - T_n(x(t))\|_{L_2[-1,1]} \leq C_0 n^{-k} \|x(t)\|_{H^k(-1,1)},$$

where C_0 is a positive constant which depends on the selected norm and is independent of $x(t)$ and n; n is the degree of Chebyshev polynomials (proof [8]). From proposition 3.1 it is concluded that approximation rate of Chebyshev polynomials is n^{-k}. If $x(t)$ is approximated by $x_N(t) = \sum_{n=0}^{N} c_n T_n(t)$, and we find $\bar{c}_n$ ($\bar{c}_n$ is an approximation of c_n and $\bar{x}_N = \sum_{n=0}^{N} \bar{c}_n T_n(t)$) then for $t \in [-1, 1]$, we have

$$\|x(t) - \bar{x}_N(t)\| \leq C_0 N^{-k} \|x(t)\| + C_2(N+1)^{1/2} N^{-k+1}.$$

From [7] and by using closed $N+1$-point Gauss-Chebeshev rule for approximation of c_n we realize [11], $|c_n - \bar{c}_n| \leq C_1 N^{-k+1}$, so it verifies the accuracy of the method. Given the truncated Chebyshev series (2.3) is an approximation of eq. (1.1), it should approximately satisfy these equations, thus for each $s_i \in [0, 1]$,

$$E(s_i) = \mathbf{C}^T \mathbf{T}(s_i) - y(s_i) - \lambda_1 \int_0^{s_i} k_1(s_i, t) f(t, \mathbf{C}^T \mathbf{T}(t)) dt - \lambda_2 \int_0^1 k_2(s_i, t) g(t, \mathbf{C}^T \mathbf{T}(t)) dt \approx 0$$

If $\max E(s_i) = 10^{-k}$ (k is any positive integer) is prescribed then the truncation limit N is increased until the difference $E(s_i)$ at each points s_i becomes smaller than the prescribed 10^{-k}. We can discuss a less strong proposition:

Proposition 3.2 *Assume that $(C(J), \|.\|)$ is the Banach space of all continuous functions on $J = [0, 1]$ with norm $\|x(s)\| = \max_{0 \leq s \leq 1} |x(s)|$ and the following conditions on K_1, K_2 and f, g for eq. (1.1) are satisfied and we define $K_s \equiv K(s, t)$ for $s, t \in [0, 1]$,*

1. $\lim_{s \to \tau} \|K_s - K_\tau\| = 0, \ \tau \in [0, 1]$,
2. $M_1 = \sup_{0 \leq s,t \leq 1} |K_1(s, t)| < \infty$, $M_2 = \sup_{0 \leq s,t \leq 1} |K_2(s, t)| < \infty$,
3. *$f(s, t)$, $g(s, t)$ are continuous in $s \in [0, 1]$ and Lipschitz continuous in $t \in (-\infty, \infty)$, i.e. there exists a constant C_1 and $C_2 > 0$ for which $|f(s, t_1) - f(s, t_2)| \leq C_1 |t_1 - t_2|$, for all $t_1, t_2 \in (-\infty, \infty)$ and $|g(s, t_1) - g(s, t_2)| \leq C_2 |t_1 - t_2|$, for all $t_1, t_2 \in (-\infty, \infty)$,*

then the solution of nonlinear equation (1.1) converges ([13],[16]).

Also in the $L_\infty[0, 1]$ we can propose as follow: Let $(C[0, 1], \|.\|)$ is the Banach space of all continuous functions on interval $[0, 1]$ with $\|x(t)\|_\infty = \max_{t \in [0,1]} |x(t)|$. Assume $|K_1(s, t)| \leq M_1$ and $|K_2(s, t)| \leq M_2$ and suppose the nonlinear terms $f(t, x(t)) = F(t)$ and $g(t, x(t)) = G(t)$ are satisfied in Lipschitz conditions:

$$|F(u) - F(v)| \leq L_1 |u - v|,$$
$$|G(u) - G(v)| \leq L_2 |u - v|.$$

Moreover define $\alpha = |\lambda_1| M_1 L_1 + |\lambda_2| M_2 L_2$. If $x(s)$ and $x_N(s)$ show respectively the exact and approximate solutions of eq. (1.1), we have

Theorem 3.1 *The solution of general nonlinear Fredholm-Volterra Integral equation (1.1) by using Chebyshev polynomials converges if $\alpha \geq 1$; in other words $\lim_{N \to \infty} \|x(s) - x_N(s)\| = 0$.*

Proof:

$$\|x(s) - x_N(s)\|_\infty = \max_{s \in [0,1]} |x(s) - x_N(s)| = \max_{s \in [0,1]} |\lambda_1 \int_0^s K_1(s, t)(F(x(t)) - F(x_N(t)))dt| + \max_{s \in [0,1]} |\lambda_2 \int_0^s K_2(s, t) (G(x(t)) - G(x_N(t)))dt| \leq |\lambda_1| M_1 L_1 s \|x(s) - x_N(s)\|_\infty + |\lambda_2| M_2 L_2 \|x(s) - x_N(s)\|_\infty \Rightarrow \|x(s) - x_N(s)\|_\infty \leq \alpha \|x(s) - x_N(s)\|_\infty.$$

so the proof is completed.

4 Illustrative Examples

In this section we consider some nonlinear Fredholm and Volterra integral equations which have been solved with other early methods such as operational matrix approach and solve them by introduced method.

Example 4.1 *Consider the Fredholm integral equation*

$$x(s) = y(s) + \int_0^s (s - t) x^2(t) dt + \int_0^1 (s + t) x(t) dt, \tag{4.9}$$

with the exact solution $x(s) = s^2 - 2$ and $y(s) = \frac{-1}{30} s^6 + \frac{1}{3} s^4 - s^2 + \frac{5}{3} s - \frac{5}{4}$. We use five methods for this example. Table (1) shows the mean absolute error $\|x - x_N\|_2$ for equal spaced points of interval in each methods and N stands for the number of basis functions and times are in arbitrary unit.

In this table, $\widetilde{CPU}$ stands for CPU time of a problem with the same complexity as this example. However a closer inspection of the results shows that the other methods behave rather differently as N increases and our fast algorithm due to its low operation count yields much smaller round of errors than the other methods.

Example 4.2 *Consider Volterra-Fredholm Hammerstein integral equation given in [12],*

$$x(s) = 2\cos(s) - 2 + 3\int_0^s \sin(s-t)x^2(t)dt + \frac{6}{7-6\cos 1}\int_0^1 (1-t)\cos^2(s)(t+x(t))dt, \quad (4.10)$$

with the exact solution $x(s) = \cos(s)$. Table (2) shows the absolute error $|x - x_N|$ in some points of $[0,1]$ where x_N is the approximate solution and N stands for the number of basis functions in the approximate solution.
Table (2) shows that by this fast method we can obtain the same results with lowest operations than do by method of [17].

Example 4.3 *Consider nonlinear Fredholm integral equation given in [6],*

$$x(s) = exp(1)s + 1 - \int_0^1 (s+t)e^{x(t)}dt, \quad (4.11)$$

which has the exact solution $x(t) = t$. As in preceding examples Table (3) shows the absolute error $\|x - x_N\|_2$ in some points of $[0,1]$.

Example 4.4 *Consider the following boundary value problem*

$$\begin{array}{l} x''(t) - e^{x(t)} = 0, \quad 0 \leq t \leq 1, \\ x(0) = x(1) = 0, \end{array} \quad (4.12)$$

which is of grate interest in hydrodynamics [14] with exact solution

$$x(t) = -\ln(2) + \ln(\lambda(t)),$$

where

$$\lambda(t) = \left(\frac{c}{\cos(\frac{1}{2}c(t-0.5))}\right)^2.$$

Here c is the root of the equation

$$\left(\frac{c}{\cos(\frac{c}{4})}\right)^2 = 2.$$

Problem (4.12) can be reformulated as the integral equation

$$x(s) = \int_0^1 k(s,t)e^{x(t)}dt, \quad 0 \leq t \leq 1,$$

where

$$k(s,t) = \begin{cases} -t(1-s), & t \leq s, \\ -s(1-t), & s \leq t. \end{cases}$$

Table 4 shows maximum error for the method of [14] and our fast method and their operation counts (OC). This table also shows that the fast method is a factor of 10 better.

5 Conclusion

As shown by numerical examples, the method introduced here can be simply implemented to general nonlinear integral equations of the second kind. The advantages are much less implementations and fast computations which is comparable with all huge cost early methods with simple nonlinear terms. We also have shown the convergence and the rate of the convergence.

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