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Solution of fuzzy differential equations

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Abstract

Hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behavior. The hybrid differential equations have a wide range of applications in science and engineering. The hybrid systems are devoted to modeling, design, and validation of interactive systems of computer programs and continuous systems. Hybrid fuzzy differential equations (HFDEs) is considered by Kim et al. [11]. In the present paper it is shown that the example presented by Kim et al. in the Case I is not very accurate and in the Case II, is incorrect. Namely, the exact solution proposed by the authors in the Case II are not solutions of the given HFDE. The correct exact solution is also presented here, together with some results for characterizing solutions of FDEs under Hukuhara differentiability by an equivalent system of ODEs. Then, the homotopy analysis method (HAM) is applied to obtained the series solution of the HFDEs. Finally, we illustrate our approach by a numerical example.

Keywords : Fuzzy differential equations; Homotopy analysis method; Approximate solution.

1 Introduction

The concept of fuzzy derivative was introduced by Dubois and Prade [8] who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [22] and by Goetschel and Voxman [9]. Fuzzy differential equations were first formulated by Kaleva [10] and Seikkala [23] in time dependent form. Kaleva had formulated fuzzy differential equations, in terms of Hukuhara derivative [10]. Also, the fuzzy initial value problem have been studied by several authors [3, 4, 5, 7, 17, 18].

Hybrid system is a dynamic system that exhibits both continuous and discrete dynamic behavior. The hybrid systems are devoted to modeling, design, and validation of interactive systems of computer programs and continuous systems. The differential equations containing fuzzy value functions and interaction with a discrete time controller are named as hybrid fuzzy differential equations (HFDEs) [19]. In the recent paper [11], Kim et al. proposed numerical solutions of HFDEs based on Hukuhara or Seikkala derivative, by using improved predictor corrector method. The authors of [11] also present one example to illustrate their methods.

In 1992, Liao [13] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely HAM, [14, 15, 16]. Abbasbandy and Allahviranloo [1, 2, 6] applied homotopy perturbation method (HPM), which is a special case of HAM, to solve Riccati differential equation. Also he used HAM for solving quadratic Riccati differential equation [24] and nonlinear Fredholm integral equations.

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2 Preliminaries

Definition 2.1 *A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semicontinuous membership function of bounded support.*

Definition 2.2 *[10] A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r); 0 \leq r \leq 1$ which satisfy the following requirements:*

i. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function on $(0, 1]$ and right continuous at 0.

ii. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function on $(0, 1]$ and right continuous at 0.

iii. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

This fuzzy number space can be embedded into the Banach space where the metric is given by the Hausdorff distance.

Definition 2.3 *Let $u, v \in E^1$. If there exists $w \in E^1$ such that $u = v + w$ then w is called the H-difference of u, v and it is denoted by $u - v$.*

Definition 2.4 *A function $f : (a, b) \longrightarrow E^1$ is called H-differentiable at $\hat{x} \in (a, b)$ if, for $h > 0$ sufficiently small, there exist the H-differences $f(\hat{x} + h) - f(\hat{t}), f(\hat{x}) - f(\hat{x} - h)$, and an element $f'(\hat{x}) \in E^1$ such that:*

$$lim_{h\longrightarrow 0^+} D(\frac{f(\hat{x}+h)-f(\hat{x})}{h}, f'(\hat{x})) =$$

$$lim_{h\longrightarrow 0^+} D(\frac{f(\hat{x})-f(\hat{x}-h)}{h}, f'(\hat{x})) = 0.$$

Then $f'(\hat{x})$ is called the fuzzy derivative of f at $\hat{x}$.

3 Hybrid fuzzy differential equations

In this paper, we will study the HFDE

$$\begin{array}{l} y'(x) = f(x, y(x), \lambda_k(y_k)), \\ x \in [x_k, x_{k+1}], \ k = 0, 1, 2, \cdots, \\ y(a) = y_0, \end{array} \tag{3.1}$$

where y is a fuzzy function of x, y_k denotes $y(x_k)$, $f : [x_0, \infty) \times E \times E \to E$ is continuous, each $\lambda_k : E \to E$ is continuous and $\{t_k\}_{k=0}^{\infty}$ is strictly increasing and unbounded. A solution to Eq.(3.1) will be a fuzzy function $y : [x_0, \infty) \to E$ satisfying Eq.(3.1). For $k = 0, 1, 2, \cdots$, let $f_k : [x_k, x_{k+1}] \times E \to E$, where $f_k(x, y_k(x)) = f(x, y_k(x), \lambda_k(y_k))$. A solution of (3.1) can be expressed as

$$y(x) = \begin{cases} y_0(x), \ x_0 \leq x \leq x_1, \\ y_1(x), \ x_1 \leq x \leq x_2, \\ \vdots \\ y_k(x), \ x_k \leq x \leq x_{k+1}, \\ \vdots \end{cases} \tag{3.2}$$

A solution y of (3.1) will be continuous and piecewise differentiable over $[x_0, \infty)$ and differentiable in each interval (x_k, x_{k+1}) for $k = 0, 1, 2, \cdots$.

Theorem 3.1 *[20] Consider the HFDE (3.1) where for $k = 0, 1, 2, \cdots$, each $f_k : [x_k, x_{k+1}] \times E \to E$ is such that*

(i) $f_k(x, y) = (\underline{f_k}(x, \underline{y}(r), \overline{y}(r)), \overline{f_k}(x, \underline{y}(r), \overline{y}(r)))$,
(ii) $\underline{f_k}$ and $\overline{f_k}$ are equicontinuous and uniformly bounded on any bounded set,
(iii) There exists an $L_k > 0$ such that

$$|\underline{f_k}(x, y_1, z_1) - \underline{f_k}(x, y_2, z_2)| \leq$$

$$L_k max\{|y_2 - y_1|, |z_2 - z_1|\},$$

$$|\overline{f_k}(x, y_1, z_1) - \overline{f_k}(x, y_2, z_2)| \leq$$

$$L_k max\{|y_2 - y_1|, |z_2 - z_1|\}.$$

Then (3.1) and the hybrid system of ODEs

$$\underline{y'_k}(x) = \underline{f_k}(x, \underline{y_k}, \overline{y_k}),$$

$$\overline{y'_k}(x) = \overline{f_k}(x, \underline{y_k}, \overline{y_k}),$$

$$\underline{y_k}(x_k) = \underline{y_{k-1}}(x_k) \ if \ k > 0, \underline{y_0}(x_0) = \underline{y_0},$$

$$\overline{y_k}(x_k) = \overline{y_{k-1}}(x_k) \ if \ k > 0, \overline{y_0}(x_0) = \overline{y_0},$$

are equivalent.

4 The Homotopy analysis method

Let us consider the following system of differential equation

$$N_i[u_1(x; r), \ldots, u_n(x; r)] = 0, \quad i = 1, 2, \ldots, n, \tag{4.3}$$

subject to the following initial conditions:

$$u_i(x_0;r) = u_i, \quad i = 1, 2, \ldots, n,$$

where N_i are nonlinear operators that represent the whole equations, and x denote the independent variable and $u_i(x;r)$ are unknowns function respectively. By means of generalizing the traditional HAM, Liao [12] constructed the so-called zero-order deformation equations for $i = 1, 2, \ldots, n$,

$$(1-q)L_i[\phi_i(x;r;q) - u_{i0}(x;r)] =$$

$$qh_iH_i(x;r)N_i[\phi_1(x;r;q), \ldots, \phi_n(x;r;q)], \tag{4.4}$$

where $q \in [0,1]$ is the embedding parameter, $h_i \neq 0$ are non-zero auxiliary parameters for $H_i(x;r) \neq 0$ are non-zero auxiliary functions, $L_i = D_x^{\alpha_i}(n-1 < \alpha < n)$ are auxiliary linear operator with the following property for $i = 1, 2, \ldots, n$,

$$L_i[\phi_i(x;r)] = 0 \quad when \quad \phi_i(x;r) = 0.$$

$u_{i0}(x;r)$ are initial guess of $u_i(x;r), u_i(x;r;q)$ are unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi_i(x;r;0) = u_{i0}(x;r) phi_i(x;r;1) = u_i(x;r),$$

$$i = 1, 2, \ldots, n,$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi_i(x;r;q)$ varies from the initial guesses $u_{i0}(x;r)$ to the solution $u_i(x;r)$. Expanding $\phi_i(x;r;q)$ in Taylor series with respect to q, we have

$$\phi_i(x;r;q) = u_{i0}(x;r) + \textstyle\sum_{m=1}^{\infty} u_{im}(x;r)q^m, \quad i = 1, 2, \ldots, n, \tag{4.5}$$

where

$$u_{im}(x;r) = \frac{1}{m!}\frac{\partial^m \phi_i(x;r;q)}{\partial q^m}|_{q=0},$$

$$i = 1, 2, \ldots, n.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series Eq. (4.5) converges at $q = 1$, then we have

$$u_{im}(x;r) = u_{i0}(x;r) + \textstyle\sum_{m=1}^{\infty} u_{im}(x;r), \tag{4.6}$$

$$i = 1, 2, \ldots, n.$$

Define the vector

$$\overrightarrow{u}_{in} = \{u_{i0}(x;r), u_{i1}(x;r), \ldots, u_{in}(x;r)\},$$

$$i = 1, 2, \ldots, n.$$

Differentiating Eq. (4.4) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we obtain the mth-order deformation equation for $i = 1, 2, \ldots, n$,

$$L_i[u_{im}(x;r) - \chi_m u_{im-1}(x;r)] =$$

$$h_iH_iR_{im}(\overrightarrow{u}_{1m-1}, \ldots, \overrightarrow{u}_{nm-1}, x;r),$$

where

$$R_{im}(\overrightarrow{u}_{1m-1}, \ldots, \overrightarrow{u}_{nm-1}, x;r) = \frac{1}{(m-1)!}$$

$$\frac{\partial^{m-1} N_i[\phi_1(x;r;q), \ldots, \phi_n(x;r;q)]}{\partial q^{m-1}}|_{q=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases}$$

5 Examples

In this Section, we apply HAM to one example.

Example 5.1 *Kim and Sakthivel [11] numerically solved the example below by using the improved predictor-corrector method in the setting of Hukuhara or Seikkala differentiability. A similar example was considered in [21] for fuzzy hybrid systems using Adams-Bashforth method and Adams-Moulton method. In case I:, the method implemented in this paper gives better approximation and in case II: we give the exact solution, then, we solve by HAM. For this example, we choose $H(x) = I$ and for each fuzzy numbers, we use $r = 0, 0.2, \ldots, 1$.*

Consider the following hybrid fuzzy initial value problem

$$\begin{cases} y'(x) = -y(x) + m(x)\lambda_k(y(x_k)), \\ x \in [x_k, x_{k+1}], \ x_k = k, \ k = 0, 1, 2, \cdots, \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), \\ 0 \leq r \leq 1, \end{cases} \tag{5.7}$$

where $m(x) = |sin(\pi x)|, \ k = 0, 1, \ldots,$ and

$$\lambda_k(\mu) = \begin{cases} 0 & if\ k = 0, \\ \mu & if\ k \in \{1, 2, \cdots\}. \end{cases}$$

Case I: When $k = 0$, the solution of problem (5.7) in the interval $[0, 1]$:

When $k = 0$, the hybrid fuzzy initial value problem (5.7) becomes

$$\begin{cases} y'(x) = -y(x), \quad x \in [0, 1], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), \\ 0 \le r \le 1. \end{cases} \tag{5.8}$$

The authors in [11], assert that the exact solution at $x = 0.1$ is given by

$$y(0.1) = (\underline{y}(0.1; r), \overline{y}(0.1; r)) = ((-0.1875+ 0.1875r)e^{0.1} + (0.9375 + 0.0625r)e^{-0.1}, - (-0.1875 + 0.1875r)e^{0.1} + (0.9375+ 0.0625r)e^{-0.1}). \tag{5.9}$$

This value is obtained by assuming that the solution takes the form This value is obtained by assuming that the solution takes the form

$$y(x) = (\underline{y}(x; r), \overline{y}(x; r)) = ((-0.1875+ 0.1875r)e^{x} + (0.9375 + 0.0625r)e^{-x}, - (-0.1875 + 0.1875r)e^{x} + (0.9375+ 0.0625r)e^{-x}). \tag{5.10}$$

it in convenient to choose $y_0(x; r) = (0.75 + 0.25r, 1.125 - 0.125r)$ as the initial approximate of Eq. (5.15). The h-curves of 4th-order are drawn in Fig. 1. Since -0.99 is a valid value of h, thus, for HAM solution, we obtain approximate solution with $h = -0.99$. Comparison between the exact solution and the approximate solution given by HAM are drawn in Figs. 2 and 3. Also, Comparison between the improved predictor-corrector method [11] and the approximate solution given by HAM are given in Table 1 and Table 2. Addition for HAM,we have $D(y(0.1), y_{approx[4]}(0.1)) = 7.13315986e - 7e^{-8}$. Tables 1, 2, 3, 4 shows the errors in estimating for these two methods.

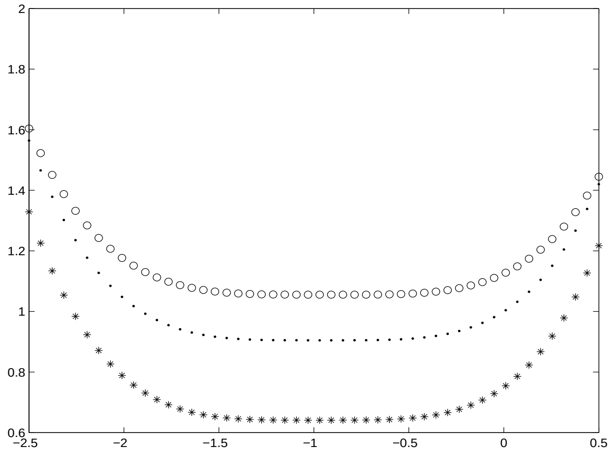

Figure 1: The h-curves of 4th-order of approximation solution.

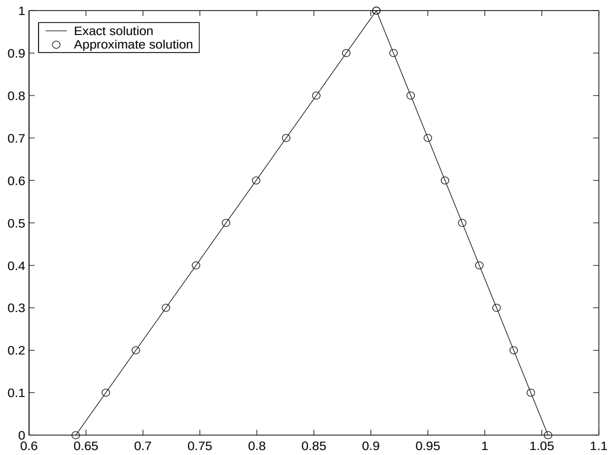

Figure 2: Comparison between the exact solution and approximate solution at $x = 0.1$.

Case II: When $k = 1$, the solution of problem (5.7) in the interval $[1, 2]$: The authors in [11], the above fuzzy problem (5.7) for $k = 1$ considered as:

$$\begin{cases} y'(x) = -y(x) + sin(\pi x)\lambda_1(y(x)), \\ x \in [1, 2], \\ y(0) = (0.75 + 0.25r, 1.125 - 0.125r), \\ 0 \le r \le 1, \end{cases} \tag{5.11}$$

and assert that the exact solution is:

$$(\underline{y}(x; r), \overline{y}(x; r)) = (e^{-0.31830988cos(\pi x)} ((-1.54665250 + 0.17185027r)sinh(x)+ (1.03110167 + 0.34370055r)cosh(x)), -e^{-0.31830988cos(\pi x)}((-1.54665250+ 0.17185027r)cosh(x) + (1.03110167+ 0.34370055r)sinh(x))). \tag{5.12}$$

Table 1: Improved Predictor-corrector method and error.

r	Estimation $\underline{y}(0.1; r)$	
	Impro.Pre-Cor	Error
0	0.64532301	0.42574e-2
0.2	0.69722586	0.34059e-2
0.4	0.74912871	0.25544e-2
0.6	0.80103156	0.17029e-2
0.8	0.85293441	0.85137e-3
1	0.90483726	0.15340e-6

Table 2: HAM method and error.

r	Estimation $\underline{y}(0.1; r)$	
	HAM	Error
0	0.64106559	0.668361282e-7
0.2	0.69381996	0.529893859e-7
0.4	0.74657432	0.391426433e-7
0.6	0.79932868	0.252959007e-7
0.8	0.85208305	0.114491578e-7
1	0.90483741	0.239758413e-8

Table 3: Improved Predictor-corrector method and error.

r	Estimation $\overline{y}(0.1; r)$	
	Impro.Pre-Cor	Error
0	1.04954471	0.59599e-2
0.2	1.02060322	0.47679e-2
0.4	0.99166173	0.35760e-2
0.6	0.96272024	0.23840e-2
0.8	0.93377875	0.11921e-3
1	0.90483726	0.15340e-6

Table 4: HAM method and error.

r	Estimation $\overline{y}(0.1; r)$	
	HAM	Error
0	1.055504555	0.713315986e-7
0.2	1.025371127	0.575447960e-7
0.4	0.995237699	0.4375799322e-7
0.6	0.965104271	0.2997119008e-7
0.8	0.934970843	0.1618438805e-7
1	0.90483741	0.23975843532e-8

This solution is not satisfy in Eq. (5.7). Indeed by using of [20] and (5.7), when $k = 1$, the hybrid fuzzy initial value problem (5.7) becomes

$$\begin{cases} y'(x) = -y(x) - sin(\pi x)y(x_1),\ \ x \in [1, 2], \\ y(1) = ((-0.1875 + 0.1875r)e^1 + (0.9375 \\ +0.0625r)e^{-1}, \\ -(-0.1875 + 0.1875r)e^1 + (0.9375+ \\ 0.0625r)e^{-1}),\ \ 0 \le r \le 1. \end{cases} \tag{5.13}$$

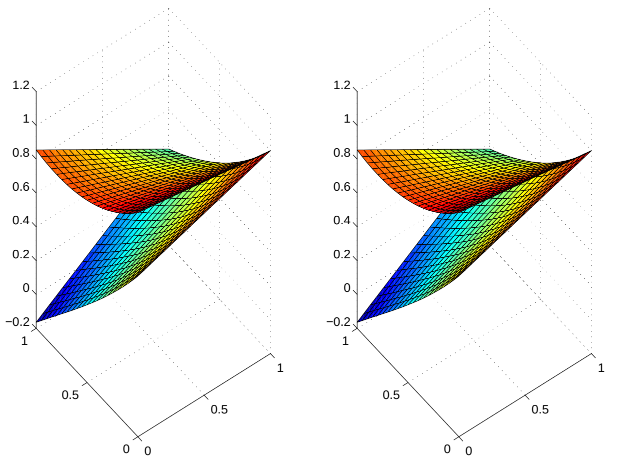

Figure 3: Comparison between the exact solution and approximate solution, left: The exact solution and right: The approximate solution.

By using the results of the Section 3 we can solve the above FDE. Indeed, by Theorem 3.1, we have

$$\begin{cases} \underline{y}'(x;r) = -\overline{y}(x;r) - sin(\pi x)\underline{y}(x_1;r), \\ x \in [1,2], \\ \overline{y}'(x;r) = -\underline{y}(x;r) - sin(\pi x)\overline{y}(x_1;r), \\ \underline{y}(1;r) = (-0.1875 + 0.1875r)e^1 + \\ (0.9375 + 0.0625r)e^{-1}, \\ \overline{y}(1;r) = -(-0.1875 + 0.1875r)e^1 + \\ (0.9375 + 0.0625r)e^{-1}, \ 0 \leq r \leq 1. \end{cases} \tag{5.14}$$

Therefore, the exact solution in $x \in [1,2]$ becomes

$$(\underline{y}(x;r), \overline{y}(x;r)) = ((0.9375 + 0.0625r)A - (0.1875 - 0.1875r)B, (0.9375 + 0.0625r)A + (0.1875 - 0.1875r)B), \tag{5.15}$$

where

$$A = e^{-x} - \frac{1}{1+\pi^2}(e^{-1}(sin(\pi x) - \pi cos(\pi x)) - \pi e^{-x}),$$

$$B = e^{x} + \frac{1}{1+\pi^2}(e(sin(\pi x) + \pi cos(\pi x)) + \pi e^{x}).$$

Results are shown in Figs. 4-5 and in Tables 5 and 6.

6 Conclusion

In this paper, we presented one theorem for the solutions of hybrid fuzzy differential equations which allow us to translate a hybrid fuzzy differential equations into a system of ODEs. The main advantage of HAM is compute the series pattern solution of hybrid fuzzy differential equations. The results show that the proposed method is a promising tool for this type of hybrid fuzzy differential equations. The HAM is more suitable than another analytic methods, because this method provided us with a convenient way to control the convergence of an approximating series.

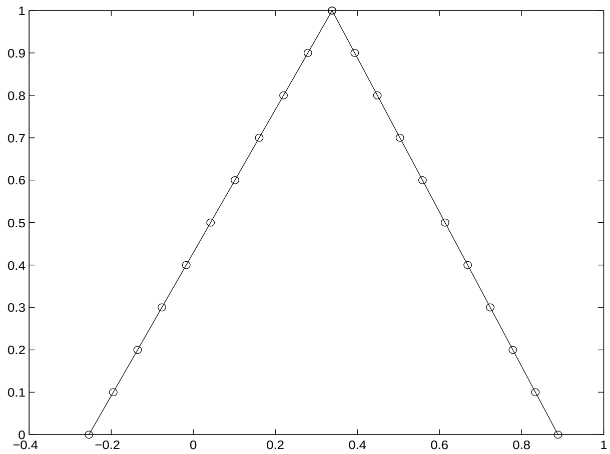

Figure 4: Comparison between the correct exact solution and approximate solution by HAM at $x = 1.1$.

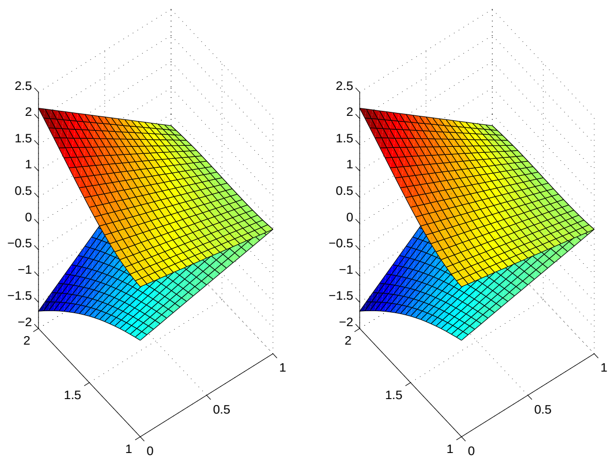

Figure 5: Comparison between the correct exact solution and approximate solution by HAM, left: The correct exact solution and right: The approximate solution.

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Table 5: Exact solution and approximate at $\underline{y}(1.1)$.

r	Exact solution	HAM
0	-0.25422941	-0.25422843
0.2	-0.13570045	-0.13570122
0.4	-0.17171498	-0.17171332
0.6	0.10135746	0.10135651
0.8	0.21988642	0.21988785
1	0.33841538	0.33841622

Table 6: Exact solution and approximate solution at $\overline{y}(1.1)$.

r	Exact solution	HAM
0	0.88875825	0.88875653
0.2	0.77868968	0.77868862
0.4	0.66862110	0.66862257
0.6	0.55855253	0.55855345
0.8	0.44848395	0.44848231
1	0.33841538	0.33841622

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