Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 8, No. 1, 2016 Article ID IJIM-00674, 7 pages
Research Article

Science and Research Branch (IAU)

Variance analysis of control variate technique and applications in Asian option pricing

B. Fathi Vajargah *†, A. Salimipour ‡, S. Salahshour §

Received Date: 2014-01-01 Revised Date: 2015-02-05 Accepted Date: 2015-10-07

Abstract

This paper presents an analytical view of variance reduction by control variate technique for pricing arithmetic Asian options as a financial derivatives. In this paper, the effect of correlation between two random variables is shown. We propose an efficient method for choose suitable control in pricing arithmetic Asian options based on the control variates (CV). The numerical experiment shows the productivity of the proposed method.

Keywords : Monte Carlo simulation; Arithmetic Asian options; Variance reduction technique; Control variates; Correlation.

1 Introduction

For some simple derivatives there exists closed form solution for pricing them and these problems are not our problem. For example : standard options and geometric Asian options. Some payo structures cannot be evaluated using closed analytical formulas. Monte Carlo simulation (Kemna and Vorst [13], Boyle and Emanuel [1], Boyle, Broadie, and Glasserman [2]) and numerical finite-difference PDE methods (Vecer [19] and [20], Rogers and Shi [17], Zvan, Forsyth and Vetzal [23]) are between the numerical approaches to the pricing Asian options. Monte Carlo method used widely to determine these prices. However the Monte Carlo method is financially well to do, it consumes time to simulate a price inside reasonable boundaries of accuracy for sure contracts.for reduce computational time, some variance reduction techniques have been suggested. For example, control variates, antithetic variates and importance sampling. Control variate method have been famously used for calculative finance as a method of variance reduction. Kemna and Vorst [13] employed a discounted geometric average Asian option payoff less than its price as a control. This technique is effective because the geometric Asian option has a analytical solution form and the correlation amonge the arithmetic and the geometric average is high. This study focus on the analysis of control variates, that is one of the most popular and eicient methods used. This method takes advantage of random variables with positively correlated with the variable under consideration and known expected value.

*Corresponding author. behrouz.fathi@gmail.com
†Department of Statistics, Faculty of Mathematical Science, University of Guilan, Rasht, Iran.
‡Department of Applied Mathematics, Faculty of Mathematical Science, University of Guilan, Rasht, Iran.
§Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Iran.

2 Option pricing and Asian options

Option is a contract between a seller and a buyer that gives option buyer this right to buy or sell the underlying asset in agreed price at a later time.There are two types of options: the call option and the put option. Call option gives the buyer this right to buy the underlying asset, while put option gives the buyer to sell that. The agreed price in the option contract is known as the strike price; the time in the contract is known as the expiry date. The traditional of options are European and American options. European options give to buyer or seller the right to buy or sell on the expiry date, on which the option expires or matures. American options give to buyer or seller the right to buy or sell at any time prior to or at expiry. There are many other types of options so-called exotic (no standard) options such as barrier options; Bermudan options; Asian options; or look back options. Asian option is the option that obtained from the average price over a time period. For this reason, these options have a lower volatility and hence those prices are cheaper relative to their European options. Asian options are ussually traded on commodity and currencies products which have lower trading volumes. Banker's Trust Tokyo office used them at first, in 1987 for pricing average options on crude oil contracts. hence known "Asian" option. [16] Let S(t) be the stock price at time [0, T]. Assume that the initial stock price is S0 = S(0), the final expiration date is T , and the strike price is K. The BlackScholes model characterize the assessment of the stock price by way of the stochastic differential equation as follows

$$dS_t = rS_t dt + S_t dW_t \quad (2.1)$$

where W_t is a standard Brownian motion and the parameters and r are the volatility of the stock price and the mean rate of return, relatively. Under the risk-neutral measure, the mean rate of return to be the same as the interest rate r [16]. the solution of Equation (2.1) is as follows [8]:

$$S(T) = S(0)exp((r - \sigma^2/2)T + W(T)) \quad (2.2)$$

In Asian options, the arithmetic mean of the stock price indicated by

$$A(t_m) = \frac{1}{m+1}\Sigma_{i=0}^{m} S(iT/m)$$

Under the risk-neutral measure Q, arithmetic Asian call options at time 0 is

$$C^a = E^Q[e^{-rT}(A(t_m) - K)^+]$$

and the geometric mean of the stock price indicated by [10]

$$G(t_m) = (\prod_{i=0}^{m+1} S(iT/m))^{\frac{1}{m+1}}$$

and geometric Asian call options at time 0 is

$$C^g = E^Q[e^{-rT}(G(t_m) - K)^+]$$

The distribution of the sum of lognormal prices has no explicit representation. Therefore, arithmetic Asian options do not have a analitically price formulas. Thus, solutions for the arithmetic Asian options are more complex. In next section we first discuss the role of variance reduction in evolution the computational efficiency of Monte Carlo simulation. Then we converse about control variate technique and demonstrate their application to arithmetic Asian options.

3 Evolution of efficiency by variance reduction

The variance reduction appears so clearly conversable that the exact argument for its profit is sometimes connived. We shortly survey the underlying explanation for reduction of variance and look at it from the point of view to improve calculative efficient. For example, Suppose that-parameter is the price of a derivative security and we want to compute . By Monte Carlo simulation we can generate an i.i.d. sequence $\{Y_i, i = 1, 2, \ldots\}$, where each Y_i has expectation and variance (μ, σ^2). The sample mean shown below is a natural and common estimator of μ with n replications

$$1/n \sum_{i=1}^{n} Y_i$$

using the central limit theorem, for large n, above sample mean has normal distribution, with mean and variance $(\mu, \sigma^2/n)$. For example, confidence interval for options which underlying asset has log-normal distribution is

$$\mu - \frac{1.96\sigma}{\sqrt{n}} < OptionPrice < \mu + \frac{1.96\sigma}{\sqrt{n}}$$

and show that the error in the estimator is equivalent to $\frac{\sigma}{\sqrt{n}}$. So, decrease of the variance σ^2 by a factor of n, when other parameters are constant, is equivalent to increase the number of samples by a factor of n^2. Suppose that we can estimate by two kind of unbiased Monte Carlo estimators $\{Y_i^1, i = 1, 2, \ldots\}$ and $\{Y_i^2, i = 1, 2, \ldots\}$, then $E[Y_i^1] = E[Y_i^2] = \mu$, but $\sigma_1 < \sigma_2$, where $\sigma_j^2 = Var[Y^j], j = 1, 2$. considering earlier observations it becomes that for estimating sample mean Y^1 gives a more exact estimate than sample mean Y^2 (both of n replications). this analysis is very simplifies to comparison of the two estimators, because it fails to capture possible differences in the computational accurancy of them. However variance of Y^1 is less that the Y^2 but it may be generate Y^1 consume more time than generate Y^2. smaller variance is not enough criterion for preferenc first estimator over another. To compare estimators with different computational requirements as well as different variances, Boyle, Broadie, and Glasserman [2] claim as follows. Suppose required to generate one replication of Y^j is a constant $\{b_j, j = 1, 2, \ldots\}$. With computing time t, the number of replications of Y^j that can be generated is $\lfloor \frac{t}{b_j} \rfloor$.for simplicity, we fall the $\lfloor . \rfloor$ and deal with the ratios t/b_j as though they were integers. The two estimators available with computing time t are, therefore,

$$\frac{b_1}{t} \sum_{i=1}^{t/b_1} Y_i^1 \quad and \quad \frac{b_2}{t} \sum_{i=2}^{t/b_2} Y_i^2$$

For large t, these are approximately normally distributed with mean μ and with standard deviations

$$\sigma_1 \sqrt{b_1/t} \quad and \quad \sigma_2 \sqrt{b_2/t}$$

Thus, for large t, the initial estimator should be favored over the other if

$$\sigma_1^2 b_1 < \sigma_2^2 b_2 \qquad (3.3)$$

Eq. (3.3) supply a base for computational requirements and estimator variance. If we consider efficiency as a base for comparison of estimators, the lower variance estimator should be favored only if the variance ratio σ_1^2/σ_2^2 is smaller than the work ratio b_2/b_1. By the same argument, a higher variance estimator may in fact be more desirable if it takes much less time to generate. In its simplest form, the principle declared in Eq. (3.3) dates at least to Hammersley and Handscomb [10]. Glynn and Whitt [9] extended this idea. They consider efficiency in the behavior of bias and also allow the work per run to be random.

4 Control variates

Control variates method is one of the most widely applicable, effective and easiest to use of the variance reduction techniques. the fundamental of this method discribed in section (3). The most simple execution of control variates replaces unknown expectation value with the difference between the unknown value and known value that is expectation value. Kemna and Vorst [13] and Boyle and Emanuel [1] analyze a specific explanation of Asian options. Suppose C^a be the arithmetic Asian option price and C^g be the geometric Asian option price. C^a is of much greater practical value becuase Most options based on averages use arithmetic averaging, but whereas C^a is not solvable in close form, C^g can be often evaluated in analytical form. We can compute a better approximation of C^a by knowledge C^g value. Control variate method help us for implement this work. Let $C^a = E[\hat{C^a}]$ and $C^g = E[\hat{C^g}]$, where $\hat{C^a}$ and $\hat{C^g}$ are the discounted option payoffs for a one simulated path of the underlying asset. Then

$$C^a = C^g + E[\hat{C^a} - \hat{C^g}]$$

in the other hand, C^a can be declare as the known price C^g and the expected difference between $\hat{C^a}$ and $\hat{C^g}$. Therefore, an unbiased estimator of $\hat{C^a}$ is given by

$$\hat{C^a}_{CV} = \hat{C^a} + (C^g - \hat{C^g}) \qquad (4.4)$$

This representation proposed a partly different explanation. $\hat{C^a}_{CV}$ adjusts the straightforward estimator $\hat{C^a}$ according to the difference between the known value C^g and the observed value$\hat{C^g}$. The known error $(C^g - \hat{C^g})$ is used as a control in the estimation of C^a [1]. Because

$$Var(\hat{C^a}_{CV}) = Var(\hat{C^a}) + Var(\hat{C^g}) - 2Cov(\hat{C^a}, \hat{C^g})$$

The control variates method is effecient if the covariance between $\hat{C^a}$ and $\hat{C^g}$ be high. Other control variates for Asian options investigated by Fu, Madan, and Wang [7] that based on Laplace transform values. These seem to be lessstrongly correlated with the option price. A closer experience show that this estimator does not make optimal solurtion for option prices. regard the family of unbiased estimators

$$\hat{C^a}_{\beta} = \hat{C^a} + \beta(C^g - \hat{C^g}) \quad (4.5)$$

parameterized by the scalar . We have

$$Var(\hat{C^a}_{\beta}) = Var(\hat{C^a}) + \beta^2 Var(\hat{C^g}) - 2\beta Cov(\hat{C^a}, \hat{C^g}) \quad (4.6)$$

Therefore, the variance minimizing β is

$$\beta^* = \frac{Cov(\hat{C^a}, \hat{C^g})}{Var(\hat{C^g})}$$

According to the request, β^* might be or mighty not be close to 1. If we use an estimator of the form Eq. (4.4), our chance for variance reduction be less. In fact, whereas Eq. (4.4) might decrease or increase variance, an estimator based on β^* is assurance not to increase variance, and outcome in a strict decrease in variance until $\hat{C^a}$ and $\hat{C^g}$ are not uncorrelated.

Substituting β^* in Eq. (4.6) and simplifying, we find that the ratio of the variance of the estimator having optimal control to that of the uncontrolled estimator is

$$\frac{Var(\hat{C^a}_{\beta})}{Var(\hat{C^a})} = \frac{Var[\hat{C^a} + \beta(C^g - \hat{C^g})]}{Var(\hat{C^a})} = 1 - \rho^2_{\hat{C^a},\hat{C^g}} \quad (4.7)$$

Where

$$\rho_{\hat{C^a},\hat{C^g}} = \frac{Cov(\hat{C^a}, \hat{C^g})}{\sqrt{Var(\hat{C^a})Var(\hat{C^g})}} = Corr(\hat{C^a}, \hat{C^g})$$

A few observations follow from this expression: With the optimal coefficient β^*, the efficiency of a control variate, as measured by the variance reduction ratio Eq. (4.7), is determined by the strength of the correlation between the Quantity of $\hat{C^a}, \hat{C^g}$ The variance reduction factor (VRF) $1/(1 - \rho^2_{\hat{C^a},\hat{C^g}})$ increases sharply as $|\rho_{\hat{C^a},\hat{C^g}}|$ approaches 1 and, accordingly, it drop off quickly as $|\rho_{\hat{C^a},\hat{C^g}}|$ decreases away from 1.
In practice, we infrequently know β^* because we infrequently know $Cov(\hat{C^a}, \hat{C^g})$. However, given n independent replications $(C^{ai}, C^{gi}), i = 1, n$ of the pairs $(\hat{C^a}, \hat{C^g})$ we can estimate β^* via sample estimators as follow:

$$S_{C^gC^g} = \frac{1}{n-1}\sum_{i=1}^{n}(C^{gi} - \bar{C^g})^2$$

$$S_{C^aC^g} = \frac{1}{n-1}\sum_{i=1}^{n}(C^{gi} - \bar{C^g})(C^{ai} - \bar{C^a})$$

Where

$$\bar{C^a} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} C^{ai} \quad , \quad \bar{C^g} = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} C^{gi}$$

Gives theβ^* estimator as

$$\hat{\beta} = S_{XY}S_{XX}^{-1} \quad (II)$$

The strong law of large numbers assure that $\hat{\beta}$ converges to $\hat{\beta}^*$ with probability 1.

Using all n replications to compute an estimate $\hat{\beta}$ of $\hat{\beta}^*$ shows a bias in the estimator

$$\frac{1}{n}\sum_{i=1}^{n} C^{ai} + \hat{\beta}(C^g - \frac{1}{n}\sum_{i=1}^{n} C^{gi})$$

and its estimated standard error because of the dependence between $\hat{\beta}$ and the C^{gi}.

The benefit of working with Eq. (4.5) instead Eq. (4.4) becomes even more significant when the introduced controls are farther. For example, when the asset price is simulated under risk-neutral probability measure, the present value $e^{-rT}E[S_T]$ of the terminal price must equal the current price S_0. We can, thus, form the estimator

$$\hat{C^a} + \beta_1(C^g - \hat{C^g}) + \beta_2(S_0 - e^{-rT}S_T)$$

The variance minimizing coefficients (β_1^*, β_2^*) are easily found by multiple regression. This

Table 1: Get good speed-ups only if ρ is very close to 1

ρ	0.5	0.6	0.7	0.8	0.9	0.95	0.99	0.999
VRF	1.3	1.6	2	2.8	5.3	10.3	50.3	500

optimization step appears especially critical in this case; for whereas one may guess that β_1^* is close to 1, it appears unlikely that β_2^* would be. The expression in (II) is the slop of the lest-squares regression line through the points $\{(C^{gi}, C^{ai}), i = 1, \dots, n\}$. Figure 1 shows a scatter plot of simulation outputs (C^{gi}, C^{ai}) and the estimated regression line for these points, which passes through the point $(\bar{C}^g, \bar{C}^a)$. In the figure, $\bar{C}^g < E[C^g]$, indicating that the n replications have underestimated $E[C^g]$. If the (C^{gi}, C^{ai}) are positively correlated, this suggests that the simulation estimate$\bar{C}^a$ likely underestimates $E[C^g]$. This further suggests that we should adjust the estimator upward. The regression line determines the magnitude of the adjustment; in particular, $\mu(\hat{\beta})$ is the value fitted by the regression line at the point $E[C^a]$.

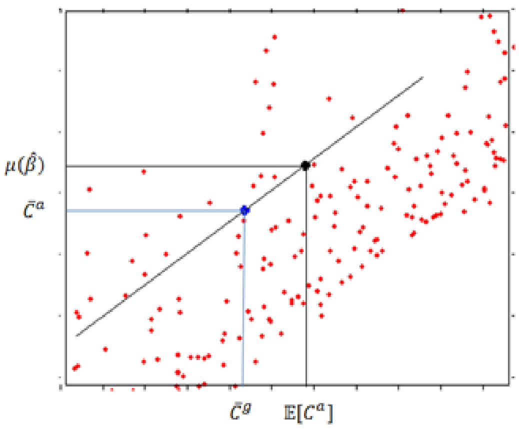

Figure 1: Regression interpretation of control variate method. The regression line through the points (C^{gi}, C^{ai}) has slope   and passes through $(\bar{C}^g, \bar{C}^a)$. The control variate estimator$\mu(\hat{\beta})$ is the value fitted by the line at $E[C^a]$. In the figure, the sample mean $\bar{C}^g$ underestimates $E[C^g]$ and $\bar{C}^a$ is adjusted upward accordingly

5 Numerical simulation

In this section we will show that when correlation between $\hat{C}^a$ and control variable is high, this control is suitable for reducing the variance of $\hat{C}^a$. In the following we will consider an arithmetic Asian options with $S_0 = 100, r = 0.05, \sigma = 0.2, T = 1(year), K = 90, m = 10$, and NRepl is the number of independent replications. From the results in Table 1, we can see that the effect of correlation between $\hat{C}^a$ and control variables. Also we see that the simulation error reduced very well when correlation is very close to 1.

Figure 2, shows the correlation between control variates presented in Table 2 and simulated quantity of arithmetic Asian option pricing. In this figure we can see that, with assumptions $S_0 = 100, r = 0.05, \sigma = 0.2, T = 1(year), K = 90, m = 10$, in condition of higher correlation intensity the simulated points fit on the regression line with slope β^*.

For example, if we consider geometric average

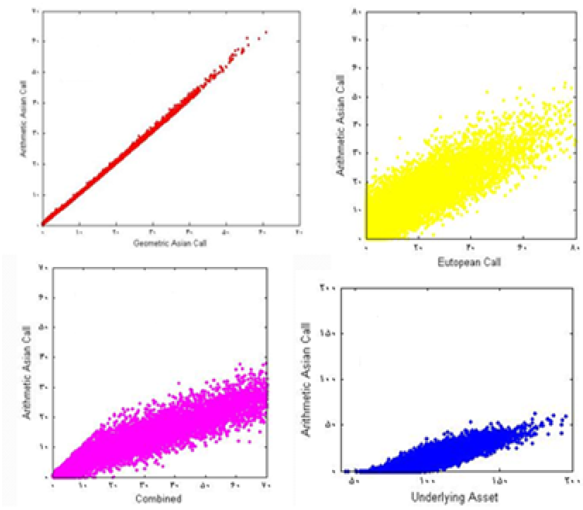

Figure 2: Shows the correlation between control variates presented in Table 2 and simulated quatity of arithmetic Asian option pricing.

Asian option as control variate, then $\rho = 0.9997$. according to the Figure 2, we see the ratio of simulated points approximately fit on the regression line passes through the points $(\bar{C}^g, \bar{C}^a)$ and $(E[C^a], \mu(\beta^*))$, where $, \mu(\beta^*) = \hat{C}^a + \beta^*(C^g - \hat{C}^g)$.

Table 2: The simulation results with parameters: $S0 = 100, K = 90, r = 0.05, T = 1(year), \sigma = 0.2, NRepl = 10000$

Row	Control	option value	ρ	Error	β^*
1	$(\hat{C}^a)$NMC	12.4507	-	0.1012	-
2	Eruopean call	12.5543	0.8740	0.0495	0.5169
3	Underlying asset	12.5365	0.8774	0.0488	0.4467
4	Geometric asian call	12.5394	0.9997	0.0026	1.0250
5	Combination of 2,4	12.5470	0.9493	0.0320	0.3674
6	Combination of 3,4	12.5346	0.9447	0.0334	0.3308

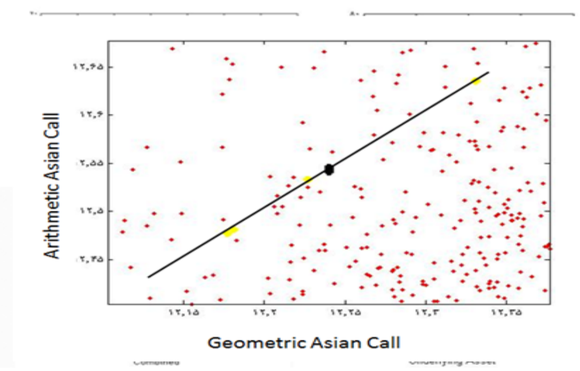

Figure 3: With simulating consecutive for each presented control variate in condition of higher correlation intensity, the regression lines with slope β^*, become closer together.

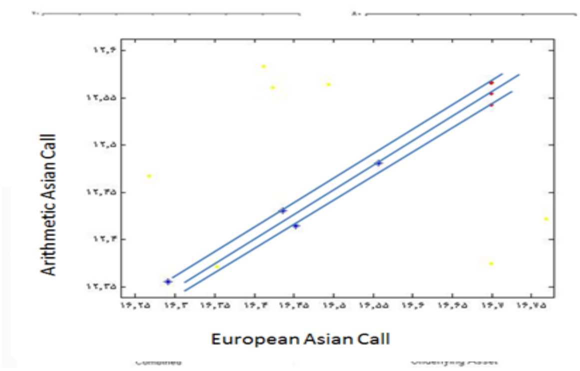

Figure 4: Considering the same assumptions told before, we see that the regression lines have deviation, because of reduced correlation intensity

6 Conclusion

In this paper, using control variate technique we find out, to obtain a suitable control variate, first one can exploit the correlation intensity between this control variate and essential estimator, if the correlation intensity be very close to 1, this control will be suitable. in other word, obtaining correlation is the necessary condition to choose a control variate.

References

[1] P. Boyle, M. Broadie, P. Glasserman, *Monte Carlo methods for security pricing*, Journal of Economic Dynamics and Control 21 (1997) 1267-1321.

[2] P. Boyle, D. Emanuel, *Options on the General Mean*, Working paper, University of British Columbia 1980.

[3] KD. Dinge, W. Hrmann, *A general control variate method for option pricing under Lvy processes*, Eur. J. Oper. Res. 221 (2012) 368-377.

[4] D. Duffie, P. Glynn, *Efficient Monte Carlo simulation of security prices*, Ann Appl Probab 5 (1995) 897-905.

[5] D. Dufresne, F. Vasquez-Abad, *Accelerated simulation for pricing Asian options*, Proceedings of the 1998 winter simulation conference (1993) Washington, DC.

[6] J. Fouque, C. Han, *Variance reduction for Monte Carlo methods to evaluate option prices under multi-factor stochastic volatility models*, Quant Finance 2 (2004) 597-606.

[7] M. Fu, D. Madan, T. Wong, *Pricing continuous time Asian options: a comparison of analytical and Monte Carlo methods*, Working paper, College of Business and Management, University of Maryland, College Park, MD. (1995).

[8] P. Glasserman , *Monte Carlo methods in financial engineering*. NewYork: Springer-Verlag, (2003).

[9] P. W. Glynn, W. Whitt, *The asymptotic efficiency of simulation estimators*, Operations Research 40 (1992) 505-520.

[10] J. M. Hammersley, D. C. Handscomb, *Monte Carlo Methods*, Chapman Hall, London. (1964).

[11] PJ. Huber, *Robust statistics*, NewYork: John Wiley Sons; (1981).

[12] A. Jasra, P. Del Moral, *Sequential Monte Carlo methods for option pricing*, Stochast Anal Appl 29 (2011) 292-316.

[13] A. Kemna, A. Vorst, *A Pricing Method for Options Based on Average Values*, Journal of Banking and Finance 14 (1990) 113-129.

[14] V. Linetsky, *Exact Pricing of Asian Options: An Application of Spectral Theory*, working paper, IEMS Department, Northwestern University (2001).

[15] J. Ma, C. Xu, *An efficient control variate method for pricing variance derivatives*, J Comput Appl Math 235 (2010) 108-119.

[16] F. Mehrdoust, *A new hybrid Monte Carlo simulation for Asian options pricing*, Journal of Statistical Computation and Simulation (2013).

[17] L. Rogers, Z. Shi, *The value of an Asian option*, J. Appl. Probab. 32 (1995) 1077-1088.

[18] RY. Rubinstein, *Simulation and the Monte Carlo method*, NewYork: John Wiley Sons (1981).

[19] J. Vecer, *A New PDE Approach for Pricing Arithmetic Asian Options*, Journal of Computational Finance, forthcoming (2000).

[20] J. Vecer, *New Pricing of Asian Options*, working paper, Columbia University (2001).

[21] H. Zhang, *Pricing Asian Options using Monte Carlo Methods*, U. U. D. M. Project Report 7 (2009).

[22] WWY. Hsu, YD. Lyuu, *Efficient pricing of discrete Asian options*, Appl Math Comput 217 (2011) 9875-9894.

[23] R. Zvan, P. Forsyth, K. Vetzal, Robust Numerical Methods for PDE Models of Asian Options, Journal of Computational Finance 1 (1997) 39-78.

Behrouz Fathi vajargah has got PhD degree from University of Reading, England in 2001 and now he is the Associate Professor in University of Guilan, Guilan, Iran. He has published more than 100 papers in international journals and conferences. Now, he is working on Computations based on Simulation and Monte Carlo Methods.

Since September 2011, Ayoob Salimipour is a Ph.D student in applied mathematics, stochastic processes field at University of Guilan. He has got his M.Sc degree in applied mathematics, operational research field from Sistan Balouchestan University. His research focus on solving financial stochastic differential equations by Monte Carlo methods.

S. Salahshour is working as a Assistant Professor in the Department of Mathematics, Mobarakeh Branch, IAU. His current area of research is Fuzzy fractional differential equations, Fuzzy Expert Systems and Fuzzy System of Equations.