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Research Article

Error estimation of fuzzy Newton-Cotes method for Integration of fuzzy functions

N. Ahmady *†, E. Ahmady ‡

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Abstract

Fuzzy Newton-Cotes method for integration of fuzzy functions that was proposed by Ahmady in [1]. In this paper we construct error estimate of fuzzy Newton-Cotes method such as fuzzy Trapezoidal rule and fuzzy Simpson rule by using Taylor's series. The corresponding error terms are proven by two theorems. We prove that the fuzzy Trapezoidal rule is accurate for fuzzy polynomial of degree one and fuzzy Simpson rule is accurate for polynomial of degree three. The accuracy of fuzzy Trapezoidal rule and fuzzy Simpson rule for integration of fuzzy functions are illustrated by two examples.

Keywords : Fuzzy integration; Fuzzy Newton-Cotes method; Fuzzy trapezoidal's rule; Fuzzy Simpson's rule.

1 Introduction

The fuzzy integration problem plays a major role in various areas such as mathematics, physics, statistics and engineering. The Newton Cotes methods with positive coefficient for integration of fuzzy function by Allahviranloo [2] were discussed. Bede and Gal, [5] proposed quadrature rules for integrals of fuzzy-number-valued, they introduced some quadrature rules for the Henstock integral of fuzzy-number-valued mappings by giving error bounds for mappings of bounded variation and of Lipschitz type. They considered special class of quadrature rules for the Henstock integrals , such as midpoint-type, trapezoidal and three-point-type quadrature. Ahmady introduced fuzzy Newton-Cotes formula, such as fuzzy trapezoidal method and fuzzy Simpson method for integration of fuzzy functions [1]. In this paper error estimation of Newton-Cotes formula is discussed. This paper is organized as follows: In Section 2, some basic definitions and results which will be used later are brought. Error Term of Fuzzy Trapezoidal Rule and Fuzzy Simpson's rule for solving fuzzy integral are introduced in Section 3 and 4. Examples are presented in Section 5, and the final Section contains a conclusion.

2 Preliminaries

First, we introduce the notation that will be used in this paper.

2.1 Notation and definitions

A fuzzy number u is a fuzzy subset of the real line with a normal, convex and upper semicontinuous membership function of bounded support. The

*Corresponding author. n.ahmadi@iauvaramin.ac.ir
†Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran.
‡Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.

family of fuzzy numbers will be denoted by R_F. An arbitrary fuzzy number u is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ that, satisfies the following requirements:

- $\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0,1], with respect to any r.
- $\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0,1], with respect to any r.
- $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Then the r-level set

$$[u]^r = \{s|\ u(s) \geq r\}, \quad 0 < r \leq 1,$$

is a closed bounded interval, denoted by

$$[u]^r = [\underline{u}(r), \overline{u}(r)].$$

Let I be a real interval. A mapping $y : I \rightarrow R_F$ is called a fuzzy process, and its $r-$level set is denoted by

$$[y(t)]^r = [\underline{y}(t,r), \overline{y}(t,r)], \quad t \in I, \quad r \in (0,1].$$

Definition 2.1 *[7] Distance between two fuzzy numbers* $u = (\underline{u}(r), \overline{u}(r))$ *and* $v = (\underline{v}(r), \overline{v}(r))$ *is given by*

$$d(u,\ v) = \{\int_0^1 \{(\underline{u}(r) - \underline{v}(r))^2 + (\overline{u}(r) - \overline{v}(r))^2\}dr\Big\}^{\frac{1}{2}}.$$

Definition 2.2 *A fuzzy-valued function* $f : [a,b] \rightarrow R_F$ *is said to be continuous at* $t_0 \in [a,b]$ *if for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that* $d(f(t), f(t_0)) < \epsilon$, *whenever* $t \in [a,b]$ *and* $|t - t_0| < \delta$. *We say that* f *is fuzzy continuous on* $[a,b]$ *if* f *is continuous at each* $t_0 \in [a,b]$ *such that the continuity is one-sided at endpoints a, b.*

Theorem 2.1 *[4] Let* f *is continuous function from the open set* $U \subseteq R^n$, $n \in N$ *in to* R_F, *then* $\underline{f}, \overline{f}$ *are continuous functions from* U *into* R, *for all* $r \in [0,1]$.

In this paper $C_F[a,b]$ is the space of all continuous fuzzy-valued function on $[a,b]$.

3 Error Term of Fuzzy Trapezoidal Rule

Fuzzy Trapezoidal rule for approximating $\int_a^b f(x)dx$, $x_i = a, x_{i+1} = b, h = b - a$, was introduced by [1] as follows

$$\int_{x_i}^{x_{i+1}} \underline{f}(x,r)dx = \frac{h}{2}\{\underline{f}(x_i,r) + \underline{f}(x_{i+1},r)\}$$

$$\int_{x_i}^{x_{i+1}} \overline{f}(x,r)dx = \frac{h}{2}\{\overline{f}(x_i,r) + \overline{f}(x_{i+1},r)\}$$

The composite fuzzy trapezoidal rule is obtained by applying the fuzzy trapezoidal rule in each subinterval $[x_i, x_{i+1}]$, $i = 0, \cdots, n-1$, if $x_i = a + ih$, where $h = \frac{b-a}{n}$, the fuzzy trapezial rule for fuzzy function $f(x)$ is obtain as follows:

$$\int_a^b \underline{f}(x,r)dx = \frac{h}{2}\{\underline{f}(a) + \sum_{i=1}^{n-1} \underline{f}(a+ih) + \underline{f}(b)\},$$

$$\int_a^b \overline{f}(x,r)dx = \frac{h}{2}\{\overline{f}(a) + \sum_{i=1}^{n-1} \overline{f}(a+ih) + \overline{f}(b)\},$$

Consider approximating $\int_{x_i}^{x_{i+1}} f(x)dx$, where $x_i = x_0 + ih$ and $h = x_i - x_{i-1}$, error terms of trapezoidal rule $\underline{E}_i(r)$ and $\overline{E}_i(r)$ are defined as follows:

$$\underline{E}_i(r) = \int_{x_i}^{x_{i+1}} \underline{f}(x,r)dx - \{\frac{h}{2}\{\underline{f}(x_i,r) + \underline{f}(x_{i+1},r)\}\} \tag{3.1}$$

$$\overline{E}_i(r) = \int_{x_i}^{x_{i+1}} \overline{f}(x,r)dx - \{\frac{h}{2}\{\overline{f}(x_i,r) + \overline{f}(x_{i+1},r)\}\} \tag{3.2}$$

Let us start with Taylor's series for $\underline{f}(x,r)$ and $\overline{f}(x,r)$ around x_i, where the first two derivative $\underline{f}(x,r)$ and $\overline{f}(x,r)$ are continuous on (a,b), and $h = \frac{b-a}{n}$,

$$\begin{aligned} &\frac{h}{2}\{\underline{f}(x_i,r) + \underline{f}(x_{i+1},r)\} \\ &= \frac{h}{2}\{\underline{f}(x_i,r) + \{\underline{f}(x_i,r) + h\underline{f}'(x_i,r) + \frac{h}{2}\underline{f}''(\eta_i,r)\} \\ &= h\underline{f}(x_i,r) + \frac{h^2}{2}\underline{f}'(x_i,r) + \frac{h^3}{4}\underline{f}''(\eta_i,r). \end{aligned} \tag{3.3}$$

Also integrating from x_i to x_{i+1} gives:

$$\int_{x_i}^{x_{i+1}} \underline{f}(x,r)dx$$
$$= \int_{x_i}^{x_{i+1}} \{\underline{f}(x_i,r) + (x-x_i)\underline{f}'(x_i,r)$$
$$+\frac{(x-x_i)^2}{2!}\underline{f}''(\eta_i,r),$$
$$= x.\underline{f}(x_i,r) + \frac{(x-x_i)^2}{2}\underline{f}'(x_i,r)$$
$$+\frac{(x-x_i)^3}{3!}\underline{f}''(\eta_i,r)|_{x_i}^{x_{i+1}}.$$

Therefore

$$\int_{x_i}^{x_{i+1}} \underline{f}(x,r)dx \quad (3.4)$$
$$= h.\underline{f}(x_i,r) + \frac{h^2}{2}\underline{f}'(x_i,r) + \frac{h^3}{3!}\underline{f}''(\eta_i,r),$$

by subtracting (3.4) and (3.3) and cancels the similar terms, obtain:

$$\underline{E}_i(r) = -\frac{h^3}{12}\underline{f}''(\eta_i,r), \quad \eta_i \in [x_i,x_{i+1}], \quad (3.5)$$

By similar way

$$\overline{E}_i(r) = -\frac{h^3}{12}\overline{f}''(\eta_i,r), \quad \eta_i \in [x_i,x_{i+1}], \quad (3.6)$$

Now we want to put fuzzy error $E_i(r)$ to zero. For this propose let $d(E_i,0) = 0$, thus:

$$\int_0^1 (\underline{E}_i(r) + \overline{E}_i(r))^2 dr = 0, \quad (3.7)$$

this means

$$\underline{E}_i(r) + \overline{E}_i(r) = 0$$

therefore for $\eta_i \in [x_i, x_{i+1}]$, and $i = 0,\cdots,n-1$, the error of fuzzy trapezoidal rule for each interval $[x_i,x_{i+1}]$ denoted by $E_i(FT(h))$ is obtained as follows

$$E_i(FT(h)) = -\frac{h^3}{12}(\underline{f}''(\eta_i,r) + \overline{f}''(\eta_i,r)). \quad (3.8)$$

Theorem 3.1 *Let $f \in C^2([a,b],R_F)$ the error term for the composite fuzzy trapezoidal rule for $h = \frac{b-a}{n} = x_{i+1} - x_i$, is $E(FT(h)) = -\frac{(b-a)}{24}h^2\{\underline{f}''(\eta,r) + \overline{f}''(\eta,r)\}$, where $\eta \in [a,b]$.*

Proof : The error of composite fuzzy trapezoidal rule is obtained by applying the error of fuzzy trapezoidal rule in each subinterval $[x_i,x_{i+1}]$, for $i = 0,\cdots,n-1$, i.e

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx =$$
$$\int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx,$$

therefore

$$\begin{aligned} E(FT(h)) &= -\frac{h^3}{12}\{(\underline{f}''(\eta_1,r) + \overline{f}''(\eta_1,r)) \\ &+(\underline{f}''(\eta_2,r) + \overline{f}''(\eta_3,r)) + \cdots \\ &+(\underline{f}''(\eta_n,r) + \overline{f}''(\eta_n,r))\}, \end{aligned}$$

f'' is continuous by theorem (2.1), $\underline{f}''$ is continuous, therefore there exist m_1 and M_1 such that $m_1 \le \underline{f}''(\eta_i,r) \le M_1$ for $\eta_i \in [x_i,x_{i+1}]$ therefore

$$\begin{aligned} n.m_1 &\le \underline{f}''(\eta_1,r) + \underline{f}''(\eta_2,r) \\ &+\cdots + \underline{f}''(\eta_n,r) \le n.M_1, \end{aligned}$$

$$m_1 \le \frac{\underline{f}''(\eta_1,r) + \underline{f}''(\eta_2,r) + \cdots + \underline{f}''(\eta_n,r)}{n} \le M_1,$$

by using continuity conditions there exist $\underline{f}''(\eta,r)$, such that

$$min_{x\in[a,b]}\underline{f}(x,r) \le \underline{f}''(\eta,r) \le max_{x\in[a,b]}\underline{f}(x,r)$$

therefore

$$\begin{aligned} \underline{f}''(\eta,r) &= \frac{\underline{f}''(\eta_1,r) + \underline{f}''(\eta_2,r) + \cdots + \underline{f}''(\eta_n,r)}{n} \\ n\underline{f}''(\eta,r) &= \underline{f}''(\eta_1,r) + \underline{f}''(\eta_2,r) + \cdots + \underline{f}''(\eta_n,r), \end{aligned}$$

by similar way we obtain:

$$\begin{aligned} \overline{f}''(\eta,r) &= \frac{\overline{f}''(\eta_1,r) + \overline{f}''(\eta_2,r) + \cdots + \overline{f}''(\eta_n,r)}{n}, \\ n\overline{f}''(\eta,r) &= \overline{f}''(\eta_1,r) + \overline{f}''(\eta_2,r) + \cdots + \overline{f}''(\eta_n,r), \end{aligned}$$

Therefore for error term we have

$$\begin{aligned} E(FT(h)) &= -\frac{h^3}{12}\{(\underline{f}''(\eta_1,r) + \overline{f}''(\eta_1,r)) \\ &+ (\underline{f}''(\eta_2,r) + \overline{f}''(\eta_2,r)) \\ &+ \cdots + (\underline{f}''(\eta_n,r) + \overline{f}''(\eta_n,r))\}, \end{aligned}$$

thus for $\eta \in [a,b]$,

$$E(FT(h)) =$$
$$-\frac{n.h^3}{12}\{n.\underline{f}''(\eta,r) + \overline{f}''(\eta,r)\},$$
$$= -\frac{(b-a)}{24}h^2\{\underline{f}''(\eta,r) + \overline{f}''(\eta,r)\},$$

Remark 3.1 *Since the error term for the fuzzy trapezoidal rule involve $(\underline{f}'', \overline{f}'')$, the rule gives the $d(E,0)=0$ when applied to any function whose second derivatives is identically zero.*

4 Error Term of Fuzzy Simpson Rule

Consider approximating $\int_{x_i}^{x_{i+2}} f(x)dx$, where $x_i = x_0 + ih$ and $h = x_i - x_{i-1}$, fuzzy Simpson rule was introduced by [1] as follows:

$$\begin{aligned}\int_{x_i}^{x_{i+2}} & \underline{f}(x,r)dx \\ = & \frac{h}{3}((\frac{5}{4}\underline{f}_i(x,r) - \frac{1}{4}\overline{f}_i(x,r)) \\ + & 4\underline{f}_{i+1}(x,r) \\ + & (\frac{5}{4}\underline{f}_{i+2}(x,r) - \frac{1}{4}\overline{f}_{i+2}(x,r)),\end{aligned}$$

$$\begin{aligned}\int_{x_i}^{x_{i+2}} & \overline{f}(x,r)dx \\ = & \frac{h}{3}((\frac{5}{4}\overline{f}_i(x,r) - \frac{1}{4}\underline{f}_i(x,r)) \\ + & 4\overline{f}_{i+1}(x,r) \\ + & (\frac{5}{4}\overline{f}_{i+2}(x,r) - \frac{1}{4}\underline{f}_{i+2}(x,r)),\end{aligned}$$

The composite fuzzy Simpson rule is obtained by applying the fuzzy Simpson rule in each subinterval $[x_i, x_{i+1}]$, $i = 0, \cdots, n-1$, i.e.,

$$\begin{aligned}&\int_{x_0}^{x_n} \underline{f}(x,r)dx \qquad (4.9)\\ &= \frac{h}{3}\{(\frac{5}{4}\underline{f}(x_0,r) - \frac{1}{4}\overline{f}(x_0,r)) \\ &+4(\underline{f}(x_1,r) + \underline{f}(x_3,r) + \cdots + \underline{f}(x_{n-1},r)) \\ &+2[(\frac{5}{4}\underline{f}(x_2,r) - \frac{1}{4}\overline{f}(x_2,r)) \\ &+(\frac{5}{4}\underline{f}(x_4,r) - \frac{1}{4}\overline{f}(x_4,r)) \\ &+\cdots+(\frac{5}{4}\underline{f}(x_{n-2},r) - \frac{1}{4}\overline{f}(x_{n-2},r))] \\ &+(\frac{5}{4}\underline{f}(x_n,r) - \frac{1}{4}\overline{f}(x_n,r))\},\end{aligned}$$

and

$$\begin{aligned}&\int_{x_0}^{x_n} \overline{f}(x,r)dx \qquad (4.10)\\ &= \frac{h}{3}\{(\frac{5}{4}\overline{f}(x_0,r) - \frac{1}{4}\underline{f}(x_0,r)) \\ &+4(\overline{f}(x_1,r) + \overline{f}(x_3,r) + \cdots + \overline{f}(x_{n-1},r)) \\ &+2[(\frac{5}{4}\overline{f}(x_2,r) - \frac{1}{4}\underline{f}(x_2,r)) \\ &+(\frac{5}{4}\overline{f}(x_4,r) - \frac{1}{4}\underline{f}(x_4,r)) \\ &+\cdots+(\frac{5}{4}\overline{f}(x_{n-2},r) - \frac{1}{4}\underline{f}(x_{n-2},r))] \\ &+(\frac{5}{4}\overline{f}(x_n,r) - \frac{1}{4}\underline{f}(x_n,r))\}.\end{aligned}$$

For estimating the error of fuzzy Simpson rule for approximating $\int_{x_i}^{x_{i+2}} f(x)dx$, where $x_i = x_0 + ih$ and $h = x_i - x_{i-1}$, error terms $\underline{E}_i(r)$ and $\overline{E}_i(r)$ are defined as follows:

$$\begin{aligned}&\underline{E}_i(r) = \\ &\int_{x_i}^{x_{i+2}} \underline{f}(x,r)dx - \{\frac{h}{3}((\frac{5}{4}\underline{f}_i(x,r) - \frac{1}{4}\overline{f}_i(x,r)) \\ &+4\underline{f}_{i+1}(x,r) + (\frac{5}{4}\underline{f}_{i+2}(x,r) - \frac{1}{4}\overline{f}_{i+2}(x,r))\},\end{aligned}$$

$$\begin{aligned}&\overline{E}_i(r) \\ &= \int_{x_i}^{x_{i+2}} \overline{f}(x,r)dx - \{\frac{h}{3}((\frac{5}{4}\overline{f}_i(x,r) - \frac{1}{4}\underline{f}_i(x,r)) \\ &+4\overline{f}_{i+1}(x,r) + (\frac{5}{4}\overline{f}_{i+2}(x,r) - \frac{1}{4}\underline{f}_{i+2}(x,r))\}.\end{aligned}$$

By using Taylor's series for $\underline{f}(x,r)$ and $\overline{f}(x,r)$ around x_i, where the first four derivative $\underline{f}(x,r)$ and $\overline{f}(x,r)$ are continuous on $(a,b), h = \frac{b-a}{n}$. Then we observe that

$$\begin{aligned}&\int_{x_i}^{x_{i+2}} \underline{f}(x,r)dx = \\ &\int_{x_i}^{x_{i+2}} \{\underline{f}(x_i,r) + (x-x_i)\underline{f}'(x_i,r) \\ &+\frac{(x-x_i)^2}{2!}\underline{f}''(x_i,r) \\ &+\frac{(x-x_i)^3}{3!}\underline{f}^{(3)}(x_i,r) \\ &+\frac{(x-x_i)^4}{4!}\underline{f}^{(4)}(\eta_i,r)\}dx, \\ &= 2h.\underline{f}(x_i,r) + \frac{(2h)^2}{2}\underline{f}'(x_i,r) \\ &+\frac{(2h)^3}{3!}\underline{f}''(x_i,r) + \frac{(2h)^4}{4!}\underline{f}^{(3)}(x_i,r) \\ &+\frac{(2h)^5}{5!}\underline{f}^{(4)}(\eta_i,r),\end{aligned}$$

Again we have

$$\begin{aligned}
&\tfrac{h}{3} \quad ((\frac{5}{4}\underline{f}(x_i,r) - \frac{1}{4}\overline{f}_i(x_i,r)) \\
&+ \quad 4\underline{f}_{i+1}(x_i,r) \\
&+ \quad (\frac{5}{4}\underline{f}_{i+2}(x_i,r) - \frac{1}{4}\overline{f}_{i+2}(x_i,r)) \\
&= \quad \frac{h}{3}((\frac{5}{4}\underline{f}(x_i,r) - \frac{1}{4}\overline{f}(x_i,r)) \\
&+ \quad 4\{\underline{f}(x_i,r) + h\underline{f}'(x_i,r) + \frac{h^2}{2}\underline{f}''(x_i,r) \\
&+ \quad \frac{h^3}{3!}\underline{f}^{(3)}(x_i,r) + \frac{h^4}{4!}\underline{f}^{(4)}(\eta_i,r)\} \\
&+ \quad (\frac{5}{4}\{\underline{f}(x_i,r) + 2h\underline{f}'(x_i,r) \\
&+ \quad \frac{(2h)^2}{2}\underline{f}''(x_i,r) + \frac{(2h)^3}{3!}\underline{f}^{(3)}(x_i,r) \\
&+ \quad \frac{(2h)^4}{4!}\underline{f}^{(4)}(x_i,r)\} \\
&- \quad \frac{1}{4}\{\overline{f}(x_i,r) + 2h\overline{f}'(x_i,r) + \frac{(2h)^2}{2}\overline{f}''(x_i,r) \\
&+ \quad \frac{(2h)^3}{3!}\overline{f}^{(3)}(x_i,r) + \frac{(2h)^4}{4!}\overline{f}^{(4)}(\eta_i,r)\}
\end{aligned}$$

Hence

$$\begin{aligned}
\underline{E}_i(r) &= -\frac{h}{6}\underline{f}(x_i,r) - \frac{h^2}{6}\underline{f}'(x_i,r) - \frac{h^3}{6}\underline{f}''(x_i,r) \\
&- \frac{h^4}{9}\underline{f}^{(3)}(x_i,r) - \frac{h^5}{15}\underline{f}^{(4)}(\eta_i,r) \\
&+ \frac{h}{6}\overline{f}(x_i,r) + \frac{h^2}{6}\overline{f}'(x_i,r) + \frac{h^3}{6}\overline{f}''(x_i,r) \\
&+ \frac{h^4}{9}\overline{f}^{(3)}(x_i,r) + \frac{4h^5}{3.4!}\overline{f}^{(4)}(\eta_i,r),
\end{aligned}$$

$$\begin{aligned}
\overline{E}_i(r) &= -\frac{h}{6}\overline{f}(x_i,r) - \frac{h^2}{6}\overline{f}'(x_i,r) \\
&- \frac{h^3}{6}\overline{f}''(x_i,r) - \frac{h^4}{9}\overline{f}^{(3)}(x_i,r) - \frac{h^5}{15}\overline{f}^{(4)}(\eta_i,r) \\
&+ \frac{h}{6}\underline{f}(x_i,r) + \frac{h^2}{6}\underline{f}'(x_i,r) + \frac{h^3}{6}\underline{f}''(x_i,r) \\
&+ \frac{h^4}{9}\underline{f}^{(3)}(x_i,r) + \frac{4h^5}{3.4!}\underline{f}^{(4)}(\eta_i,r),
\end{aligned}$$

Now we want to equal error term to zero, therefore

$$d(E,0) = 0,$$

this means

$$\underline{E}_i(r) + \overline{E}_i(r) = 0$$

and finally for $\eta_i \in [x_i, x_{i+1}]$ the error term for fuzzy Simpson rule is denote by $E_i(FS(h))$ is obtained as follows:

$$E_i(FS(h)) = -\frac{h^5}{90}(\underline{f}^{(4)}(\eta_i,r) + \overline{f}^{(4)}(\eta_i,r)) \quad (4.11)$$

Theorem 4.1 *Let $f \in C^4([a,b], R_F)$ the error term for the composite fuzzy Simpson rule for $h = \frac{b-a}{n} = x_{i+1} - x_i$, is $E(FS(h)) = -\frac{(b-a)}{180}h^4\{\underline{f}^{(4)}(\eta,r) + \overline{f}^{(4)}(\eta,r)\}, \eta \in [a,b]$.*

Proof :The error of composite fuzzy Simpson rule is obtained by applying the error of fuzzy trapezoidal rule in each subinterval $[x_i, x_{i+2}]$, for $i = 0, \cdots, n-2$, i.e

$$\begin{aligned}
\textstyle\int_a^b f(x)dx &= \\
\textstyle\int_{x_0}^{x_n} f(x)dx &= \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx \\
&+ \cdots + \int_{x_{n-2}}^{x_n} f(x)dx,
\end{aligned}$$

we observe that

$$\begin{aligned}
&E(FS(h)) = \\
- &\frac{h^5}{90}\{(\underline{f}^{(4)}(\eta_1,r) + \overline{f}^{(4)}(\eta_1,r)) \\
+ &(\underline{f}^{(4)}(\eta_3,r) + \overline{f}^{(4)}(\eta_3,r)) \\
+ &\cdots + (\underline{f}^{(4)}(\eta_{n-1},r) + \overline{f}^{(4)}(\eta_{n-1},r))\}
\end{aligned}$$

$f^{(4)}$ is continuous, by theorem (2.1), $\underline{f}^{(4)}$ is continuous, therefore there exist m_1 and M_1 such that $m_1 \leq \underline{f}^{(4)}(\eta_i,r) \leq M_1$ for $\eta_i \in [x_i, x_{i+2}]$, then:

$$\begin{aligned}
\frac{n}{2}m_1 &\leq \underline{f}^{(4)}(\eta_1,r) + \underline{f}^{(4)}(\eta_3,r) \\
&+ \cdots + \underline{f}^{(4)}(\eta_{n-1},r) \leq \frac{n}{2}M_1,
\end{aligned}$$

therefore by using continuity conditions there exist $\underline{f}^{(4)}(\eta,r)$,

$$\begin{aligned}
min_{x\in[a,b]}\underline{f}^{(4)}(x,r) &\leq \underline{f}^{(4)}(\eta,r) \\
&\leq max_{x\in[a,b]}\underline{f}^{(4)}(x,r),
\end{aligned}$$

where and

$$\begin{aligned}
\frac{n}{2}\underline{f}^{(4)}(\eta,r) &= \underline{f}^{(4)}(\eta_1,r) + \underline{f}^{(4)}(\eta_3,r) \\
&+ \cdots + \underline{f}^{(4)}(\eta_{n-1},r),
\end{aligned}$$

By similar way

$$\frac{n}{2}\overline{f}^{(4)}(\eta, r) = \overline{f}^{(4)}(\eta_1, r) + \overline{f}^{(4)}(\eta_3, r) + \cdots + \overline{f}^{(4)}(\eta_{n-1}, r)$$

therefore the error term for fuzzy Simpson rule is obtained as follows

$$\begin{aligned} &E(FS(h)) \\ &= -\frac{h^5}{90}\{(\underline{f}^{(4)}(\eta_1, r) + \overline{f}^{(4)}(\eta_1, r)) \\ &+(\underline{f}^{(4)}(\eta_3, r) + \overline{f}^{(4)}(\eta_3, r)) \\ &+ \cdots + (\underline{f}^{(4)}(\eta_{n-1}, r) + \overline{f}^{(4)}(\eta_{n-1}, r))\} \end{aligned}$$

Thus for $\eta \in [a, b]$,

$$\begin{aligned} &E(FS(h)) \\ &= -\frac{h^5}{90}\{\frac{n}{2}\underline{f}^{(4)}(\eta, r) + \frac{n}{2}\overline{f}^{(4)}(\eta, r)\} \\ &= -\frac{(b-a)}{180}h^4\{\underline{f}^{(4)}(\eta, r) + \overline{f}^{(4)}(\eta, r)\}. \end{aligned}$$

Remark 4.1 *Since the error term for the fuzzy Simpson rule involve* $(\underline{f}^{(4)}, \overline{f}^{(4)})$*, the rule gives the* $d(E, 0) = 0$ *when applied to any function whose forth derivatives is identically zero.*

5 Numerical Example

Example 5.1 *Consider the following fuzzy integral,*

$$\int_0^1 f(x)dx, \quad f(x, r) = (\frac{2r}{1+x^2}, \frac{4-2r}{1+x^2})$$

the exact solution is $\frac{1}{4}\pi(2r, 4-2r)$. By $h = 0.1$ and fuzzy trapezoidal method we obtain:

$$\begin{aligned} &\int_0^1 (\frac{2r}{1+x^2}, \frac{4-2r}{1+x^2})dx \\ &= (1.469962994r + 0.1, \\ &3.039925988 - 1.469962994r), \end{aligned}$$

and the error of fuzzy trapezoidal rule is:

$$\begin{aligned} &E(FT(h))) \\ &= (0.100833333r - 0.1, \\ &0.101666667 - 0.100833333r), \end{aligned}$$

obviously

$$d(E(FT(h), 0) = 0.001666667,$$

By using fuzzy Simpson rule and h=0.1, we obtain:

$$\begin{aligned} &\int_0^1 (\frac{2r}{1+x^2}, \frac{4-2r}{1+x^2})dx \\ &= (1.964053351r - 0.3932570440 \\ &3.534849657 - 1.964053351r), \end{aligned}$$

and

$$\begin{aligned} &E(FS(h)) \\ &= (-0.393257024r + 0.3932570440 \\ &, -0.393257002 + 0.393257024r), \end{aligned}$$

therefore for fuzzy Simpson rule we have:

$$d(ES(h), 0) = 4.2 \times 10^{-8}.$$

Example 5.2 *Consider the following fuzzy integral,*

$$\int_{-1}^{1.4} f(x)dx,$$

where

$$f(x, r) = ((0.5 + 0.5r)x^3, (1.5 - 0.5r)x^3).$$

the exact solution is

$$0.7104(0.5 + 0.5r, 1.5 - 0.5).$$

By fuzzy trapezoidal rule and $h = 0.1$ we obtain:

$$\begin{aligned} &\int_{-1}^{1.4} ((0.5 + 0.5r)x^3, (1.5 - 0.5r)x^3)dx \\ &= (0.1 + 0.62r, 1.34 - 0.62r), \end{aligned}$$

the error term for fuzzy Trapezoidal rule is:

$$\begin{aligned} &E(FT(h)) \\ &= (0.2552 - .2648r, -0.2744 + 0.2648r), \end{aligned}$$

and distance of error to zero is:

$$d(E(FT(h)), 0) = -0.0192,$$

By using fuzzy Simpson rule and $h = 0.1$, we have:

$$\begin{aligned} &\int_{-1}^{1.4} ((0.5 + 0.5r)x^3, (1.5 - 0.5r)x^3)dx \\ &= (-0.0056 + .716r, 1.426400000 - 0.716r), \end{aligned}$$

the error term is obtain as follows

$$\begin{aligned} &E(FS(h)) \\ &= (0.3608 - 0.3608r, -0.3608 + .36080r), \end{aligned}$$

and the distance to zero is:

$$d(E(FS(h)), 0) = 0,$$

Clearly we see the fuzzy Simpson rule is accurate for fuzzy polynomial of degree three.

6 Conclusion

In this paper the error term for fuzzy Trapezoidal and fuzzy Simpson rules was discussed. We proved that the fuzzy Trapezoidal rule is accurate for fuzzy polynomial of degree one and fuzzy Simpson rule is accurate for polynomial of degree three.

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Nazanin Ahmady has got PhD degree from Islamic Azad University Science and Research Branch in 2008 She has been member of the faculty in Islamic Azad University Varamin-Pishva branch since 2005. Main research interest include: Fuzzy differential equations, Fuzzy Data Envelopment Analysis, Ranking and fuzzy systems.

Elham Ahmady has got PhD degree from Islamic Azad University Science and Research Branch in 2008 She has been member of the faculty in Islamic Azad University Shahr-e-Qods branch since 2005. Main research interest include: Fuzzy differential equations, Ranking and fuzzy systems.