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Research Article

Bessel multipliers on the tensor product of Hilbert C^*- modules

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Abstract

In this paper, we first show that the tensor product of a finite number of standard g-frames (resp. fusion frames, frames) is a standard g-frame (resp. fusion frame, frame) for the tensor product of Hilbert C^*-modules and vice versa, then we consider tensor products of g-Bessel multipliers, Bessel multipliers and Bessel fusion multipliers in Hilbert C^*- modules. Moreover, we obtain some results for the tensor product of duals using Bessel multipliers.

Keywords : G-frames; Bessel multipliers; tensor products; Hilbert C^*- modules.

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1 Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [7] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [6]. Frames are very useful in characterization of function spaces and other fields of applications such as filter bank theory, sigma-delta quantization, signal and image processing and wireless communications. Fusion frames [5] and g-frames [23] are important generalizations of frames.

Hilbert C^*–modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*–algebra rather than in the field of complex numbers. Hilbert C^*-modules are used in the study of locally compact quantum groups, completely positive maps between C^*-algebras, non-commutative geometry and KK-theory.

Frank and Larson presented a general approach to the frame theory in Hilbert C^*–modules (see [8]). Also A. Khosravi and B. Khosravi introduced fusion frames and g-frames in Hilbert C^*–modules (see [12]).

Bessel multipliers in Hilbert spaces were introduced by Balazs in [3]. Bessel fusion multipliers and g-Bessel multipliers in Hilbert spaces were introduced in [17] and [21], respectively. Also multipliers were introduced for p-Bessel sequences in Banach spaces (see [22]). Recently the present author and A. Khosravi generalized Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers to Hilbert C^*-modules (see [15]).

Tensor products of frames, fusion frames and g-frames in Hilbert spaces have been studied by some authors recently, see [4, 11, 13]. Also tensor products of g-frames were considered in Hilbert C^*-modules, see [11, 12, 10, 20]. Tensor products have important applications, for example tensor products are useful in the approximation of multi-variate functions of combinations of univariate ones. In this paper, we investigate tensor products of g-frames, fusion frames and frames in Hilbert C^*-modules and we consider their multipliers.

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2 Frames, fusion frames and g-frames in Hilbert C^*-modules

Suppose that $\mathfrak{A}$ is a C^*-algebra and E is a left $\mathfrak{A}-$module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}-$module if E is equipped with an $\mathfrak{A}-$valued inner product $\langle .,. \rangle : E \times E \longrightarrow \mathfrak{A}$, such that

(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;

(ii) $\langle ax, y\rangle = a\langle x, y\rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$, for each $x, y \in E$;

(iv) $\langle x, x\rangle \geq 0$, for each $x \in E$ and if $\langle x, x\rangle = 0$, then $x = 0$.

For each $x \in E$, we define $|x|= \langle x, x\rangle^{\frac{1}{2}}$ and $\|x\|= \|\langle x, x\rangle\|$ $\frac{1}{2}.IfEiscompletewith\|.\|$, it is called a *Hilbert* $\mathfrak{A}-$*module* or a *Hilbert* C^*-*module* over $\mathfrak{A}$. We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the *center* of $\mathfrak{A}$. Let E_1 and E_2 be Hilbert $\mathfrak{A}$–modules. The operator $T : E_1 \longrightarrow E_2$ is called *adjointable* if there exists an operator $T^* : E_2 \longrightarrow E_1$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$, for each $x \in E_1$ and $y \in E_2$. Every adjointable operator $T : E_1 \longrightarrow E_2$ is bounded and $\mathfrak{A}$–linear (that is, $T(ax) = aT(x)$ for each $x \in E_1$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E_1 into E_2 by $\mathfrak{L}_{\mathfrak{A}}(E_1, E_2)$. Note that $\mathfrak{L}_{\mathfrak{A}}(E_1, E_1)$ is a C^*-algebra which is denoted by $\mathfrak{L}_{\mathfrak{A}}(E_1)$, for more details see [16].

A Hilbert $\mathfrak{A}$–module E is *finitely generated* if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$–linear combination $x = \sum_{i=1}^{n} a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}-$module E is *countably generated* if there exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of $\mathfrak{A}$–linear hull of $\{x_i\}_{i\in I}$.

Let E be a Hilbert $\mathfrak{A}-$module. A family $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ is a *frame* for E, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A.\langle x, x\rangle \leq \sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle \leq B.\langle x, x\rangle, \qquad (2.1)$$

i.e., there exist real constants $0 < A \leq B < \infty$, such that the series $\sum_{i\in I}\langle x, f_i\rangle\langle f_i, x\rangle$ converges in the ultraweak operator topology to some element in the universal enveloping Von Neumann algebra of $\mathfrak{A}$ such that the inequality (2.1) holds, for each $x \in E$. The numbers A and B are called the lower and upper bound of the frame, respectively. In this case we call it an (A, B) *frame*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (2.1) converges in norm, the frame is called *standard*. If $\mathcal{F} = \{f_i\}_{i\in I}$ is a standard Bessel sequence, then the operator $S_{\mathcal{F}}$ is defined on E by $S_{\mathcal{F}}x = \sum_{i\in I}\langle x, f_i\rangle f_i$. $S_{\mathcal{F}}$ is an adjointable and positive operator and if $\mathcal{F}$ is a standard frame, then $S_{\mathcal{F}}$ is invertible. For more results about frames in Hilbert C^*–modules, see [8, 1].

A closed submodule M of E is *orthogonally complemented* if $E = M \oplus M^{\perp}$. In this case $\pi_M \in \mathfrak{L}_{\mathfrak{A}}(E, M)$, where $\pi_M : E \longrightarrow M$ is the projection onto M.

Suppose that $\{\omega_i : i \in I\} \subseteq \mathfrak{A}$ is a family of weights, i.e., each ω_i is a positive, invertible element from the center of $\mathfrak{A}$, and $\{W_i : i \in I\}$ is a family of orthogonally complemented submodules of E. Then $\{(W_i, \omega_i)\}_{i\in I}$ is a *fusion frame* if there exist positive numbers A and B such that

$$A.\langle x, x\rangle \leq \sum_{i\in I}\omega_i^2\langle \pi_{W_i}(x), \pi_{W_i}(x)\rangle \leq B.\langle x, x\rangle,$$

for each $x \in E$. If we only require to have the upper bound, then $\{(W_i, \omega_i)\}_{i\in I}$ is called a *Bessel fusion sequence* with upper bound B.

Let $\{E_i\}_{i\in I}$ be a sequence of Hilbert $\mathfrak{A}$–modules. A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that for each $x \in E$,

$$A.\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq B.\langle x, x\rangle.$$

If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*. Standard g-frames and fusion frames are defined similar to frames.

If $W = \{(W_i, \omega_i)\}_{i\in I}$ is a standard Bessel fusion sequence, then the operator $S_W : E \longrightarrow E$ which is defined by $S_W x = \sum_{i\in I}{\omega_i}^2\pi_{W_i}x$ is adjointable and called the *operator* of W. For a standard g-Bessel sequence Λ, the operator $S_\Lambda : E \longrightarrow E$ which is defined by $S_\Lambda(x) = \sum_{i\in I}\Lambda_i^*\Lambda_i(x)$ is adjointable and it is called the *operator* of Λ. If Λ is a standard (A, B) g-frame, then $A.Id_E \leq S_\Lambda \leq$

$B.Id_E$. For more results about fusion frames and g-frames in Hilbert C^*-modules, see [12, 24]. Also note that fusion frames have been introduced in Hilbert modules over pro-C^*-algebras (see [2]).
In this paper all C^*-algebras are unital and Hilbert C^*-modules are finitely or countably generated. All frames, fusion frames, g-frames and Bessel sequences are standard.

Throughout this paper I and I_k, for each $1 \leq k \leq n$, are subsets of $\mathbb{N}$. $\mathfrak{A}_k$ is a unital C^*-algebra, E, E_k and $E_{i(k)}$ are finitely or countably generated Hilbert C^*-modules, for each $k \in \{1, \dots, n\}$ and $i(k) \in I_k$.

3 Tensor products of Bessel multipliers

First we recall the definitions of Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers from [15].

As usual $\ell^\infty(I, \mathfrak{A})$ is the set $\Big\{\{a_i\}_{i\in I} \subseteq \mathfrak{A} : sup\{\|a_i\|: i \in I\} < \infty\Big\}$, and in this note m is always a sequence $\{m_i\}_{i\in I} \in \ell^\infty(I, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$. Each sequence with these properties is called a *symbol*.

Definition 3.1 *Let E_1 and E_2 be Hilbert $\mathfrak{A}$-modules, and let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E_1$ and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E_2$ be standard Bessel sequences. The operator $S_{m\mathcal{G}\mathcal{F}} : E_1 \longrightarrow E_2$ defined by $S_{m\mathcal{G}\mathcal{F}}(x) = \sum_{i\in I} m_i\langle x, f_i\rangle g_i$, is adjointable and it is called the* Bessel multiplier *for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$.*

Recall from Example 3.1 in [12] that if $W = \{(W_i, \omega_i)\}_{i\in I}$ is a standard Bessel fusion sequence (resp. standard fusion frame) for E, then $\Lambda_W = \{\omega_i\pi_{W_i}\}_{i\in I}$ is a standard g-Bessel sequence (resp. standard g-frame) for E with respect to $\{W_i\}_{i\in I}$.

Definition 3.2 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be standard g-Bessel sequences for E with respect to $\{E_i\}_{i\in I}$. Then the operator $S_{m\Gamma\Lambda} : E \longrightarrow E$ which is defined by $S_{m\Gamma\Lambda}(x) = \sum_{i\in I} m_i\Gamma_i^*\Lambda_i(x)$ is adjointable and it is called the* g-Bessel multiplier *for the g-Bessel sequences Λ and Γ. Also if $W = \{(W_i, \omega_i)\}_{i\in I}$ and $V = \{(V_i, \upsilon_i)\}_{i\in I}$ are standard Bessel fusion sequences for E, we call the operator $S_{mVW}(x) = S_{m\Lambda_V\Lambda_W}(x) = \sum_{i\in I} m_i\upsilon_i\omega_i\pi_{V_i}\pi_{W_i}(x)$, the* Bessel fusion multiplier *for W and V.*

Recall that if $\mathfrak{A}_k$ is a C^*-algebra, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n\mathfrak{A}_k$ is a C^*-algebra with the spatial norm and for each $a_k \in \mathfrak{A}_k$, we have $\|a_1 \otimes \dots \otimes a_n\| = \Pi_{k=1}^n\|a_k\|$. The multiplication and involution on simple tensors are defined by $(\otimes_{k=1}^n a_k)(\otimes_{k=1}^n b_k) = \otimes_{k=1}^n(a_k b_k)$ and $(\otimes_{k=1}^n a_k)^* = \otimes_{k=1}^n a_k^*$, respectively. As we know if $a_k \geq 0$, for each $1 \leq k \leq n$, then $\otimes_{k=1}^n a_k \geq 0$.

Now if E_k is a Hilbert $\mathfrak{A}_k-$module, for each $1 \leq k \leq n$, then the (Hilbert C^*-module) tensor product $\otimes_{k=1}^n E_k = E_1 \otimes \dots \otimes E_n$ is a Hilbert $(\otimes_{k=1}^n\mathfrak{A}_k)-$module. The module action and inner product for simple tensors are defined by

$$\begin{aligned}(\otimes_{k=1}^n a_k)(\otimes_{k=1}^n x_k) &= (a_1x_1) \otimes \dots \otimes (a_nx_n)\\ &= \otimes_{k=1}^n(a_kx_k),\end{aligned}$$

and

$$\begin{aligned}&\langle \otimes_{k=1}^n x_k, \otimes_{k=1}^n y_k\rangle\\ =\ &\langle x_1, y_1\rangle \otimes \dots \otimes \langle x_n, y_n\rangle\\ =\ &\otimes_{k=1}^n\langle x_k, y_k\rangle,\end{aligned}$$

respectively, where $a_k \in \mathfrak{A}_k$ and $x_k, y_k \in E_k$. If U_k is an adjointable operator on E_k, then the tensor product $\otimes_{k=1}^n U_k$ is an adjointable operator on $\otimes_{k=1}^n E_k$. Also $(\otimes_{k=1}^n U_k)^* = \otimes_{k=1}^n U_k^*$ and $\|\otimes_{k=1}^n U_k\| = \Pi_{k=1}^n\|U_k\|$. Note that if M_k is an orthogonally complemented submodule of E_k, for each $1 \leq k \leq n$, then it is easy to see that $\otimes_{k=1}^n M_k$ is an orthogonally complemented submodule of $\otimes_{k=1}^n E_k$ and $\pi_{\otimes_{k=1}^n M_k} = \otimes_{k=1}^n\pi_{M_k}$. For more results, see [19, 16].
In this paper $\mathcal{F}^{(k)} = \{f_{i(k)}\}_{i(k)\in I_k}$ and $\mathcal{G}^{(k)} = \{g_{i(k)}\}_{i(k)\in I_k}$ are sequences in E_k and $\otimes_{k=1}^n\mathcal{F}^{(k)}$ is defined by $\{f_{i(1)}\otimes\dots\otimes f_{i(n)}\}_{(i(1),\dots,i(n))\in(I_1\times\dots\times I_n)}$. $\Phi^{(k)} = \{\Lambda_{i(k)} \in \mathfrak{L}_{\mathfrak{A}_k}(E_k, E_{i(k)})\}_{i(k)\in I_k}$, $\Psi^{(k)} = \{\Gamma_{i(k)} \in \mathfrak{L}_{\mathfrak{A}_k}(E_k, E_{i(k)}) : i(k) \in I_k\}$, $\mathcal{W}^{(k)} = \{(W_{i(k)}, \omega_{i(k)})\}_{i(k)\in I_k}$ $\mathcal{V}^{(k)} = \{(V_{i(k)}, \upsilon_{i(k)}) : i(k) \in I_k\}$, where $W_{i(k)}$ and $V_{i(k)}$ are orthogonally complemented submodules of E_k and $\omega_{i(k)}$ and $\upsilon_{i(k)}$ are weights in $\mathfrak{A}_k$, for each $1 \leq k \leq n$. $\otimes_{k=1}^n\Phi^{(k)}$ and $\otimes_{k=1}^n\mathcal{W}^{(k)}$ are

$$\begin{aligned}&\{\Lambda_{i(1)} \otimes \dots \otimes \Lambda_{i(n)} \in\\ &\mathfrak{L}_{(\mathfrak{A}_1\otimes\dots\otimes\mathfrak{A}_n)}(\otimes_{k=1}^n E_k, E_{i(1)} \otimes \dots \otimes E_{i(n)})\\ &, (i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)\},\end{aligned}$$

$$\begin{aligned}&\{(W_{i(1)} \otimes \dots \otimes W_{i(n)}, \omega_{i(1)} \otimes \dots \otimes \omega_{i(n)})\\ &: (i(1), \dots, i(n)) \in (I_1 \times \dots \times I_n)\},\end{aligned}$$

respectively. Also $m^{(k)} = \{m_{i(k)}\}_{i(k)\in I_k}$ is a symbol in $\ell^\infty(I_k, \mathfrak{A}_k)$ and $\otimes_{k=1}^n m^{(k)}$ is the set $\{m_{i(1)} \otimes \ldots \otimes m_{i(n)}\}_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}$.

The following theorem is a generalization of [13, Theorem 2.1 (i)] to Hilbert C^*-modules and also generalizes the results obtained for tensor products of g-frames in [12], [20] and [10].

Theorem 3.1 (i) *If $\Phi^{(k)}$ is a g-Bessel sequence, for each $1 \le k \le n$, then $\otimes_{k=1}^n \Phi^{(k)}$ is a g-Bessel sequence. Moreover, $\Phi^{(k)}$ is a g-frame, for each $1 \le k \le n$ if and only if $\otimes_{k=1}^n \Phi^{(k)}$ is a g-frame.*

(ii) *If $\Phi^{(k)}$'s and $\Psi^{(k)}$'s are g-Bessel sequences, then the operator $S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n \Psi^{(k)})(\otimes_{k=1}^n \Phi^{(k)})}$ is well-defined and is equal to $\otimes_{k=1}^n S_{m^{(k)}\Psi^{(k)}\Phi^{(k)}}$.*

Proof. (i) *It is enough to prove the theorem for $n = 2$. Let B_1 and B_2 be upper bounds of $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, $I_1 = \{i_{11}, \ldots, i_{1p}, \ldots\}$ and $I_2 = \{i_{21}, \ldots, i_{2q}, \ldots\}$. Then define $S_{1p}x = \sum_{r=1}^p \Lambda^*_{i_{1r}}\Lambda_{i_{1r}}x$ and $S_{2q}y = \sum_{t=1}^q \Lambda^*_{i_{2t}}\Lambda_{i_{2t}}y$, for each $x \in E_1$ and $y \in E_2$. Now $\|S_{1p}\| \le \|S_{\Phi^{(1)}}\|$ and $\|S_{2q}\| \le \|S_{\Phi^{(2)}}\|$, for each $p, q \in \mathbb{N}$ and since $\Phi^{(1)}$ and $\Phi^{(2)}$ are standard $g-$Bessel sequences, then $0 \le S_{\Phi^{(k)}} \le B_k.Id_{E_k}$, for each $k \in \{1, 2\}$ and consequently $0 \le S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}} \le B_1B_2.Id_{(E_1\otimes E_2)}$. Therefore by Lemma 4.1 in [16], for each $z \in E_1 \otimes E_2$ and $p, q \in \mathbb{N}$, we have*

$$\langle (S_{1p} \otimes S_{2q})z, z\rangle \le$$

$$\langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z\rangle \le B_1B_2.\langle z, z\rangle. \tag{3.2}$$

It is also easy to see that $\lim_{p,q}(S_{1p} \otimes S_{2q})z = (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z$, for each $z = \sum_{l=1}^m x_l \otimes y_l \in E_1 \otimes_{alg} E_2$. Now if $z \in E_1 \otimes E_2$, then by an appropriate choice of $z_0 \in E_1 \otimes_{alg} E_2$, and the inequality

$$\begin{aligned} & \|(S_{1p} \otimes S_{2q})z - (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z\| \\ \le\ & \|S_{\Phi^{(1)}}\|\|S_{\Phi^{(2)}}\|\|z - z_0\| \\ +\ & \|(S_{1p} \otimes S_{2q})z_0 - (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z_0\| \\ +\ & B_1B_2\|z - z_0\|, \end{aligned}$$

we get $\lim_{p,q}(S_{1p} \otimes S_{2q})z = (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z$. This means that the series $\sum_{(i(1),i(2))\in I_1\times I_2}\langle(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z, (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\rangle$ converges in norm and by (3.2), we have

$$\sum_{(i(1),i(2))\in I_1\times I_2} \langle(\Lambda_{i(1)} \otimes \Lambda_{i(2)})z, (\Lambda_{i(1)} \otimes \Lambda_{i(2)})z\rangle$$

$$= \langle (S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})z, z\rangle \le B_1B_2.\langle z, z\rangle. \tag{3.3}$$

This shows that $\Phi^{(1)} \otimes \Phi^{(2)}$ is a standard g-Bessel sequence with upper bound B_1B_2.
Now suppose that $\Phi^{(1)}$ and $\Phi^{(2)}$ are g-frames with lower bounds A_1 and A_2, respectively. Since

$$\begin{aligned} & A_1A_2.Id_{E_1\otimes E_2} \\ \le\ & (\|S^{-1}_{\Phi^{(1)}}\|^{-1}\|S^{-1}_{\Phi^{(2)}}\|^{-1}).Id_{E_1\otimes E_2} \\ =\ & \|(S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}})^{-1}\|^{-1}.Id_{E_1\otimes E_2} \\ \le\ & S_{\Phi^{(1)}} \otimes S_{\Phi^{(2)}}, \end{aligned}$$

using (3.2) and (3.3), we obtain that $\otimes_{k=1}^2 \Phi^{(k)}$ is a standard g-frame with lower bound A_1A_2.
Conversely let $\otimes_{k=1}^2 \Phi^{(k)}$ be a standard g-frame with upper bound B and $x \in E_1$. Since $\otimes_{k=1}^2 \Phi^{(k)}$ is a standard g-Bessel sequence, it is clear that the series $\sum_{i(1)\in I_1}\langle\Lambda_{i(1)}x, \Lambda_{i(1)}x\rangle$ converges in norm and for each $y \in E_2$,

$$\begin{aligned} & \left\| \sum_{i(1)\in I_1} \langle\Lambda_{i(1)}x, \Lambda_{i(1)}x\rangle \right\| \times \\ & \left\| \sum_{i(2)\in I_2} \langle\Lambda_{i(2)}y, \Lambda_{i(2)}y\rangle \right\| \\ =\ & \Bigg\| \sum_{(i(1),i(2))\in I_1\times I_2} \langle(\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y), \\ & (\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y)\rangle \Bigg\| \\ \le\ & B\|x \otimes y\|^2 = B\|x\|^2\|y\|^2. \end{aligned}$$

Let $y \in E_2$ with $\|y\| = 1$. Since $\otimes_{k=1}^2 \Phi^{(k)}$ is a g-frame,
$C = \left\| \sum_{i(2)\in I_2}\langle\Lambda_{i(2)}y, \Lambda_{i(2)}y\rangle \right\|$ is a positive number, so we have

$$\left\| \sum_{i(1)\in I_1} \langle\Lambda_{i(1)}x, \Lambda_{i(1)}x\rangle \right\| \le \frac{B}{C}\|x\|^2.$$

Therefore by [24, Theorem 3.1], $\Phi^{(1)}$ is a standard g-Bessel sequence with upper bound $\frac{B}{C}$.
Now let A be a lower bound for $\otimes_{k=1}^2 \Phi^{(k)}$ and $x \in E_1$. If $y \in E_2$ with $\|y\| = 1$ and $C = \left\| \sum_{i(2)\in I_2}\langle\Lambda_{i(2)}y, \Lambda_{i(2)}y\rangle \right\|$, then it is easy to see that

$$\frac{A}{C}\|x\|^2 \le \left\| \sum_{i(1)\in I_1} \langle\Lambda_{i(1)}x, \Lambda_{i(1)}x\rangle \right\|.$$

Hence $\Phi^{(1)}$ is a standard g-frame and a similar proof shows that $\Phi^{(2)}$ is also a standard g-frame.

(ii) *By part* (i), $\otimes_{k=1}^{n}\Phi^{(k)}$ *and* $\otimes_{k=1}^{n}\Psi^{(k)}$ *are g-Bessel sequences. Now let* $\otimes_{k=1}^{n}a_k$ *be a simple tensor in* $\otimes_{k=1}^{n}\mathfrak{A}_k$*. Since* $m_{i(k)} \in \mathcal{Z}(\mathfrak{A}_k)$*, for each* $1 \leq k \leq n$*, we have*

$$\begin{aligned} & (\otimes_{k=1}^{n}a_k)(\otimes_{k=1}^{n}m_{i(k)}) = \otimes_{k=1}^{n}(a_k m_{i(k)}) \\ = \ & \otimes_{k=1}^{n}(m_{i(k)}a_k) \\ = \ & (\otimes_{k=1}^{n}m_{i(k)})(\otimes_{k=1}^{n}a_k). \end{aligned}$$

Because the above equality holds for simple tensors, $N(\otimes_{k=1}^{n}m_{i(k)}) = (\otimes_{k=1}^{n}m_{i(k)})N$*, for each* $N \in \otimes_{k=1}^{n}\mathfrak{A}_k$*. Therefore* $\otimes_{k=1}^{n}m_{i(k)} \in \mathcal{Z}(\otimes_{k=1}^{n}\mathfrak{A}_k)$ *and the relation* $\|\otimes_{k=1}^{n}m_{i(k)}\| = \Pi_{k=1}^{n}\|m_{i(k)}\| \leq \Pi_{k=1}^{n}\|m^{(k)}\|$ *yields that* $\otimes_{k=1}^{n}m^{(k)}$ *is a symbol, so* $S_{(\otimes_{k=1}^{n}m^{(k)})(\otimes_{k=1}^{n}\Psi^{(k)})(\otimes_{k=1}^{n}\Phi^{(k)})}$ *is well-defined. Now let* $n = 2$ *and* $x \otimes y \in E_1 \otimes E_2$*. Then we have*

$$\begin{aligned} & S_{(\otimes_{k=1}^{2}m^{(k)})(\otimes_{k=1}^{2}\Psi^{(k)})(\otimes_{k=1}^{2}\Phi^{(k)})}(x \otimes y) = \\ & \sum_{(i(1),i(2)) \in I_1 \times I_2} (m_{i(1)} \otimes m_{i(2)}) \\ & (\Gamma_{i(1)} \otimes \Gamma_{i(2)})^{*}(\Lambda_{i(1)} \otimes \Lambda_{i(2)})(x \otimes y) \\ = \ & \left(\sum_{i(1) \in I_1} m_{i(1)}\Gamma_{i(1)}^{*}\Lambda_{i(1)}x\right) \otimes \\ & \left(\sum_{i(2) \in I_2} m_{i(2)}\Gamma_{i(2)}^{*}\Lambda_{i(2)}y\right) \\ = \ & (S_{m^{(1)}\Psi^{(1)}\Phi^{(1)}} \otimes S_{m^{(2)}\Psi^{(2)}\Phi^{(2)}})(x \otimes y), \end{aligned}$$

and since the operators are bounded, we have

$$\begin{aligned} & S_{(m^{(1)} \otimes m^{(2)})(\Psi^{(1)} \otimes \Psi^{(2)})(\Phi^{(1)} \otimes \Phi^{(2)})} \\ = \ & S_{m^{(1)}\Psi^{(1)}\Phi^{(1)}} \otimes S_{m^{(2)}\Psi^{(2)}\Phi^{(2)}}, \end{aligned}$$

and the result follows.

Now we get the following result which is a generalization of [13, Theorem 2.1 (ii)], [13, Corollary 2.6] and [4, Theorem 4.1] to Hilbert C^*-modules:

Corollary 3.1 (i) *If* $\mathcal{W}^{(k)}$ *is a Bessel fusion sequence, for each* $1 \leq k \leq n$*, then* $\otimes_{k=1}^{n}\mathcal{W}^{(k)}$ *is a Bessel fusion sequence. Moreover,* $\mathcal{W}^{(k)}$ *is a fusion frame, for each* $1 \leq k \leq n$ *if and only if* $\otimes_{k=1}^{n}\mathcal{W}^{(k)}$ *is a fusion frame. If* $\mathcal{W}^{(k)}$*'s and* $\mathcal{V}^{(k)}$*'s are Bessel fusion sequences, then the operator* $S_{(\otimes_{k=1}^{n}m^{(k)})(\otimes_{k=1}^{n}\mathcal{W}^{(k)})(\otimes_{k=1}^{n}\mathcal{V}^{(k)})}$ *is well-defined and equals* $\otimes_{k=1}^{n}S_{m^{(k)}\mathcal{W}^{(k)}\mathcal{V}^{(k)}}$*.*

(ii) *If* $\mathcal{F}^{(k)}$ *is a Bessel sequence, for each* $1 \leq k \leq n$*, then* $\otimes_{k=1}^{n}\mathcal{F}^{(k)}$ *is a Bessel sequence. Moreover,* $\mathcal{F}^{(k)}$ *is a frame for each* $1 \leq k \leq n$ *if and only if* $\otimes_{k=1}^{n}\mathcal{F}^{(k)}$ *is a frame for* $\otimes_{k=1}^{n}E_k$*. If* $\mathcal{F}^{(k)}$*'s and* $\mathcal{G}^{(k)}$*'s are Bessel sequences, then* $S_{(\otimes_{k=1}^{n}m^{(k)})(\otimes_{k=1}^{n}\mathcal{F}^{(k)})(\otimes_{k=1}^{n}\mathcal{G}^{(k)})}$ *is well-defined and is equal to* $\otimes_{k=1}^{n}S_{m^{(k)}\mathcal{F}^{(k)}\mathcal{G}^{(k)}}$*.*

Proof. (i) *We can get the result using the above theorem, part* (a) *of Example 3.1 in* [12] *and the fact that* $\Phi^{(k)} = \{\omega_{i(k)}\pi_{W_{i(k)}}\}_{i(k) \in I_k}$ *is a standard g-frame for each* $1 \leq k \leq n$ *if and only if* $\otimes_{k=1}^{n}\Phi^{(k)} = \{(\omega_{i(1)} \otimes \ldots \otimes \omega_{i(n)})\pi_{(W_{i(1)} \otimes \ldots \otimes W_{i(n)})}\}_{(i(1),\ldots,i(n)) \in (I_1 \times \ldots \times I_n)}$ *is a standard g-frame.*
(ii) *The result follows from Theorem* 3.1 *and part* (b) *of Example 3.1 in* [12]*.*

Recall that if $\Lambda = \{\Lambda_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i)\}_{i \in I}$ and $\Gamma = \{\Gamma_i \in \mathfrak{L}_{\mathfrak{A}}(E, E_i)\}_{i \in I}$ are standard $g-$Bessel sequences such that $\sum_{i \in I}\Gamma_i^{*}\Lambda_i x = x$ or equivalently $\sum_{i \in I}\Lambda_i^{*}\Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a *$g-$dual* of Λ (resp. Γ). We define the operator $S_{\Gamma\Lambda}$ on E by $S_{\Gamma\Lambda} = S_{m\Gamma\Lambda}$, where $m = \{m_i\}_{i \in I}$ is a symbol with $m_i = 1_{\mathfrak{A}}$, for each $i \in I$. Then Γ is a $g-$dual of Λ if and only if $S_{\Gamma\Lambda} = Id_E$. The *canonical g-dual* for an (A, B) standard g-frame $\Lambda = \{\Lambda_i\}_{i \in I}$ is defined by $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i \in I}$, where $\tilde{\Lambda}_i = \Lambda_i S_{\Lambda}^{-1}$ which is an $(\frac{1}{B}, \frac{1}{A})$ standard g-frame and for each $x \in E$, we have

$$x = \sum_{i \in I}\Lambda_i^{*}\tilde{\Lambda}_i x = \sum_{i \in I}\tilde{\Lambda}_i^{*}\Lambda_i x.$$

If $\mathcal{F} = \{f_i\}_{i \in I}$ and $\mathcal{G} = \{g_i\}_{i \in I}$ are standard Bessel sequences in E, then we say that $\mathcal{G}$ (resp. $\mathcal{F}$) is a *dual* of $\mathcal{F}$ (resp. $\mathcal{G}$), if $x = \sum_{i \in I}\langle x, f_i\rangle g_i$ or equivalently $x = \sum_{i \in I}\langle x, g_i\rangle f_i$, for each $x \in E$. If $\mathcal{F}$ is an (A, B) standard frame, then $\widetilde{\mathcal{F}} = \{S$
F^-1f i}i∈ I is an $(\frac{1}{B}, \frac{1}{A})$ standard frame with $x = \sum_{i \in I}\langle x, S$
F^-1f i$\rangle f_i = \sum_{i \in I}\langle x, f_i\rangle S$
F^-1f i, *foreachx*∈ E. Hence $\widetilde{\mathcal{F}} = \{S$
F^-1f i}i∈ I is a dual of $\mathcal{F}$ called the *canonical dual* of $\mathcal{F}$.
Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a standard Bessel fusion sequence with upper bound B and $V = \{(V_i, \upsilon_i)\}_{i \in I}$ be a (C, D) standard fusion frame for E. Since $S_V^{-2} \leq \frac{1}{C^2}.Id_E$, by Lemma 4.1 in [16] and the fact that $\upsilon_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$,

we have

$$\begin{aligned}&\langle m_i\upsilon_i S_V^{-1}\pi_{V_i}x, m_i\upsilon_i S_V^{-1}\pi_{V_i}x\rangle\\ =\ & m_i m_i^* \upsilon_i^2\langle S_V^{-2}\pi_{V_i}x,\pi_{V_i}x\rangle\\ \leq\ & \frac{\|m\|_\infty^2}{C^2}.\langle \upsilon_i\pi_{V_i}x,\upsilon_i\pi_{V_i}x\rangle.\end{aligned}$$

Now for each finite subset $\Omega\subseteq I$, using the Cauchy-Schwarz inequality for Hilbert C^*-modules, we obtain that

$$\begin{aligned}&\Big\|\sum_{i\in\Omega} m_i\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x\Big\|\\ =\ & \sup_{\|y\|=1}\Big\|\sum_{i\in\Omega}\langle m_i\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x,y\rangle\Big\|\\ =\ & \sup_{\|y\|=1}\Big\|\sum_{i\in\Omega}\langle m_i\upsilon_i S_V^{-1}\pi_{V_i}x,\omega_i\pi_{W_i}y\rangle\Big\|\\ \leq\ & \Big(\frac{\|m\|_\infty}{C}\Big\|\sum_{i\in\Omega}|\upsilon_i\pi_{V_i}x|^2\Big\|^{\frac{1}{2}}\Big)\times\\ & \Big(\sup_{\|y\|=1}\Big\|\sum_{i\in\Omega}|\omega_i\pi_{W_i}y|^2\Big\|^{\frac{1}{2}}\Big)\\ \leq\ & \frac{\sqrt{B}\|m\|_\infty}{C}\Big\|\sum_{i\in\Omega}\langle\upsilon_i\pi_{V_i}x,\upsilon_i\pi_{V_i}x\rangle\Big\|^{\frac{1}{2}}.\end{aligned}$$

Since V is standard, the series $\sum_{i\in I} m_i\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x$ converges in E and

$$\begin{aligned}&\Big\|\sum_{i\in I} m_i\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x\Big\|\\ \leq\ & \frac{\sqrt{BD}\|m\|_\infty}{C}\|x\|.\end{aligned}$$

Now it is easy to see that the operator $S_{m\mathcal{V}_\mathcal{W}}$ which is defined on E by

$$S_{m\mathcal{V}_\mathcal{W}}x=\sum_{i\in I} m_i\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x,$$

is adjointable.

Proposition 3.1 *Let $\mathcal{W}^{(k)}$ be a Bessel fusion sequence and $\mathcal{V}^{(k)}$ be a fusion frame, for each $1\leq k\leq n$. Then*

$$\begin{aligned}&S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n\mathcal{V}^{(k)})_{(\otimes_{k=1}^n\mathcal{W}^{(k)})}}\\ =\ & \otimes_{k=1}^n S_{m^{(k)}\mathcal{V}^{(k)}_{\mathcal{W}^{(k)}}}.\end{aligned}$$

Proof. *It follows from Corollary 3.1 that $\otimes_{k=1}^n\mathcal{V}^{(k)}$ and $\otimes_{k=1}^n\mathcal{W}^{(k)}$ are standard fusion frame and standard Bessel fusion sequence, respectively. Now it is easy to see that*

$$\begin{aligned}&S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n\mathcal{V}^{(k)})_{(\otimes_{k=1}^n\mathcal{W}^{(k)})}}\\ =\ & \sum_{(i(1),\ldots,i(n))\in(I_1\times\ldots\times I_n)}\Big[(m_{i(1)}\otimes\ldots\\ & \otimes m_{i(n)})(\upsilon_{i(1)}\otimes\ldots\otimes\upsilon_{i(n)})(\omega_{i(1)}\otimes\\ & \ldots\otimes\omega_{i(n)})\pi_{(W_{i(1)}\otimes\ldots\otimes W_{i(n)})}\\ & S^{-1}_{\otimes_{k=1}^n\mathcal{V}^{(k)}}\pi_{(V_{i(1)}\otimes\ldots\otimes V_{i(n)})}\Big]\\ =\ & \Big(\sum_{i(1)\in I_1} m_{i(1)}\upsilon_{i(1)}\omega_{i(1)}\pi_{W_{i(1)}}S^{-1}_{\mathcal{V}^{(1)}}\pi_{V_{i(1)}}\Big)\\ & \otimes\ldots\otimes\\ & \sum_{i(n)\in I_n} m_{i(n)}\upsilon_{i(n)}\omega_{i(n)}\pi_{W_{i(n)}}S^{-1}_{\mathcal{V}^{(n)}}\pi_{V_{i(n)}}\\ =\ & \otimes_{k=1}^n S_{m^{(k)}\mathcal{V}^{(k)}_{\mathcal{W}^{(k)}}}\end{aligned}$$

and the result follows.

Now we have the following definition (see also [9]):

Definition 3.3 *Let $V=\{(V_i,\upsilon_i)\}_{i\in I}$ be a standard fusion frame and $W=\{(W_i,\omega_i)\}_{i\in I}$ be a standard Bessel fusion sequence for E. Then W is called an alternate dual of V if $x=\sum_{i\in I}\upsilon_i\omega_i\pi_{W_i}S_V^{-1}\pi_{V_i}x$, for each $x\in E$.*

The following proposition is a generalization of [13, Corollary 3.8] and [14, Proposition 2.10] to Hilbert C^*-modules and also generalizes the result obtained in [18, Proposition 3.6].

Proposition 3.2 (i) *If $\Psi^{(k)}$ is a g-dual of $\Phi^{(k)}$, for each $1\leq k\leq n$, then $\otimes_{k=1}^n\Psi^{(k)}$ is a g-dual of $\otimes_{k=1}^n\Phi^{(k)}$. If $\otimes_{k=1}^n\Psi^{(k)}$ is a g-dual of $\otimes_{k=1}^n\Phi^{(k)}$ and $\Psi^{(k)}$ is a g-dual of $\Phi^{(k)}$, for each $k\in\{1,\ldots,n-1\}$, then $\Psi^{(n)}$ is also a g-dual of $\Phi^{(n)}$.*

(ii) *If $\mathcal{W}^{(k)}$ is an alternate dual of $\mathcal{V}^{(k)}$, for each $1\leq k\leq n$, then $\otimes_{k=1}^n\mathcal{W}^{(k)}$ is an alternate dual of $\otimes_{k=1}^n\mathcal{V}^{(k)}$.*

(iii) *If $\Phi^{(k)}$'s are g-frames, then $\widetilde{\otimes_{k=1}^n\Phi^{(k)}}=\otimes_{k=1}^n\widetilde{\Phi^{(k)}}$.*

Proof. (i) *Let $m_{i(k)}=1$, for each $1\leq k\leq n$ and $i(k)\in I_k$. Then Theorem 3.1 implies that*

$$\begin{aligned}&S_{(\otimes_{k=1}^n\Psi^{(k)})(\otimes_{k=1}^n\Phi^{(k)})}\\ =\ & S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n\Psi^{(k)})(\otimes_{k=1}^n\Phi^{(k)})}\\ =\ & \otimes_{k=1}^n S_{m^{(k)}\Psi^{(k)}\Phi^{(k)}}=\otimes_{k=1}^n S_{\Psi^{(k)}\Phi^{(k)}}\\ =\ & \otimes_{k=1}^n Id_{E_k}=Id_{\otimes_{k=1}^n E_k}.\end{aligned}$$

This shows that $\otimes_{k=1}^n \Psi^{(k)}$ *is a g-dual of* $\otimes_{k=1}^n \Phi^{(k)}$. *For the rest, we have*

$$\begin{aligned} Id_{\otimes_{k=1}^n E_k} &= S_{(\otimes_{k=1}^n \Psi^{(k)})(\otimes_{k=1}^n \Phi^{(k)})} \\ &= \otimes_{k=1}^n S_{\Psi^{(k)}\Phi^{(k)}} \\ &= (\otimes_{k=1}^{n-1} Id_{E_k}) \otimes S_{\Psi^{(n)}\Phi^{(n)}}, \end{aligned}$$

so $\| Id_{E_n} - S_{\Psi^{(n)}\Phi^{(n)}} \| = \| Id_{E_1} \otimes \ldots \otimes Id_{E_{n-1}} \otimes (Id_{E_n} - S_{\Psi^{(n)}\Phi^{(n)}}) \| = 0$, *and this yields that* $S_{\Psi^{(n)}\Phi^{(n)}} = Id_{E_n}$.
(ii) *Let* $m_{i(k)} = 1_{\mathfrak{A}_k}$, *for each* $1 \leq k \leq n$. *Then Proposition 3.1 implies that*

$$\begin{aligned} & S_{(\otimes_{k=1}^n m^{(k)})(\otimes_{k=1}^n \mathcal{V}^{(k)})_{(\otimes_{k=1}^n \mathcal{W}^{(k)})}} \\ = \; & \otimes_{k=1}^n S_{m^{(k)}\mathcal{V}^{(k)}_{\mathcal{W}^{(k)}}} = \otimes_{k=1}^n Id_{E_k}, \end{aligned}$$

and the result follows.
(iii) *By Theorem 3.1,* $\otimes_{k=1}^n \Phi^{(k)}$ *is a g-frame and by considering* $m_{i(k)} = 1$, *for each* $1 \leq k \leq n$ *and* $i(k) \in I_k$, *similar to part* (i), *we get* $S_{(\otimes_{k=1}^n \Phi^{(k)})} = \otimes_{k=1}^n S_{\Phi^{(k)}}$. *This implies that* $S^{-1}_{\otimes_{k=1}^n \Phi^{(k)}} = \otimes_{k=1}^n S^{-1}_{\Phi^{(k)}}$, *then for each* $(i(1), \ldots, i(n)) \in I_1 \times \ldots \times I_n$, *we have*

$$\begin{aligned} & (\Lambda_{i(1)} \otimes \ldots \otimes \Lambda_{i(n)}) S^{-1}_{\otimes_{k=1}^n \Phi^{(k)}} \\ = \; & (\Lambda_{i(1)} S^{-1}_{\Phi^{(1)}}) \otimes \ldots \otimes (\Lambda_{i(n)} S^{-1}_{\Phi^{(n)}}). \end{aligned}$$

This shows that $\widetilde{\otimes_{k=1}^n \Phi^{(k)}} = \otimes_{k=1}^n \widetilde{\Phi^{(k)}}$.

Now we obtain the following result which is a generalization of Corollary 2.11 in [14] to Hilbert C^*-modules:

Corollary 3.2 (i) *Let* $\mathcal{F}^{(k)}$ *and* $\mathcal{G}^{(k)}$ *be Bessel sequences for* E. *If* $\mathcal{G}^{(k)}$ *is a dual of* $\mathcal{F}^{(k)}$, *for each* $1 \leq k \leq n$, *then* $\otimes_{k=1}^n \mathcal{G}^{(k)}$ *is a dual of* $\otimes_{k=1}^n \mathcal{F}^{(k)}$. *If* $\otimes_{k=1}^n \mathcal{G}^{(k)}$ *is a dual of* $\otimes_{k=1}^n \mathcal{F}^{(k)}$ *and* $\mathcal{G}^{(k)}$ *is a dual of* $\mathcal{F}^{(k)}$, *for each* $1 \leq k \leq n-1$, *then* $\mathcal{G}^{(n)}$ *is a dual of* $\mathcal{F}^{(n)}$.

(ii) *If* $\mathcal{F}^{(k)}$ *is a frame, for each* $1 \leq k \leq n$, *then* $\widetilde{\otimes_{k=1}^n \mathcal{F}^{(k)}} = \otimes_{k=1}^n \widetilde{\mathcal{F}^{(k)}}$.

Proof. *The result follows from the above proposition and Corollary 3.1 by considering* $\Phi^{(k)} = \{\Lambda_{i(k)}\}_{i(k) \in I_k}$ *and* $\Psi^{(k)} = \{\Gamma_{i(k)}\}_{i(k) \in I_k}$, *where* $\Lambda_{i(k)} x = \langle x, f_{i(k)} \rangle$ *and* $\Gamma_{i(k)} x = \langle x, g_{i(k)} \rangle$, *for each* $x \in E$.

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