Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 7, No. 4, 2015 Article ID IJIM-00649, 7 pages
Research Article

Science and Research Branch (IAU)

Implementation of Sinc-Galerkin on Parabolic Inverse problem with unknown boundary condition

J. Biazar [*†], T. Houlari [‡]

Abstract

The determination of an unknown boundary condition, in a nonlinaer inverse diffusion problem is considered. For solving these ill-posed inverse problems, Galerkin method based on Sinc basis functions for space and time will be used. To solve the system of linear equation, a noise is imposed and Tikhonove regularization is applied. By using a sensor located at a point in the domain of x, say $x = a'$, and determining $u(a', t)$ a stable solution will be achived. An illustrative example is provided to show the ability and the efficiency of this numerical approach.

Keywords : Ill-posed inverse problems; Sinc-Galerkin method; Tikhonov regularization; Unkown boundary condition.

1 Introduction

The parameter determination in a parabolic partial differential equation from the over-specified data plays an important role in applied mathematics, physics and engineering. These problems are widely encountered in the modelling of physical phenomena [5, 2, 3, 4]. In this paper we shall consider an inverse problem of finding an unknown boundary condition, $u(0, t)$, in a parabolic partial differential equation. The main problem is finding the temperature distribution, $u(x, t)$, as well as the boundary condition, $u(0, t)$, simultaneously. Let's consider the following parabolic PDE

$$pu \equiv u_t(x,t) - \kappa(t)u_{xx}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t < \infty, \qquad (1.1)$$

with the following initial and boundary conditions

$$u(x,0) = g(x), \qquad 0 \le x \le 1, \qquad (1.2)$$

$$u(0,t) = \gamma(t), \qquad 0 \le t \le \infty, \qquad (1.3)$$

$$u(1,t) = \delta(t), \qquad 0 \le t \le \infty, \qquad (1.4)$$

subject to an overspecified condition

$$u(a',t) = s(t), \qquad 0 \le t \le \infty, \qquad (1.5)$$

where g, δ, and s are known continuous or piecewise continuous functions in their domains and pu stants for an equation for determining u. These functions also satisfy the conditions $g(a') = s(0)$ and $g(1) = \delta(0)$. While the functions $u(x, t)$ and $u(0, t)$ are unknown. By employing condition 1.5, a numerical algorithm is presented for solving this inverse problem, based on the fully Sinc-Galerkin

*Corresponding author. biazar@guilan.ac.ir
†Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-1914, Guilan, Rasht, Iran.
‡Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-1914, Guilan, Rasht, Iran.

method. The Sinc-Galerkin method was first presented by Stenger in [15]. This method has been applied to a variety of partial differential equations [15, 16, 12, 11, 1, 9, 14]. References [16] and [12] provide excellent overviews of existing methods based on Sinc functions. In a Fully Sinc-Galerkin technique a Sinc function basis is used, in both space and time. This method has an exponential order of convergence [16, 12].

For the sake of simplicity by the following transformation the problem (1.1) will be change to the homogeneous overspecified and boundary conditions,

$$u(x,t) = v(x,t) + \varphi(x,t),$$

where

$$\begin{aligned}\varphi(x,t) &= s(t)[\frac{x-1}{a'-1}] + \delta(t)[\frac{a'-x}{a'-1}] \\ &+ \theta(t)g(a')[\frac{x-1}{a'-1}] \\ &- \theta(t)g(1)[\frac{a'-x}{a'-1}] + \theta(t)g(x),\end{aligned} \tag{1.6}$$

where the differentiable function $\theta(t)$ satisfies $\theta(0) = 1$, and $\theta'(0) = 1$.

In particular,

$$\theta(t) = \frac{t+1}{t^2+1}.$$

This transformation leads to the following equation with homogeneous overspecified and boundary conditions.

$$\begin{aligned}pv &\equiv v_t(x,t) - \kappa(t)v_{xx}(x,t) = f^*, \\ & \quad 0 < x < 1, \quad 0 < t < \infty, \qquad (1.7)\\ v(x,0) &= 0, 0 \le x \le 1, \qquad (1.8)\\ v(1,t) &= 0, v(a',t) = 0, 0 \le t \le \infty, \qquad (1.9)\end{aligned}$$

where

$$f^* = -[(\varphi_t(x,t)) - \kappa(t)(\varphi_{xx}(x,t))]. \tag{1.10}$$

This paper is orgonized as follows; In Section 2, an inverse problem will be considered. Some properties of Sinc function and Sinc quadrature rule will be introduced and Sinc-Galerkin method will be implemented for solving introduced inverse problem. To show the efficiency of the proposed method a numerical illustrative example is provided in Section 3. Section 4 is droted to a brief conclusion.

2 Inverse Problem

The Sinc-Galerkin method is applied to solve an inverse problem. For applying this method, one should be familiar with the Sinc function, Sinc quadrature rules and their properties.

The Sinc function is defined on $\mathbb{R}$ by

$$Sinc(x) \equiv \begin{cases} \frac{sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

To have the Sinc transform functions, for both space and time nodes, let's consider $h_x > 0$, $h_t > 0$, and define

$$\begin{aligned}S(j,h_x)(x) &\equiv Sinc(\frac{x - jh_x}{h_x}), \\ j &= 0, \pm1, \pm2, \ldots. \\ S(j,h_t)(t) &\equiv Sinc(\frac{t - jh_t}{h_t}), \\ j &= 0, \pm1, \pm2, \ldots.\end{aligned}$$

To construct approximations by using the Sinc function on the intervals $(a', 1)$ and $(0, \infty)$, we consider the conformal mappings

$$\Phi(x) = ln(\frac{x-a'}{1-x}),$$

and

$$\Upsilon(t) = ln(t).$$

Thus the appropriate Sinc functions over $(a', 1)$ and $(0, \infty)$ are given by

$$S_i(x) = S(i,h_x) \circ \Phi(x) \equiv sinc(\frac{\Phi(x) - ih_x}{h_x}), \tag{2.11}$$

and

$$S'_j(t) = S'(j,h) \circ \Upsilon(t) \equiv sinc(\frac{\Upsilon(t) - jh_t}{h_t}). \tag{2.12}$$

respectivly.

For solving Eq.(1.7) with conditions (1.8) and (1.9), the Sinc basis functions (2.11), and (2.12) are used. Let's consider an approximate solution as the following

$$v(x,t) = \sum_{j=-M_t}^{N_t} \sum_{i=-M_x}^{N_x} c_{ij} S_i(x) S'_j(t), \tag{2.13}$$

where M_x, M_t, N_x, and N_t are positive integers, and $m_x = M_x + N_x + 1, m_t = M_t + N_t + 1$. c_{ij} are unknown constant that will be determined from residual and Galerkin approach,

$$< Pu, S_k S_l' >= 0,$$

for $-M_x \leq k \leq N_x$, $-M_t \leq l \leq N_t$, may be written

$$< Pv - f^*, S_k S_l' >= 0, \tag{2.14}$$

where f^* is given by (1.10) and the inner product is defined by

$$< \eta, \zeta >= \int_0^\infty \int_a^1 \eta(x,t)\zeta(x,t)\nu(x)\omega(t)dxdt, \tag{2.15}$$

$\nu(x)\omega(t)$ is a weight function. The Sinc-Galerkin method actually requires the evaluated derivatives of sinc basis functions, $S(i,h) \circ \Phi(x)$, at the sinc nodes, $x = x_k$. The rth derivative of $S(i,h) \circ \Phi(x)$, with respect to Φ, evaluated at the nodal point x_k is denoted by

$$\frac{1}{h^n}\delta_{ik}^{(n)} \equiv \frac{d^n}{d\Phi^n}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} . \tag{2.16}$$

Theorem 2.1 *Let Φ be a conformal one-to-one map of the simply connected domain D_E onto D_s then*

$$\delta_{ik}^{(0)} = [S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} 1, & k = i, \\ 0, & k \neq i, \end{cases} \tag{2.17}$$

$$\delta_{ik}^{(1)} = h\frac{d}{d\Phi}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} 0, & k = i, \\ \frac{(-1)^{(k-i)}}{(k-i)}, & k \neq i, \end{cases} \tag{2.18}$$

and

$$\delta_{ik}^{(2)} = h^2\frac{d^2}{d\Phi^2}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & k = i, \\ \frac{-2(-1)^{(k-i)}}{(k-i)^2}, & k \neq i. \end{cases} \tag{2.19}$$

proof. See [12].

Now, suppose that the weight function in the inner product (2.15) be as

$$\omega(t)\nu(x) = \sqrt{\frac{\Upsilon'}{\Phi'}}.$$

A complete discussion on the choice of the weight function can be find in [12]. Applying the Sinc quadrature rule for double integrals is addressed, by Koonprasert and Bowers, in [9]. Substitution of (2.13) in to the (2.14), applying Sinc quadrature rule for double integrals, and replacing $v(x_i, t_j)$ by v_{ij} leads to the following discrete system

$$\begin{aligned}
&(-\sum_{q=-M_t}^{N_t} v_{iq}[\frac{1}{h_t}\delta_{jq}^{(1)}][\frac{\omega(t_q)\nu(x_i)}{\Phi'(x_i)\Upsilon'(t_q)}]) \\
&-(\sum_{p=-M_x}^{N_x} v_{pj}\frac{\kappa(t_j)\omega(t_j)}{\Phi'(x_p)\Upsilon'(t_j)}[(\Phi'^2\nu)(x_p) \\
&(\frac{1}{h_x^2}\delta_{ip}^{(2)}) + [\Phi''\nu + 2\Phi'\nu'](x_p)(\frac{1}{h_x}\delta_{ip}^{(1)})]) \\
&-v_{ij}\frac{\kappa(t_j)\omega(t_j)\nu''(x_i)}{\Phi'(x_i)\Upsilon'(t_j)} - v_{ij}\frac{\omega(t_j)\nu(x_i)}{\Phi'(x_i)\Upsilon'(t_j)} \\
&= \frac{f^*(x_i,t_j)\omega(t_j)\nu(x_i)}{\Phi'(x_i)\Upsilon'(t_j)},
\end{aligned} \tag{2.20}$$

for $i = -M_x, ..., N_x$ and $j = -M_t, ..., N_t$.

Drivatives in (2.16) can be stored in matrices: for x variable

$$I_{m_x \times m_x}^{(n)} = [\delta_{ip}^{(n)}],$$

for t variable

$$I_{m_t \times m_t}^{(n)} = [\delta_{jq}^{(n)}],$$

where $n = 0, 1, 2$.

If function g is evaluated at the sinc nodes $x = x_k$ for $-M_x \leq i \leq N_x$ then the $m_x \times m_x$ square diagonal matrix $D_{m_x \times m_x}(g)$ is written as

$$D_{m_x \times m_x}(g) = \begin{pmatrix} g(x_{-M_x}) & & & & \\ & \ddots & & & \\ & & g(x_0) & & \\ & & & \ddots & \\ & & & & g(x_{N_x}) \end{pmatrix}.$$

By this notation, the system (2.20) turnes to the following matrix form

$$\begin{aligned}
& D(\frac{-\nu}{\Phi'}).X.[\frac{1}{h_t}I_{m_t}^{(1)}D(\frac{\omega'}{\Upsilon'})]^t \\
+ \ & D(\frac{-\nu}{\Phi'}).X.D(\frac{\omega'}{\Upsilon'}) + [\frac{-1}{h_x^2}I_{m_x}^{(2)}D(\Phi'\nu) \\
+ \ & \frac{-1}{h_x}I_{m_x}^{(1)}D(\frac{\Phi''\nu}{\Phi'} + 2\nu')].X.D(\frac{\omega\kappa}{\Upsilon'}) \\
+ \ & D(\frac{-\nu''}{\Phi'}).X.D(\frac{\omega\kappa}{\Upsilon'}) = D(\frac{\nu}{\Phi'}).F.D(\frac{\omega}{\Upsilon'})
\end{aligned} \tag{2.21}$$

Table 1: The errors $\| E_s(h) \|$ for different orders, h, a' and $t_f = 1s$.

M_x	h	$a' = 0.04$	$a' = 0.1$	$a' = 0.2$
2	3.1415	1.1724×10^{-4}	2.9594×10^{-4}	6.0153×10^{-4}
4	2.2214	2.4003×10^{-4}	6.0589×10^{-4}	1.2315×10^{-3}
8	1.5707	2.7054×10^{-4}	6.8290×10^{-4}	1.3880×10^{-3}
16	1.1607	2.4003×10^{-4}	6.0589×10^{-4}	1.2315×10^{-4}

Table 2: The absolute errors of $\gamma(t)$, by 0th order Tikhonov for $M_x = 4$ and $t_f = 1s$.

t	$a' = 0.04$	$a' = 0.1$	$a' = 0.2$
0.1	2.2977×10^{-4}	5.7670×10^{-4}	1.1705×10^{-3}
0.2	2.6560×10^{-4}	6.9318×10^{-4}	1.4202×10^{-3}
0.3	8.7696×10^{-5}	2.5192×10^{-4}	5.2672×10^{-4}
0.4	3.2437×10^{-4}	7.8897×10^{-4}	1.5884×10^{-3}
0.5	9.6085×10^{-4}	2.3992×10^{-3}	4.8634×10^{-3}
0.6	1.7857×10^{-3}	4.4864×10^{-3}	9.1085×10^{-3}
0.7	2.7452×10^{-3}	6.9138×10^{-3}	1.4045×10^{-3}
0.8	3.7792×10^{-3}	9.5294×10^{-3}	1.9364×10^{-2}
0.9	48329×10^{-3}	1.2191×10^{-2}	2.4783×10^{-2}

where X is the $m_x m_t$ matrix of unknown coefficients c_{ij}. The ijth-entry of $F_{m_x m_t}$ is equal to $F(x_i, t_j)$, where $-M_x \leqslant i \leqslant N_x$ and $-M_t \leqslant i \leqslant N_t$. The system (2.21) can simplify as the following

$$A_1 X B_1 + A_2 X B_2 + A_3 X B_3 + A_4 X B_4 = C \quad (2.22)$$

where

$$\begin{aligned} A_1 &= A_2 = D(\frac{-\nu}{\Phi'}), \\ B_1 &= [\frac{1}{h_t} I_{m_t}^{(1)} D(\frac{\omega'}{\Upsilon'})]^t, \\ B_2 &= D(\frac{\omega'}{\Upsilon'}), \\ A_3 &= [\frac{-1}{h_x^2} I_{m_x}^{(2)} D(\Phi'\nu) \\ &+ \frac{-1}{h_x} I_{m_x}^{(1)} D(\frac{\Phi''\nu}{\Phi'} + 2\nu')], \\ B_3 &= B_4 = D(\frac{\omega\kappa}{\Upsilon'}), \\ A_4 &= D(\frac{-\nu''}{\Phi'}), \end{aligned}$$

and

$$C = D(\frac{\nu}{\Phi'}).F.D(\frac{\omega}{\Upsilon'}).$$

By using Kronecker sum notation and the concatenation on matrices, the system (2.22) can be written as followes, [12],

$$\varpi co(X) = co(C), \quad (2.23)$$

where ϖ is a matrix, involving Kronecker products, with the dimontion $(m_x m_t) \times (m_x m_t)$ that can be denoted as the following

$$\varpi = B_1^T \otimes A_1 + B_2^T \otimes A_2 + B_3^T \otimes A_3 + B_4^T \otimes A_4,$$

and $co(X)$ and $co(C)$ are vectors with $(m_x m_t)$ entries. Having these simplifications and notations done sinc coefficients c_{ij}. So, it will be determined from the system (2.23)

$$\varpi W = Y. \quad (2.24)$$

where

$$W = co(X), \quad and \quad Y = co(C).$$

The system (2.24), as an ill-conditioned one, is solved by Tikhonov regularization a specific package which is in matlab. ([6], [7] and [19]).

3 Numerical Result

For using this approach to solve a test problem with an unknown boundary condition in the inverse problem (1.1), some notations and relations need.

For choosing an appropriate sinc grid in space and time, we suppose that a exact solution satisfies the condition

$$|u(x,t)| \leq C x^{\alpha_s + \frac{1}{2}} (1-x)^{\beta_s + \frac{1}{2}} t^{\sigma_s + \frac{1}{2}} e^{-\varsigma t}, \quad (3.25)$$

for$(x,t) \in (a,1) \times (0,1)$, the following selections should be considered

$$N_x = [|\frac{\alpha_s}{\beta_s} M_x + 1|], \ M_t = [|\frac{\alpha_s}{\sigma_s} M_x + 1|],$$
$$N_t = [|\frac{\alpha_s}{\varsigma_s} M_x + 1|], \qquad (3.26)$$

where $[|.|]$ denotes the greatest integer operation, $h \equiv h_x = h_t$ and

$$h = (\frac{\pi d}{\alpha_s M_x})^{\frac{1}{2}}. \qquad (3.27)$$

In addition, $\parallel E_s(h) \parallel$ is defined, for reporting error on the Sinc grid points (x_i, t_j) , as the following

$$\parallel E_s(h) \parallel = max_{i,j}\{|u(x_i,t_j) - u_{m_x,m_t}(x_i,t_j)|: x_i = \frac{a' + e^{ih}}{1+e^{ih}}, t_j = e^{jh}\}.$$

Example 3.1 *Let's consider the following problem, which is a known equation and has been considered in some refrences for different proposes, for example in refrences [5, 3] inverse problem is considered for $\kappa(t)$ as an unknown function. here, we solve it for unknown boundary condition.*

$$pu \equiv u_t(x,t) - \kappa(t) u_{xx}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t < \infty,$$

where

$$\kappa(x,t) = \frac{2[6t^2 + (1+t^3)^2 cos(\frac{t}{2})]}{(1+t^3)[1+2t^3+(1+t^3)sin(\frac{t}{2})]},$$

with the following initial and boundary condition

$$u(x,0) = e^{(\frac{x}{2})}, \ 0 \le x \le 1,$$
$$u(1,t) = \frac{\sqrt{e}(1+2t^3)}{(1+t^3)} + \sqrt{e} sin(\frac{t}{2}), \quad 0 \le t \le t_f,$$

with the exact solution

$$u(x,t) = \frac{e^{(\frac{x}{2})}(1+2t^3)}{1+t^3} + e^{(\frac{x}{2})} sin(\frac{t}{2}).$$

The errors are presented at $u(0,t)$ for $\alpha_s = \beta_s = \sigma_s = \frac{1}{2}$, $\varsigma = 1$, $d = \frac{\pi}{2}$ and noisy data, (noisy data=input data+(0.001) rand (1)), for different orders, M_x, and different step lenghts,h, and different sensor locations, a' are presented in Tables 1. In Table 2 the same errores are appeared at different points and the same sensor locations, but fix, $M_x = 4$ and $t_f = 1s$, and exact solution and the results of Table 2 for $a' = 0.04$, $a' = 0.1$, $a' = 0.2$ are plotted in Figure 1.

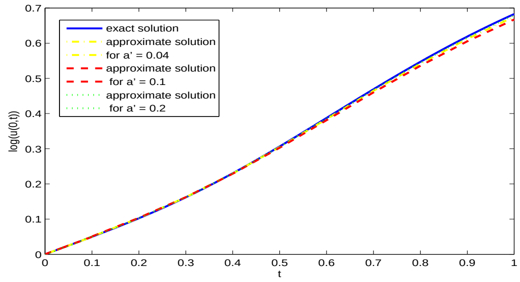


Figure 1: The comparison between the exact solutions and approximation solutions in sensor's different location.

4 Conclusion

Inverse problem has been used for parabolic partial differential equation, with unknown boundary conditions, succesfully. In this study, due to unknown boundary condition a sensor is imposed as an extra condition. The results achived in this study confirms exprimental fact i.e. when the location of the sensor is not closed to boundary position, as is in the reality, the errores still small enough to count on the method as a powerfull devise for solving inverse problems. Regarding the fact that " the closer sensor location to the boundary the more accurate results " we have considered locations 0.04, 0.1, and 0.2, for the sensor even at $a' = 0.2$, which is not closed to the boundary, the error are still small which confirms the efficiency and stability of the method.

References

[1] K. L. Bowers, T. S. Carlson, J. Lund, *Advection-diffusion equations: temporal sinc methods*, Numer Meth Partial Diff Eq. 11 (1995) 399-422.

[2] M. Dehghan, *An inverse problem of finding a source parameter in a semilinear parabolic equation*, Appl. Math. Model. 25 (2001) 743-754.

[3] M. Dehghan, *Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement*, Numer. Methods Partial Differential Equations 21 (2005) 611-622.

[4] M. Dehghan, *Parameter determination in a partial differential equation from the overspecified data*, Math. Comput. Modelling 41 (2005) 197-213.

[5] M. Dehghan, M. Lakestani, *The use of Chebyshev cardinal functions for the solution of a partial differential equation with an unknown time-dependent coefficient subject to an extra measurement*, Journal of Computational and Applied Mathematics 235 (2010) 669-678

[6] L. Elden, *A Note on the Computation of the Generalized Cross-validation Function for Ill-conditioned Least Squares Problems*, BIT 24 (1984) 467-472.

[7] G. H. Golub, M. Heath, G.Wahba, *Generalized Cross-validation as a Method for Choosing a Good Ridge Parameter*, Technometrics 21 (1979) 215-223.

[8] P.C. Hansen, *Analysis of discrete ill-posed problems by means of the L-curve*, SIAM Rev 34 (1992) 561-80.

[9] S. Koonprasert and K. L. Bowers, *The fully Sinc-Galerkin method for time-dependent boundary conditions*, Numer Meth Partial Diff Eq. 20 (2004) 494-526.

[10] C. L. Lawson, R. J. Hanson, *Solving Least Squares Problems, Philadelphia, PA: SIAM*, (1995).

[11] D. L. Lewis, J. Lund, K. L. Bowers, *The space-time Sinc-Galerkin method for parabolic problems*, Int J Numer Methods Engrg 24 (1987) 1629-1644.

[12] J. Lund and K. L. Bowers, *Sinc methods for quadrature and differential equations, SIAM*, Philadelphia, (1992).

[13] L. Martin, L. Elliott, P. J. Heggs, D. B. Ingham, D. Lesnic, Wen X., *Dual Reciprocity Boundary Element Method Solution of the Cauchy Problem for Helmholtz-type Equations with Variable Coecients*, Journal of sound and vibration 297 (2006) 89-105.

[14] A. Shidfar and R. Zolfaghari, *Determination of an unknown function in a parabolic inverse problem by Sinc-collocation method*, Numer Meth Partial Diff Eq. 27 (2011) 1584-1598.

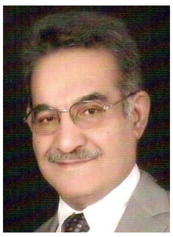
[15] F. Stenger, *A Sinc-Galerkin method of solution of boundary value problems*, Math Comp. 33 (1979) 85-109.

[16] F. Stenger, *Numerical methods based on sinc and analytic functions, Springer-Verlag*, New York (1993).

[17] A. N. Tikhonov, V. Y. Arsenin, *On the solution of ill-posed problems,Wiley*, New York, (1977).

[18] A. N. Tikhonov, V. Y. Arsenin, *Solution of Ill-Posed Problems, V. H. Winston and Sons*, Washington, DC. (1977).

[19] G. Wahba, *Spline Models for Observational Data*, CBMS-NSF Regional Conference Series in Applied Mathematics 59 (1990) SIAM, Philadelphia.



Jafar .Biazar is full professor of Applied Mathematics at University of Guilan, Rasht, Iran. His main research field is solving linear and non-linear functional equations, by new semi numerical approaches, such as ADM, HPM, HAM, VIM, FIM, EFM, and others, taking into consideration theoretical points, stability, consistency, and convergence. He has been conducting research on solving non-linear PDEs, under AIF-ACOA (PPSC) grant, with Oil and Gas research group, at Dalhousie University, Halifax, Canada, during summers for five years (2001-2005). Dr. Biazar has published more than 160, papers in ISI, ISC, and international indexed Journals. Professor Biazar is Editor in chief of Iranian Journal of Optimization (IJO), and in the list of Editorial board of eight, peered reviewed, indexed international journals, and reviewer of many ISI and ISC journals. He is recognized as among world's top 1

Tahereh Houlari received her B.Sc. degree in Applied Mathematics (2009) and her M.Sc. degree in Applied Mathematics (2012), from the Damghan University, Iran. She is a second-year Ph.D. student at University of Guilan, Iran. She is mainly interested in working numerical solution inverse problems and fuzzy logic.