Well-dispersed subsets of non-dominated solutions for MOMILP problem

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Abstract

This paper uses the weighted $L_1$ norm to propose an algorithm for finding a well-dispersed subset of non-dominated solutions of multiple objective mixed integer linear programming problem. When all variables are integer it finds the whole set of efficient solutions. In each iteration of the proposed method only a mixed integer linear programming problem is solved and its optimal solutions generates the elements of the well-dispersed subset non-dominated solutions (WDSNDSs) of MOMILP. According to the distance of non-dominated solutions from the ideal point the elements of the WDSNDSs are ranked, hence it does not need the filtering procedures. Using suitable values for the parameter of the proposed model an appropriate WDSNDSs by less computational efforts can be generated. Two numerical examples present to illustrate the applicability of the proposed method and compare it with earlier work.

Keywords: Multi-Objective Mixed Integer Linear Programming; Efficient solutions; Well-dispersed subset non-dominated solutions; $L_1$ norm.

1 Introduction

Multiple Objective Mixed Integer Linear Programming (MOMILP) problems occur frequently in many applications. Many engineering, operations, and scientific applications include a mixture of discrete and continuous decision variables and linear relationship involving the decision variables that have a pronounced effect on the set of feasible and optimal solutions in Multi Criteria Decision Making (MCDM).

In recent decades, Numerous algorithms also interactive procedures have been designed to solve Multiple Objective Linear Programming (MOLP) [2, 3, 4, 9]. MOMILP and Multiple Objective Integer Linear Programming (MOILP) are important research areas as many practical situations discrete representations have to deal with several objectives [1, 8]. Surveys considering most of methods for generating non-dominated vectors are also available [12, 14].

may not be satisfactory for problems with a large number of objective function. Sylva and Crema [11] propose a method for finding a well-dispersed subset of non-dominated solutions based on maximizing the infinity norm distance from a set of known solutions. They claim that their approach originally provides a variant of the procedure by Sylva and Crema [10]. The major drawback of this approach is the difficulty of solving the constrained problems due to increasing number of constraints and binary variables.

This paper proposes a method to find a WDSNDSs by using the weighted $L_1$-norm. When all variables are integer it finds the whole set of efficient solutions. In each iteration of the proposed algorithm only one mixed integer linear programming problem is solved, while Sylva and Crema’s method [11] needs to solve two problems in each iteration which the optimal solution of one of them necessary not be efficient. The proposed algorithm ranks the elements of the WDSNDSs, hence we do not need the filtering procedures and using suitable values for the parameter of the proposed model we can obtain an appropriate WDSNDSs by less computational efforts. It modifies the dispersal of the WDSNDSs according to the decision maker opinions.

The paper is organized as follows. Section 2 presents a brief background about MOMILP problem. Section 3 introduces some models and an algorithm to generate a WDSNDSs of an MOMILP problem. Illustration with two numerical examples are given in Section 4. Finally, the concluding results are presented.

## 2 MOMILP problem

The MOMILP with $s$-objective functions can be defined as follows:

$$\begin{align*}
\max \ & \{C_1W, \ldots, C_sW\} \\
\text{s.t.} \ & A_iW \leq b_i, \ i = 1, \ldots, m \\
& W \geq 0, w_j \in Z^+, j \in J
\end{align*}$$

(2.1)

where $C_r = (c_{r1}, \ldots, c_{rm})$ $(r = 1, \ldots, s)$, $A_i = (a_{i1}, \ldots, a_{im})$ $(i = 1, 2, \ldots, m)$, $J \subseteq \{1, \ldots, n\}$, $Z^+ = \{0, 1, 2, \ldots\}$ and $W = (w_1, \ldots, w_n)^T$. The set of feasible solutions of problem (2.1) is defined by $X = \{W \mid A_iW \leq b_i, i = 1, \ldots, m, W \geq 0, w_j \in Z^+, j \in J \}$ and is assumed to be a non-empty set. The objective vector $Z = (z_1, \ldots, z_s)^T = (C_1W, \ldots, C_sW)^T$ for $W \in X$ is said to be non-dominated vector if and only if there is no $Z^o = (z^o_1, \ldots, z^o_s)^T = (C_1W^o, \ldots, C_sW^o)^T$ for $W^o \in X$ such that $z_r \geq z^o_r$ for all $r \in \{1, \ldots, s\}$ and $z_r > z^o_r$ for at least one $r$. The set of $F = \{Z \mid Z = (C_1W, \ldots, C_sW)^T, W \in X \}$ is called the values space of objective functions in problem (2.1). Let $g_r = C_r W^*(r = 1, \ldots, s)$, where $W^*_r$ is the optimal solution of the following single objective mixed integer programming problem:

$$\begin{align*}
g_r = \max \ & C_r W \\
\text{s.t.} \ & W \in X.
\end{align*}$$

(2.2)

Let us consider $X$ be bounded and $g = (g_1, \ldots, g_s)^T = (C_1 W_1^*, \ldots, C_s W_s^*)^T$ is referred to as the ideal vector of model (2.1) [5]. As can be seen, for each $W \in X$ as a feasible solution of problem (2.1), the vector $g$ dominates the vector $Z = (C_1W_1, \ldots, C_sW_s)^T \neq g$.

## 3 Well-dispersed subsets of efficient solutions

Suppose our aim is to find a subset of efficient solutions with a desired dispersal and $\lambda \in \Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_s)^T \mid \lambda_r > 0, r = 1, \ldots, s\}$ is known, where $\lambda$ is decision maker preferences about objective functions. To obtain a member of the WDSNDSs for problem (2.1), say $CW$, we specify $W \in X$ such that $g - Z = (g_1 - C_1W, \ldots, g_s - C_sW)^T$ is minimized. To this purpose, the following MOMILP problem can be solved:

$$\begin{align*}
\min \ & \{g_1 - C_1W, \ldots, g_s - C_sW\} \\
\text{s.t.} \ & W \in X.
\end{align*}$$

(3.3)

To find the efficient solutions of model (3.3) by using the weighted $L_1$-norm, i.e., $d_\lambda(g, CW)$, and according to $g_r \geq C_r W (r = 1, \ldots, s, W \in X)$ we have:

$$\begin{align*}
\min_{W \in X} d_\lambda(g, CW) &= \min_{W \in X} \sum_{r=1}^s \lambda_r |g_r - C_r W| \\
&= \min_{W \in X} \sum_{r=1}^s \lambda_r (g_r - C_r W) \\
&= \sum_{r=1}^s \lambda_r g_r + \min_{W \in X} \sum_{r=1}^s \lambda_r (-C_r W) \\
&= \sum_{r=1}^s \lambda_r g_r - \max_{W \in X} \sum_{j=1}^n \sum_{r=1}^s \lambda_r c_{rj} w_j.
\end{align*}$$
Using the above relations, model (3.3) is converted to the following mixed integer linear programming problem.

\[
\max \sum_{r=1}^{s} \lambda_r C_r W \quad (3.4)
\]

\[
\text{s.t.} \quad W \in X.
\]

Let problem (3.4) is feasible and \(W^*\) be its optimal solution.

**Theorem 3.1** The optimal solutions of problem (3.4) are efficient solutions of model (2.1).

**Proof.** The proof is similar to that of Theorem 2.3 in [5] and is not repeated here. □

Model (3.4) is for finding some efficient solutions of the MOMILP problem, another member of the WDSNDSs for problem (2.1), say \(\overline{CW}\), is determined such that

1. the distance of the \(g\) and \(\overline{CW}\), i.e. \(d_\lambda(g, \overline{CW})\), is minimized and
2. there exists \(r \in \{1, \ldots, s\}\) such that \(|\overline{CW} - CW^*| \geq \varepsilon\).

To ward this end, some constraints and variables are added to problem (3.4) and the obtained model is solved. This process is continued and a sequence of mixed integer programming problem is attained. Let \(W_{h-1}^w\) be the optimal solution of the model of the \((h-1)\)th iteration, i.e. the model \(M_{h-1}\). Then, by adding the following constraints to the model \(M_{h-1}\), the model of the \(h\)th iteration \(M_h\) is determined.

\[
C_r W \geq C_r W_{h-1}^w + \alpha - M t_{rh}, \quad r = 1, \ldots, s
\]

\[
\sum_{r=1}^{s} t_{rh} \leq s - 1
\]

\[
\alpha \geq \varepsilon
\]

\[
t_{rh} \in \{0, 1\}, \quad r = 1, \ldots, s
\]

where \(M\) is a sufficiently large positive value and \(M = \max_{1 \leq \ell \leq s} |g_r|\) can be used as its lower bound. When \(t_{rh} = 1\), the constraint \(C_r W_h \geq C_r W_{h-1}^w + \alpha - M t_{rh}\) is redundant and the constraint \(\sum_{r=1}^{s} t_{rh} \leq s - 1\) imply that there exists \(l \in \{1, \ldots, s\}\) such that \(t_{rh} = 0\).

Using the above discussion the following model is considered:

\[
M_{h+1} : \quad \max \sum_{r=1}^{s} \lambda_r C_r W
\]

\[
\text{s.t.} \quad W \in X
\]

\[
C_r W \geq C_r W_p^* + \alpha - M t_{rp}, \quad r = 1, \ldots, s, \quad p = 1, \ldots, h
\]

\[
\sum_{r=1}^{s} t_{rp} \leq s - 1,
\]

\[
p = 1, \ldots, h
\]

\[
\alpha \geq \varepsilon
\]

\[
t_{rp} \in \{0, 1\}, \quad r = 1, \ldots, s,
\]

\[
p = 1, \ldots, h.
\]

\[
W_{1} = \max_{W \in X} \lambda^T CW
\]

Suppose that model (3.6) is feasible and \((W^*, t^*, \alpha^*)\) is its optimal solution, where \(t^* = (t_{11}^*, \ldots, t_{sh}^*)\) For \(t_{ip} = 0\) the constraints \(C_i W \geq C_i W_p^* + \alpha - M t_{ip}\) and \(\alpha \geq \varepsilon\) imply that \(C_i W^* - C_i W_p^* \geq \alpha^* \geq \varepsilon > 0\). This leads to a suitable dispersal of the elements of WDSNDSs. Figure 2 illustrates the proposed method for an MOMILP with two objective functions \((z_1, z_2) = (C_1 W, C_2 W)\). The points on the segments DB and BI are the non-dominated solutions, and \(g\) is the ideal point. \(\text{OG+OF} = (\max_{W \in X} \lambda^T CW)\) is the optimal value of model (3.4) for \(\lambda = (\lambda_1, \lambda_2) = (1, 1)\). Hence, the point \(B = CW^* = (C_1 W^*, C_2 W^*)\) is identified as a non-dominated vector by model (3.4). Let \(\varepsilon = GH = EF\). Then, model (3.6) compares \(\text{OH+ON}\) and \(\text{OK+OE}\) and identifies \(R = CW\) and \(S = \overline{CW}\) as the second element of the WDSNDSs. If

1. \(\max\{\text{OH+ON, OK+OE}\} = \text{OH+ON}\), then \(t_1^* = 0, t_2^* = 1, \alpha^* = \text{GH}, R = \overline{CW}\) in WDSNDSs, as the second element, and
2. \(\max\{\text{OH+ON, OK+OE}\} = \text{OK+OE}\), then \(t_1^* = 1, t_2^* = 0, \alpha^* = \text{EF}\) and \(S = \overline{CW}\) in WDSNDSs, as the second element.

When \(\text{OH+ON}=\text{OK+OE}\), the solutions \(\overline{W}\) and \(\overline{W}\) are the alternative optimal solutions of model (3.6) and \(\overline{CW}, \overline{CW} \in \text{WDSNDSs}\). According to the following theorem, to find the elements of WDSNDSs of model (2.1) it is enough to solve model (3.6) in each iteration of the proposed algorithm.
Theorem 3.2 The optimal solutions of problem (3.6) are efficient solutions of model (2.1).

Proof. Let \((W^*_h, t^*_r, \alpha^*_r)\) be an optimal solution of model (3.6) and assume that \(W^*_h\) is an inefficient solution of model (2.1). Therefore, there is a feasible solution of model (2.1), say \(W'\), such that

\[
C_rW' \geq C_rW^*_h, \quad r = 1, \ldots, s, \\
\exists l \in \{1, \ldots, s\}, \quad C_lW' > C_lW^*_h. 
\]  

(3.7)

But, \(W' \in X\) and \(C_rW' \geq C_rW^*_h \geq C_rW^*_h + \alpha^*_r - Mt^*_r, r = 1, \ldots, s, h = 1, \ldots, p.\) Therefore, \((W', t^*, \alpha^*)\) is a feasible solution of model (3.6). Since \(\lambda \in \Lambda\) is strictly positive, \(\sum_{r=1}^s \lambda_rC_rW' > \sum_{r=1}^s \lambda_rC_rW^*_h\), which is a contradiction. \(\square\)

In model (3.6) we assume \(\varepsilon\) is a small positive number. But, by solving the following model the upper bound of \(\varepsilon\) can be found.

\[
\varepsilon^* = \max \varepsilon \\
\text{s.t.} \quad W \in X \\
C_rW \geq C_rW^*_h + \alpha - Mt^*_r, \\
r = 1, \ldots, s, \quad p = 1, \ldots, h \\
\sum_{r=1}^s t^*_r \leq s - 1, p = 1, \ldots, h \\
\alpha \geq \varepsilon \geq 0 \\
t^*_r \in \{0,1\}, r = 1, \ldots, s, \\
p = 1, \ldots, h. 
\]  

(3.8)

It is evident when \(\varepsilon > \varepsilon^*\), model (3.6) is infeasible and the interval \([0, \varepsilon^*]\) is the assurance interval of \(\varepsilon.\) In some situations decision maker needs a WDSNDSs with \(q\) elements. To specify such well-dispersed subset from the non-dominated solutions a common \(\varepsilon\) is needed. To this end, we can assume \(\varepsilon\) is a small positive number or we can use the \(\varepsilon^*\) of the first iteration, the optimal value of the following model, to approximate the upper bound of \(\varepsilon\) as \(\varepsilon \leq \frac{\varepsilon^*}{k}, (k > q).\)

\[
\varepsilon^* = \max \varepsilon \\
\text{s.t.} \quad W \in X \\
C_rW \geq C_rW^*_h + \alpha - Mt^*_r, \\
r = 1, \ldots, s \\
\sum_{r=1}^s t^*_r \leq s - 1, \\
p = 1, \ldots, h \\
\alpha \geq \varepsilon \\
t^*_r \in \{0,1\}, r = 1, \ldots, s. 
\]  

(3.9)

Indeed, approximately model (3.6) with \(\varepsilon \leq \frac{\varepsilon^*}{k}\) as common \(\varepsilon\) after \(q\) iterations finds a suitable WDSNDSs with \(q\) elements.

Using a termination condition such as (i) Infeasibility of models (3.4) or (3.6) (ii) a given bound on the number of the founded efficient solutions (iii) a given bound on the running time and regarding to the above discussions the stepwise description to generate a WDSNDSs is stated as follows.

3.1 The proposed Algorithm

Initialization

Choose \(\lambda \in \Lambda\) and solve model (3.4). Set \(h = 0\) and specify \(WD_0 = \{W^*_h\}\) as the set of optimal solution of model (3.4). If \(WD_0 = \phi,\) stop and the set of WDSNDSs is empty, otherwise choose \(M, \varepsilon,\) a stop condition and go to step 1.

Generalization

Step 1: Solve model (3.6), and specify \(WD = \{W^*_h\}\) as the set of optimal solution of model (3.6).

Step 2: If \(WD = \phi\) stop, and put the set of \(\{CW^*_0, CW^*_1, \ldots, CW^*_h\}\), as the WDSNDSs, otherwise set \(WD_{h+1} = WD_h \cup WD\) and go to step 1.

Note that when all of the variables of model (2.1) are integer, the proposed algorithm generates the whole set of efficient solutions. In this case, we have to set \(\varepsilon = 0\) and hence model (3.6) is converted the following model:
An optimal solution to above problem is $W^*_0 = (2, 0, 1)$. Therefore, $WD_0 = \{W^*_0 = (2, 0, 1)\} \neq \phi$ and $Z_o = (z_1, z_2) = (2, 0)$. Let $M = 100$.

To estimate an $\varepsilon$ corresponding to a WDSNDSs with 7 elements the following model is solved:

\[
\begin{align*}
\max & \quad \varepsilon \\
\text{s.t.} & \quad w_1 + 2w_2 + 2w_3 \leq 4 \\
& \quad 2w_1 + w_2 - 2w_3 \leq 2 \\
& \quad w_1, w_2, w_3 \geq 0, w_3 \in \{0, 1\}.
\end{align*}
\]

Using the optimal value of the above model, $\varepsilon^* = 2$, to find assurance interval of $\varepsilon$, we have $0 < \varepsilon \leq \frac{\varepsilon^*}{k} = \frac{2}{k} (k > 7)$. By choosing $k = 10$ we obtain $\varepsilon = 0.2$ as a common $\varepsilon$ for model (3.6).

**Generalization**

**Iteration 1**

To specify $WD$ the following model is solved:

\[
\begin{align*}
\max & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 + 2w_2 + 2w_3 \leq 4 \\
& \quad 2w_1 + w_2 - 2w_3 \leq 2 \\
& \quad w_1, w_2, w_3 \geq 0, w_3 \in \{0, 1\}.
\end{align*}
\]

This problem is feasible and $W^*_1 = (0, 2, 0)$ is its optimal solution. Hence, $Z_1 = (z_1, z_2) = (0, 2)$ and $WD_1 = WD_0 \cup WD = \{(0, 2), (2, 0), (1, 2)\}$.

**Iteration 2**

By adding the new constraints, the following model is obtained:

\[
\begin{align*}
\max & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 + 2w_2 + 2w_3 \leq 4 \\
& \quad 2w_1 + w_2 - 2w_3 \leq 2 \\
& \quad w_1 + 100\alpha_{11} - \alpha \geq 2 \\
& \quad w_2 + 100\alpha_{21} - \alpha \geq 0 \\
& \quad t_{11} + t_{21} \leq 1 \\
& \quad \alpha \geq 0.2 \\
& \quad w_1, w_2, t_{11}, t_{21}, w_3 \in \{0, 1\}.
\end{align*}
\]

An optimal solution is $W^*_2 = (1.6, 0.2, 1)$ and its corresponding non-dominated solution is $Z_2 = (1.6, 0.2)$. So, $WD_2 = WD_1 \cup WD_1 \cup WD_1$.
{\{(0,2,0), (2,0,1), (1.6,0.2,1)\}}.

**Iteration 3**
The corresponding problem for \(WD_2\) is as:

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 + 2w_2 + 2w_3 \leq 4 \\
& \quad 2w_1 + w_2 - 2w_3 \leq 2 \\
& \quad w_1 + 100t_{11} - \alpha \geq 2 \\
& \quad w_2 + 100t_{21} - \alpha \geq 0 \\
& \quad w_1 + 100t_{12} - \alpha \geq 0 \\
& \quad w_2 + 100t_{22} - \alpha \geq 2 \\
& \quad w_1 + 100t_{13} - \alpha \geq 1.6 \\
& \quad w_2 + 100t_{23} - \alpha \geq 0.2 \\
& \quad t_{1h} + t_{2h} \leq 1, \ h = 1, 2, 3 \\
& \quad w_1, w_2 \geq 0, \alpha \geq 0.2 \\
& \quad t_{1h}, t_{2h}, w_3 \in \{0,1\}, \ h = 1,2,3. \\
\end{align*}
\]

An optimal solution to the above problem is \(WD_3^* = (0.2,1.6,0)\) with a non-dominated vector equal to \((0.2,1.6)\). Consequently,\
\[
WD_3 = WD_2 \cup WD = \{(0,2,0), (2,0,1), (1.6,0.2,1), (0.2,1.6,0)\}.
\]

**Iteration 4**
In order to find another member of WDSNDSs the following problem must be solved.

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 + 2w_2 + 2w_3 \leq 4 \\
& \quad 2w_1 + w_2 - 2w_3 \leq 2 \\
& \quad w_1 + 100t_{11} - \alpha \geq 2 \\
& \quad w_2 + 100t_{21} - \alpha \geq 0 \\
& \quad w_1 + 100t_{12} - \alpha \geq 0 \\
& \quad w_2 + 100t_{22} - \alpha \geq 2 \\
& \quad w_1 + 100t_{13} - \alpha \geq 1.6 \\
& \quad w_2 + 100t_{23} - \alpha \geq 0.2 \\
& \quad w_1 + 100t_{14} - \alpha \geq 0.2 \\
& \quad w_2 + 100t_{24} - \alpha \geq 1.6 \\
& \quad t_{1h} + t_{2h} \leq 1, \ h = 1,2,3,4 \\
& \quad w_1, w_2 \geq 0, \alpha \geq 0.2 \\
& \quad t_{1h}, t_{2h}, w_3 \in \{0,1\}, \ h = 1,2,3.4 \\
\end{align*}
\]

An optimal solution is \(WD_4^* = (1.2,0.4,1)\) and its corresponding non-dominated solution is \(Z_4 = (1.2,0.4)\). Therefore,\
\[
WD_4 = WD_3 \cup WD = \{(0,2,0), (2,0,1), (1.6,0.2,1), (0.2,1.6,0), (1.2,0.4,1)\}.
\]

**Iterations 5 and 6**
For the purpose of brevity, we neglect the formulation of problems, in the rest of the iterations. The optimal solutions of the 5th and 6th iterations are \(WD_5^* = (0.4,1.2,0)\) and \(WD_6^* = (0.8,0.6,1)\), respectively, and hence \(Z_5 = (0.4,1.2), Z_6 = (0.4,1.2)\) and \(WD_6 = WD_4 \cup \{W_5^*, \{W_6^*\}\} = \{(0,2,0), (2,0,1), (1.6,0.2,1), (0.2,1.6,0), (1.2,0.4,1), (0.4,1.2,0), (0.8,0.6,1)\}\).

Therefore, using \(\varepsilon = 0.2\) the set \{\(Z_0, Z_1, \ldots, Z_6\)\} is the WDSNDSs. The elements of WDSNDSs have been ranked by their distance from ideal point such that the rank of \(CW_j\) is better than the rank of \(CW_{j+1}\) for \(j = 0, \ldots, 5\).

If we choose \(\varepsilon < 0.2\), then another WDSNDSs with lower dispersal of elements are generated. For instance, if we choose \(\varepsilon = 0.1\), a WDSNDSs with further elements is generated. Column 2 of Table 1 shows the elements of the generated WDSNDSs with \(\varepsilon = 0.1\).

<table>
<thead>
<tr>
<th>(j)</th>
<th>(W_j^* = (w_{1j}^<em>, w_{2j}^</em>, w_{3j}^*))</th>
<th>(Z_j = (z_{1j}, z_{2j}))</th>
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<td>(2.0,1)</td>
<td>(2.0)</td>
</tr>
<tr>
<td>2</td>
<td>(0.2,0)</td>
<td>(0.2)</td>
</tr>
<tr>
<td>3</td>
<td>(1.8,0.1,1)</td>
<td>(1.8,0.1)</td>
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<td>(0.1,1.8)</td>
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<td>(0.5,1)</td>
</tr>
<tr>
<td>13</td>
<td>(0.8,0.6,1)</td>
<td>(0.8,0.6)</td>
</tr>
</tbody>
</table>

Table 1. The generated WDSNDSs with \(\varepsilon = 0.1\)

**Example 4.2** As the second example to illustrate the proposed algorithm the following MOMILP problem is considered [11]:

\[
\begin{align*}
\text{max} & \quad w_1 - 2w_2 \\
\text{max} & \quad -w_1 + 3w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1, w_2 \in \{0,1,2\}. \\
\end{align*}
\]

**Initialization**
Let \(\lambda = (4,3)\). To start the algorithm, the following model is considered:

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1, w_2 \in \{0,1,2\}. \\
\end{align*}
\]

The vector \(W_o^* = (2,2)\) is an optimal solution of the above problem. Therefore, \(WD_0 = \{W_o^* = (2,2)\} \neq \phi\) and \(Z_0 = (z_1, z_2) = (-2,4)\). Let \(M = 100, \varepsilon = 0\) and consider the infeasibility of model \((3.10)\) as stop condition.
Generalization
Iteration 1
Using (3.10), to specify \( WD_1 \) the following model is solved:

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1 - 2w_2 + 100t_{11} > -2 \\
& \quad -w_1 + 3w_2 + 100t_{21} > 4 \\
& \quad t_{11} + t_{21} \leq 1 \\
& \quad w_1, w_2 \in \{0, 1, 2\}, t_{11}, t_{21} \in \{0, 1\}.
\end{align*}
\]

The optimal solution of the above model is \( W_1^* = (1, 2) \) and hence \( Z_1 = (z_1, z_2) = (-3, 5) \), and \( WD_1 = WD_0 \cup \{W_1^*\} = \{(2, 2), (1, 2)\} \).

Iteration 2
By considering the new constraints the following model is obtained:

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1 - 2w_2 + 100t_{11} > -2 \\
& \quad -w_1 + 3w_2 + 100t_{21} > 4 \\
& \quad w_1 - 2w_2 + 100t_{12} > -3 \\
& \quad -w_1 + 3w_2 + 100t_{22} > 5 \\
& \quad t_{1p} + t_{2p} \leq 1, p = 1, 2 \\
& \quad w_1, w_2 \in \{0, 1, 2\}, t_{1p}, t_{2p} \in \{0, 1\}, \\
& \quad p = 1, 2.
\end{align*}
\]

An optimal solution is \( Z_2 = (2, 1) \) and hence \( WD_2 = WD_1 \cup \{W_2^*\} = \{(2, 2), (1, 2), (2, 1)\} \).

Iteration 3
In order to find another member of WDSNDSs, the following problem must be solved.

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1 - 2w_2 + 100t_{11} > -2 \\
& \quad -w_1 + 3w_2 + 100t_{21} > 4 \\
& \quad w_1 - 2w_2 + 100t_{12} > -3 \\
& \quad -w_1 + 3w_2 + 100t_{22} > 5 \\
& \quad w_1 - 2w_2 + 100t_{13} > 0 \\
& \quad -w_1 + 3w_2 + 100t_{23} > 1 \\
& \quad t_{1p} + t_{2p} \leq 1, p = 1, 2, 3 \\
& \quad w_1, w_2 \in \{0, 1, 2\}, \\
& \quad t_{1p}, t_{2p} \in \{0, 1\}, p = 1, 2, 3.
\end{align*}
\]

An optimal solution to the above problem is \( W_3^* = (0, 2) \) and hence \( Z_3 = (-4, 6) \), and \( WD_3 = WD_2 \cup \{W_3^*\} = \{(2, 2), (1, 2), (2, 1), (0, 2)\} \).

Iteration 4
To generate another member of WDSNDSs the following problem is solved.

\[
\begin{align*}
\text{max} & \quad w_1 + w_2 \\
\text{s.t.} & \quad w_1 - 2w_2 \leq 0 \\
& \quad w_1 - 2w_2 + 100t_{11} > -2 \\
& \quad -w_1 + 3w_2 + 100t_{21} > 4 \\
& \quad w_1 - 2w_2 + 100t_{12} > -3 \\
& \quad -w_1 + 3w_2 + 100t_{22} > 5 \\
& \quad w_1 - 2w_2 + 100t_{13} > 0 \\
& \quad -w_1 + 3w_2 + 100t_{23} > 1 \\
& \quad t_{1p} + t_{2p} \leq 1, p = 1, 2, 3 \\
& \quad w_1, w_2 \in \{0, 1, 2\}, \\
& \quad t_{1p}, t_{2p} \in \{0, 1\}, p = 1, 2, 3.
\end{align*}
\]

The vector \( W_4^* = (1, 1) \) is an optimal solution of the above model. Therefore, \( Z_4 = (-1, 2) \) and \( WD_4 = WD_3 \cup \{W_4^*\} = \{(2, 2), (1, 2), (2, 1), (0, 2), (1, 1)\} \).

Iteration 5
The model of this iteration is infeasible and the algorithm is terminated. Therefore, the set \( \text{WDSNDSs} = \{Z_0, Z_1, \ldots, Z_4\} \) is the whole set of non-dominated solutions. This example has been solved in [11]. To find a \( Z \in \text{WDSNDSs} \), Sylva and Creme [11] solve two problems while our method solves only one problem in each iteration.

5 Conclusion
This paper proposed an algorithm to find a WDSNDSs of an MOMILP problem. In each iteration of the proposed algorithm, only one mixed integer linear programming problem is solved. According to the \( \bar{\lambda} \in \Lambda \), the opinions of decision maker, the rank of the optimal solution of model (3.6) in the \( p^{th} \) iteration is better than its optimal solution in the \( (p + 1)^{th} \) iteration. Hence, the elements of the WDSNDSs of an MOMILP problem are ranked according to their distance form ideal point and the generated WDSNDSs can be used without any filtering procedures. Using suitable value for the parameter of the proposed model an appropriate WDSNDSs by less computational efforts is generated.

Similar to Sylva and Creme’s method [11], corresponding to an MOMILP problem with \( s \) objective functions, in each iteration \( s + 1 \) constraints and \( s \) variables are added to the mixed integer model which is solved. This increases the computational efforts to generate the WDSNDSs and can be studied in the future. The proposed
method can be modified to solve mixed integer non-linear programming problem.

References


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