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Research Article

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Well-dispersed subsets of non-dominated solutions for MOMILP problem

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Abstract

This paper uses the weighted L_1-norm to propose an algorithm for finding a well-dispersed subset of non-dominated solutions of multiple objective mixed integer linear programming problem. When all variables are integer it finds the whole set of efficient solutions. In each iteration of the proposed method only a mixed integer linear programming problem is solved and its optimal solutions generates the elements of the well-dispersed subset non-dominated solutions (WDSNDSs) of MOMILP. According to the distance of non-dominated solutions from the ideal point the elements of the WDSNDSs are ranked, hence it does not need the filtering procedures. Using suitable values for the parameter of the proposed model an appropriate WDSNDSs by less computational efforts can be generated. Two numerical examples present to illustrate the applicability of the proposed method and compare it with earlier work.

Keywords : Multi-Objective Mixed Integer Linear Programming; Efficient solutions; Well-dispersed subset non-dominated solutions; L_1-norm.

1 Introduction

Multiple Objective Mixed Integer Linear Programming (MOMILP) problems occur frequently in many applications. Many engineering, operations, and scientific applications include a mixture of discrete and continues decision variables and linear relationship involving the decision variables that have a pronounced effect on the set of feasible and optimal solutions in Multi Criteria Decision Making (MCDM).

In recent decades, Numerous algorithm also interactive procedures have been designed to solve Multiple Objective Linear Programming (MOLP) [2, 3, 4, 9]. MOMILP and Multiple Objective Integer Linear Programming (MOILP) are important research areas as many practical situations discrete representations have to deal with several objectives [1, 8]. Surveys considering most of methods for generating non-dominated vectors are also available [12, 14].

Mavrotas and Diakoulaki [7] present a branch and bound algorithm to generate non-dominated solution of MOMILP problems. Jahanshahloo et al. [5] propose a method to find all efficient solutions of zero-one MOLP problem. In each iteration of the proposed algorithm at least one efficient solution is found. Klein and Hannan [6] develop an algorithm for the MOILP problem which uses some additional constraints to eliminate the known dominated solution based on the sequential solutions of the single objective models. Sylva and Crema [10] present an algorithm for enumerating all non-dominated vectors of MOILP problems by incorporating objective functions in a weighted function. Nevertheless, its performance

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may not be satisfactory for problems with a large number of objective function. Sylva and Crema [11] propose a method for finding a well-dispersed subset of non-dominated solutions based on maximizing the infinity norm distance from a set of known solutions. They claim that their approach originally provides a variant of the procedure by Sylva and Crema [10]. The major drawback of this approach is the difficulty of solving the constrained problems due to increasing number of constraints and binary variables.

This paper proposes a method to find a WDSNDSs by using the weighted L_1-norm. When all variables are integer it finds the whole set of efficient solutions. In each iteration of the proposed algorithm only one mixed integer linear programming problem is solved, while Sylva and Crema's method [11] needs to solve two problems in each iteration which the optimal solution of one of them necessary not be efficient. The proposed algorithm ranks the elements of the WDSNDSs, hence we do not need the filtering procedures and using suitable values for the parameter of the proposed model we can obtain an appropriate WDSNDSs by less computational efforts. It modifies the dispersal of the WDSNDSs according to the decision maker opinions.

The paper is organized as follows. Section 2 presents a brief background about MOMILP problem. Section 3 introduces some models and an algorithm to generate a WDSNDSs of an MOMILP problem. Illustration with two numerical examples are given in Section 4. Finally, the concluding results are presented.

2 MOMILP problem

The MOMILP with s-objective functions can be defined as follows:

$$\begin{array}{ll} \max & \{C_1W,\dots,C_sW\} \\ \text{s.t.} & A_iW \le b_i, \quad i=1,\dots,m \\ & W \ge 0, w_j \in Z^+, j \in J \end{array} \tag{2.1}$$

where $C_r = (c_{1r},\dots,c_{nr})$ $(r = 1,\dots,s)$, $A_i = (a_{i1},\dots,a_{in})$ $(i = 1,2,\dots,m)$, $J \subseteq \{1,\dots,n\}$, $Z^+ = \{0,1,2,\dots\}$ and $W = (w_1,\dots,w_n)^T$. The set of feasible solutions of problem (2.1) is defined by $X = \{W \mid A_iW \le b_i, i = 1,\dots,m, W \ge 0, w_j \in Z^+, j \in J\}$ and is assumed to be a non-empty set.The objective vector $Z = (z_1,\dots,z_s)^T = (C_1W,\dots,C_sW)^T$ for $W \in X$ is said to be non-dominated vector if and only if there is no $Z^o = (z_1^o,\dots,z_s^o)^T = (C_1W^o,\dots,C_sW^o)^T$ for $W^o \in X$ such that $z_r \ge z_r^o$ for all $r \in \{1,\dots,s\}$ and $z_r > z_r^o$ for at least one r. The set of $F = \{Z \mid Z = (C_1W,\dots,C_sW)^T, W \in X\}$ is called the values space of objective functions in problem (2.1). Let $g_r = C_rW_r^*(r = 1,\dots,s)$, where W_r^* is the optimal solution of the following single objective mixed integer programming problem:

$$\begin{array}{rl} g_r = \max & C_rW \\ \text{s.t.} & W \in X. \end{array} \tag{2.2}$$

Let us consider X be bounded and $g = (g_1,\dots,g_s)^T = (C_1W_1^*,\dots,C_sW_s^*)^T$ is referred to as the ideal vector of model (2.1) [5]. As can be seen, for each $W \in X$ as a feasible solution of problem (2.1), the vector g dominates the vector $Z = (C_1W,\dots,C_sW)^T \ne g$.

3 Well-dispersed subsets of efficient solutions

Suppose our aim is to find a subset of efficient solutions with a desired dispersal and $\lambda \in \Lambda = \{\lambda = (\lambda_1,\dots,\lambda_s)^T | \lambda_r > 0, r = 1,\dots,s\}$ is known, where λ is decision maker preferences about objective functions. To obtain a member of the WDSNDSs for problem (2.1), say CW, we specify $W \in X$ such that $g - Z = (g_1 - C_1W,\dots,g_s - C_sW)^T$ is minimized. To this purpose, the following MOMILP problem can be solved:

$$\begin{array}{ll} \min & \{g_1 - C_1W,\dots,g_s - C_sW\} \\ \text{s.t.} & W \in X. \end{array} \tag{3.3}$$

To find the efficient solutions of model (3.3) by using the weighted L_1-norm, i.e., $d_\lambda(g, CW)$, and according to $g_r \ge C_rW$ $(r = 1,\dots,s, W \in X)$ we have:

$$\begin{aligned} \min_{W\in X} d_\lambda(g,CW) &= \min_{W\in X} \sum_{r=1}^{s} \lambda_r |g_r - C_rW| \\ &= \min_{W\in X} \sum_{r=1}^{s} \lambda_r (g_r - C_rW) \\ &= \sum_{r=1}^{s} \lambda_r g_r + \min_{W\in X} \sum_{r=1}^{s} \lambda_r(-C_rW) \\ &= \sum_{r=1}^{s} \lambda_r g_r - \max_{W\in X} \sum_{j=1}^{n} \sum_{r=1}^{s} \lambda_r c_{rj} w_j. \end{aligned}$$

Using the above relations, model (3.3) is converted to the following mixed integer linear programming problem.

$$\begin{array}{ll} \max & \sum_{r=1}^{s} \lambda_r C_r W \\ \text{s.t.} & W \in X. \end{array} \tag{3.4}$$

Let problem (3.4) is feasible and W^* be its optimal solution.

Theorem 3.1 *The optimal solutions of problem (3.4) are efficient solutions of model (2.1).*

Proof. The proof is similar to that of Theorem 2.3 in [5] and is not repeated here. □

Model (3.4) is for finding some efficient solutions of the MOMILP problem, another member of the WDSNDSs for problem (2.1), say $C\overline{W}$, is determined such that

1. the distance of the g and $C\overline{W}$, i.e. $d_\lambda(g, C\overline{W})$, is minimized and
2. there exists $r \in \{1, \ldots, s\}$ such that $|C\overline{W} - CW^*| \geq \varepsilon$.

To ward this end, some constraints and variables are added to problem (3.4) and the obtained model is solved. This process is continued and a sequence of mixed integer programming problem is attained. Let W^*_{h-1} be the optimal solution of the model of the $(h-1)^{th}$ iteration, i.e. the model M_{h-1}. Then, by adding the following constraints to the model M_{h-1}, the model of the h^{th} iteration (M_h) is determined.

$$\begin{array}{l} C_r W \geq C_r W^*_{h-1} + \alpha - M t_{rh}, r = 1, \ldots, s \\ \sum_{r=1}^{s} t_{rh} \leq s - 1 \\ \alpha \geq \varepsilon \\ t_{rh} \in \{0, 1\}, r = 1, \ldots, s \end{array} \tag{3.5}$$

where M is a sufficiently large positive value and $\underline{M} = \max_{1 \leq r \leq s} |g_r|$ can be used as its lower bound. When $t_{rh} = 1$, the constraint $C_r W_h \geq C_r W^*_{h-1} + \alpha - M t_{rh}$ is redundant and the constraint $\sum_{r=1}^{s} t_{rh} \leq s - 1$ imply that there exists $l \in \{1, \ldots, s\}$ such that $t_{lh} = 0$.

Using to the above discussion the following model is considered:

$$\begin{array}{lll} M_{h+1}: & \max & \sum_{r=1}^{s} \lambda_r C_r W \\ & \text{s.t.} & W \in X \\ & & C_r W \geq C_r W^*_p + \alpha - M t_{rp}, \\ & & r = 1, \ldots, s, p = 1, \ldots, h \\ & & \sum_{r=1}^{s} t_{rp} \leq s - 1, \\ & & p = 1, \ldots, h \\ & & \alpha \geq \varepsilon \\ & & t_{rp} \in \{0, 1\}, r = 1, \ldots, s, \\ & & p = 1, \ldots, h. \end{array} \tag{3.6}$$

Suppose that model (3.6) is feasible and (W^*, t^*, α^*) is its optimal solution, where $t^* = (t^*_{11}, \ldots, t^*_{sh})$. For $t_{lp} = 0$ the constraints $C_l W \geq C_l W^*_p + \alpha - M t_{lp}$ and $\alpha \geq \varepsilon$ imply that $C_l W^* - C_l W^*_p \geq \alpha^* \geq \varepsilon > 0$. This leads to a suitable dispersal of the elements of WDSNDSs. Figure 1 illustrates the proposed method for an MOMILP with two objective functions $(z_1, z_2) = (C_1 W, C_2 W)$. The points on the segments DB and BI are the non-dominated solutions, and g is the ideal point. OG+OF $(= \max_{W \in X} \lambda^T CW)$ is the optimal value of model (3.4) for $\lambda = (\lambda_1, \lambda_2) = (1, 1)$. Hence, the point B=$CW^* = (C_1 W^*, C_2 W^*)$ is identified as a non-dominated vector by model (3.4). Let ε=GH=EF. Then, model (3.6) compares OH+ON and OK+OE and identifies R=$C\widehat{W}$ or S=$C\overline{W}$ as the second member of the WDSNDSs. If

1. max{OH+ON, OK+OE}=OH+ON, then $t^*_1 = 0, t^*_2 = 1$, α^*=GH, R=$C\widehat{W} \in$ WDSNDSs, as the second element, and
2. max{OH+ON,OK+OE}=OK+OE, then $t^*_1 = 1, t^*_2 = 0$, α^* =EF and S=$C\overline{W} \in$ WDSNDSs, as the second element.

When OH+ON=OK+OE, the solutions $\overline{W}$ and $\widehat{W}$ are the alternative optimal solutions of model (3.6) and $C\overline{W}, C\widehat{W} \in$ WDSNDSs. According to the following theorem, to find the elements of WDSNDSs of model (2.1) it is enough to solve model (3.6) in each iteration of the proposed algorithm.

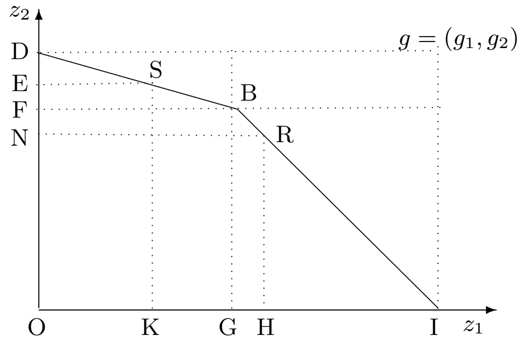

Figure 1. Illustration of the proposed method in the objective functions space

Theorem 3.2 *The optimal solutions of problem (3.6) are efficient solutions of model (2.1).*

Proof. Let (W_h^*, t^*, α^*) be an optimal solution of model (3.6) and assume that W_h^* is an inefficient solution of model (2.1). Therefore, there is a feasible solution of model (2.1), say W', such that

$$\begin{array}{l} C_rW' \geq C_rW_h^*, r=1,\ldots,s, \\ \exists\, l \in \{1,\ldots,s\}, \quad C_lW' > C_lW_h^*. \end{array} \tag{3.7}$$

But, $W' \in X$ and $C_rW' \geq C_rW_h^* \geq C_rW_h^* + \alpha^* - Mt_{rh}^*, r=1,\ldots,s, h=1,\ldots,p$. Therefore, (W', t^*, α^*) is a feasible solution of model (3.6). Since $\lambda \in \Lambda$ is strictly positive, $\sum_{r=1}^s \lambda_r C_r W' > \sum_{r=1}^s \lambda_r C_r W_h^*$, which is a contradiction. □

In model (3.6) we assume ε is a small positive number. But, by solving the following model the upper bound of ε can be found.

$$\begin{array}{rl} \varepsilon^* = \max & \varepsilon \\ \text{s.t.} & W \in X \\ & C_rW \geq C_rW_p^* + \alpha - Mt_{rp}, \\ & r=1,\ldots,s, p=1,\ldots,h \\ & \sum_{r=1}^{s} t_{rp} \leq s-1, p=1,\ldots,h \\ & \alpha \geq \varepsilon \geq 0 \\ & t_{rp} \in \{0,1\}, r=1,\ldots,s, \\ & p=1,\ldots,h \end{array} \tag{3.8}$$

It is evident when $\varepsilon > \varepsilon^*$, model (3.6) is infeasible and the interval $[0, \varepsilon^*]$ is the assurance interval of ε. In some situations decision maker needs a WDSNDSs with q elements. To specify such well-dispersed subset from the non-dominated solutions a common ε is needed. To this end, we can assume ε is a small positive number or we can use the ε^* of the first iteration, the optimal value of the following model, to approximate the upper bound of ε as $\varepsilon \leq \frac{\varepsilon^*}{k}, (k > q)$.

$$\begin{array}{rl} \varepsilon^* = \max & \varepsilon \\ \text{s.t.} & W \in X \\ & C_rW \geq C_rW_1^* + \alpha - Mt_{r1}, \\ & r=1,\ldots,s \\ & \sum_{r=1}^{s} t_{r1} \leq s-1, \\ & p=1,\ldots,h \\ & \alpha \geq \varepsilon \\ & t_{r1} \in \{0,1\}, r=1,\ldots,s. \end{array} \tag{3.9}$$

Indeed, approximately model (3.6) with $\varepsilon \leq \frac{\varepsilon^*}{k}$ as common ε after q iterations finds a suitable WDSNDSs with q elements.

Using a termination condition such as (i) Infeasibility of models (3.4) or (3.6) (ii) a given bound on the number of the founded efficient solutions (iii) a given bound on the running time and regarding to the above discussions the stepwise description to generate a WDSNDSs is stated as follows.

3.1 The proposed Algorithm

Initialization
Choose $\lambda \in \Lambda$ and solve model (3.4). Set $h=0$ and specify $WD_0 = \{W_0^*\}$ as the set of optimal solution of model (3.4). If $WD_0 = \phi$, stop and the set of WDSNDSs is empty, otherwise choose M, ε, a stop condition and go to step 1.
Generalization
Step 1: Solve model (3.6), and specify $WD = \{W_h^*\}$ as the set of optimal solution of model (3.6).
Step 2: If $WD = \phi$ stop, and put the set of $\{CW_0^*, CW_1^*, \ldots, CW_h^*\}$, as the WDSNDSs, otherwise set $WD_{h+1} = WD_h \cup WD$ and go to step 1.

Note that when all of the variables of model (2.1) are integer, the proposed algorithm generates the whole set of efficient solutions. In this case, we have to set $\varepsilon = 0$ and hence model (3.6) is converted the following model:

$$
\begin{array}{ll}
\max & \sum_{r=1}^{s} \lambda_r C_r W \\
\text{s.t.} & W \in X \\
 & C_r W > C_r W_p^* - M t_{rp}, r = 1, \ldots, s, \\
 & p = 1, \ldots, h \\
 & \sum_{r=1}^{s} t_{rp} \leq s - 1, p = 1, \ldots, h \\
 & t_{rp} \in \{0,1\}, r = 1, \ldots, s, p = 1, \ldots, h.
\end{array} \tag{3.10}
$$

Indeed, in this case the proposed algorithm in this paper is the general case of the proposed algorithm in [5].

Theorem 3.3 *By using model (3.10) instead of model (3.6) the proposed algorithm generates the whole set of efficient solution of MOILP problem.*

Proof. The proof is similar to that of Theorem 2.5 in [5]. □

4 Numerical Examples

This section examines two numerical examples to verify the validity and the effectiveness of the proposed algorithm in comparison with Sylva and Crema's method [11].

Example 4.1 *Consider the following MOMILP problem [11]:*

$$
\begin{array}{ll}
\max & w_1 \\
\max & w_2 \\
\text{s.t.} & w_1 + 2w_2 + 2w_3 \leq 4 \\
 & 2w_1 + w_2 - 2w_3 \leq 2 \\
 & w_1, w_2 \geq 0, w_3 \in \{0,1\}.
\end{array}
$$

Suppose that we need a WDSNDSs with 7 elements. In this case, we can estimate an upper bound for ε.

Initialization

Let $\lambda = (1,1)$. To start the algorithm the following model is solved:

$$
\begin{array}{ll}
\max & w_1 + w_2 \\
\text{s.t.} & w_1 + 2w_2 + 2w_3 \leq 4 \\
 & 2w_1 + w_2 - 2w_3 \leq 2 \\
 & w_1, w_2 \geq 0, w_3 \in \{0,1\}.
\end{array}
$$

An optimal solution to above problem is $W_0^* = (2,0,1)$. Therefore, $WD_0 = \{W_o^* = (2,0,1)\} \neq \phi$ and $Z_o = (z_1, z_2) = (2,0)$. Let $M = 100$.

To estimate an ε corresponding to a WDSNDSs with 7 elements the following model is solved:

$$
\begin{array}{ll}
\max & \varepsilon \\
\text{s.t.} & w_1 + 2w_2 + 2w_3 \leq 4 \\
 & 2w_1 + w_2 - 2w_3 \leq 2 \\
 & w_1 + 100t_{11} - \alpha \geq 2 \\
 & w_2 + 100t_{21} - \alpha \geq 0 \\
 & t_{11} + t_{21} \leq 1 \\
 & \alpha \geq \varepsilon \\
 & \varepsilon \geq 0, w_1, w_2 \geq 0, t_{11}, t_{21}, w_3 \in \{0,1\}.
\end{array}
$$

Using the optimal value of the above model, $\varepsilon^* = 2$, to find assurance interval of ε, we have $0 < \varepsilon \leq \frac{\varepsilon^*}{k} = \frac{2}{k}, (k > 7)$. By choosing $k = 10$ we obtain $\varepsilon = 0.2$ as a common ε for model (3.6).

Generalization

Iteration 1

To specify WD the following model is solved:

$$
\begin{array}{ll}
\max & w_1 + w_2 \\
\text{s.t.} & w_1 + 2w_2 + 2w_3 \leq 4 \\
 & 2w_1 + w_2 - 2w_3 \leq 2 \\
 & w_1 + 100t_{11} - \alpha \geq 2 \\
 & w_2 + 100t_{21} - \alpha \geq 0 \\
 & t_{11} + t_{21} \leq 1 \\
 & \alpha \geq 0.2 \\
 & w_1, w_2 \geq 0, t_{11}, t_{21}, w_3 \in \{0,1\}.
\end{array}
$$

This problem is feasible and $W_1^* = (0,2,0)$ is its optimal solution. Hence, $Z_1 = (z_1, z_2) = (0,2)$ and $WD_1 = WD_0 \cup WD = \{(0,2,0),(2,0,1)\}$.

Iteration 2

By adding the new constraints, the following model is obtained:

$$
\begin{array}{ll}
\max & w_1 + w_2 \\
\text{s.t.} & w_1 + 2w_2 + 2w_3 \leq 4 \\
 & 2w_1 + w_2 - 2w_3 \leq 2 \\
 & w_1 + 100t_{11} - \alpha \geq 2 \\
 & w_2 + 100t_{21} - \alpha \geq 0 \\
 & w_1 + 100t_{12} - \alpha \geq 0 \\
 & w_2 + 100t_{22} - \alpha \geq 2 \\
 & t_{1h} + t_{2h} \leq 1, h = 1,2 \\
 & w_1, w_2 \geq 0, \alpha \geq 0.2 \\
 & t_{1h}, t_{2h}, w_3 \in \{0,1\}, h = 1,2.
\end{array}
$$

An optimal solution is $W_2^* = (1.6, 0.2, 1)$ and its corresponding non-dominated solution is $Z_2 = (1.6, 0.2)$. So, $WD_2 = WD_1 \cup WD =$

$\{(0,2,0),(2,0,1),(1.6,0.2,1)\}$.

Iteration 3

The corresponding problem for WD_2 is as

$$\begin{array}{ll} \max & w_1+w_2 \\ \text{s.t.} & w_1+2w_2+2w_3 \leq 4 \\ & 2w_1+w_2-2w_3 \leq 2 \\ & w_1+100t_{11}-\alpha \geq 2 \\ & w_2+100t_{21}-\alpha \geq 0 \\ & w_1+100t_{12}-\alpha \geq 0 \\ & w_2+100t_{22}-\alpha \geq 2 \\ & w_1+100t_{13}-\alpha \geq 1.6 \\ & w_2+100t_{23}-\alpha \geq 0.2 \\ & t_{1h}+t_{2h} \leq 1, h=1,2,3 \\ & w_1,w_2 \geq 0, \alpha \geq 0.2 \\ & t_{1h},t_{2h},w_3 \in \{0,1\}, h=1,2,3. \end{array}$$

An optimal solution to the above problem is $W_3^*=(0.2,1.6,0)$ with a non-dominated vector equal to $(0.2,1.6)$. Consequently,

$WD_3 = WD_2 \cup WD = \{(0,2,0),(2,0,1),(1.6,0.2,1),(0.2,1.6,0)\}$.

Iteration 4

In order to find another member of WDSNDSs the following problem must be solved.

$$\begin{array}{ll} \max & w_1+w_2 \\ \text{s.t.} & w_1+2w_2+2w_3 \leq 4 \\ & 2w_1+w_2-2w_3 \leq 2 \\ & w_1+100t_{11}-\alpha \geq 2 \\ & w_2+100t_{21}-\alpha \geq 0 \\ & w_1+100t_{12}-\alpha \geq 0 \\ & w_2+100t_{22}-\alpha \geq 2 \\ & w_1+100t_{13}-\alpha \geq 1.6 \\ & w_2+100t_{23}-\alpha \geq 0.2 \\ & w_1+100t_{14}-\alpha \geq 0.2 \\ & w_2+100t_{24}-\alpha \geq 1.6 \\ & t_{1h}+t_{2h} \leq 1, h=1,2,3,4 \\ & w_1,w_2 \geq 0, \alpha \geq 0.2 \\ & t_{1h},t_{2h},w_3 \in \{0,1\}, h=1,2,3,4. \end{array}$$

An optimal solution is $W_4^*=(1.2,0.4,1)$ and its corresponding non-dominated solution is $Z_4=(1.2,0.4)$. Therefore,

$WD_4 = WD_3 \cup WD = \{(0,2,0),(2,0,1),(1.6,0.2,1),(0.2,1.6,0),(1.2,0.4,1)\}$.

Iterations 5 and 6

For the purpose briefness, we neglect the formulation of problems, in the rest of iterations. The optimal solutions of the 5th and 6th iterations are $W_5^*=(0.4,1.2,0)$ and $W_6^*=(0.8,0.6,1)$, respectively, and hence $Z_5=(0.4,1.2)$, $Z_6=(0.4,1.2)$ and $WD_6=WD_4 \cup \{W_5^*\} \cup \{W_6^*\}=\{(0,2,0),(2,0,1),(1.6,0.2,1),(0.2,1.6,0),(1.2,0.4,1),(0.4,1.2,0),(0.8,0.6,1)\}$.

Therefore, using $\varepsilon=0.2$ the set $\{Z_0,Z_1,\ldots,Z_6\}$ is the WDSNDSs. The elements of WDSNDSs have been ranked by their distance from ideal point such that the rank of CW_j is better than the rank of CW_{j+1} for $j=0,\ldots,5$.

If we choose $\varepsilon<0.2$, then another WDSNDSs with lower dispersal of elements are generated. For instance, if we choose $\varepsilon=0.1$, a WDSNDSs with further elements is generated. Column 2 of Table 1 shows the elements of the generated WDSNDSs with $\varepsilon=0.1$.

j	$W_j^*=(w_{1j}^*,w_{2j}^*,w_{3j}^*)$	$Z_j^*=(z_{1j}^*,z_{2j}^*)$
1	(2,0,1)	(2,0)
2	(0,2,0)	(0,2)
3	(1.8,0.1,1)	(1.8,0.1)
4	(0.1,1.8,0)	(0.1,1.8)
5	(1.6,0.2,1)	(1.6,0.2)
6	(0.2,1.6,0)	(0.2,1.6)
7	(1.4,0.3,1)	(1.4,0.3)
8	(0.3,1.4,0)	(0.3,1.4)
9	(1.2,0.4,1)	(1.2,0.4)
10	(0.4,1.2,0)	(0.4,1.2)
11	(1,0.5,1)	(1,0.5)
12	(0.5,1,0)	(0.5,1)
13	(0.8,0.6,1)	(0.8,0.6)

Table 1. The generated WDSNDSs with $\varepsilon=0.1$

Example 4.2 *As the second example to illustrate the proposed algorithm the following MOMILP problem is considered [11]:*

$$\begin{array}{ll} \max & w_1-2w_2 \\ \max & -w_1+3w_2 \\ \text{s.t.} & w_1-2w_2 \leq 0 \\ & w_1,w_2 \in \{0,1,2\}. \end{array}$$

Initialization

Let $\lambda=(4,3)$. To start the algorithm, the following model is considered:

$$\begin{array}{ll} \max & w_1+w_2 \\ \text{s.t.} & w_1-2w_2 \leq 0 \\ & w_1,w_2 \in \{0,1,2\}. \end{array}$$

The vector $W_o^*=(2,2)$ is an optimal solution of the above problem. Therefore, $WD_0=\{W_o^*=(2,2)\} \neq \phi$ and $Z_o=(z_1,z_2)=(-2,4)$. Let $M=100$, $\varepsilon=0$ and consider the infeasibility of model (3.10) as stop condition.

Generalization

Iteration 1

Using (3.10), to specify WD_1 the following model is solved:

$$\begin{array}{ll} \max & w_1 + w_2 \\ \text{s.t.} & w_1 - 2w_2 \leq 0 \\ & w_1 - 2w_2 + 100t_{11} > -2 \\ & -w_1 + 3w_2 + 100t_{21} > 4 \\ & t_{11} + t_{21} \leq 1 \\ & w_1, w_2 \in \{0,1,2\}, t_{11}, t_{21} \in \{0,1\}. \end{array}$$

The optimal solution of the above model is $W_1^* = (1,2)$ and hence $Z_1 = (z_1, z_2) = (-3,5)$, and $WD_1 = WD_0 \cup \{W_1^*\} = \{(2,2),(1,2)\}$.

Iteration 2

By considering the new constraints the following model is obtained:

$$\begin{array}{ll} \max & w_1 + w_2 \\ \text{s.t.} & w_1 - 2w_2 \leq 0 \\ & w_1 - 2w_2 + 100t_{11} > -2 \\ & -w_1 + 3w_2 + 100t_{21} > 4 \\ & w_1 - 2w_2 + 100t_{12} > -3 \\ & -w_1 + 3w_2 + 100t_{22} > 5 \\ & t_{1p} + t_{2p} \leq 1, p = 1,2 \\ & w_1, w_2 \in \{0,1,2\}, t_{1p}, t_{2p} \in \{0,1\}, \\ & p = 1,2. \end{array}$$

An optimal solution is $W_2^* = (2,1)$ and hence $Z_2 = (0,1)$, and $WD_2 = WD_1 \cup \{W_2^*\} = \{(2,2),(1,2),(2,1)\}$.

Iteration 3

In order to find another member of WDSNDSs, the following problem must be solved.

$$\begin{array}{ll} \max & w_1 + w_2 \\ \text{s.t.} & w_1 - 2w_2 \leq 0 \\ & w_1 - 2w_2 + 100t_{11} > -2 \\ & -w_1 + 3w_2 + 100t_{21} > 4 \\ & w_1 - 2w_2 + 100t_{12} > -3 \\ & -w_1 + 3w_2 + 100t_{21} > 5 \\ & w_1 - 2w_2 + 100t_{13} > 0 \\ & -w_1 + 3w_2 + 100t_{23} > 1 \\ & t_{1p} + t_{2p} \leq 1, p = 1,2,3 \\ & w_1, w_2 \in \{0,1,2\}, \\ & t_{1p}, t_{2p} \in \{0,1\}, p = 1,2,3. \end{array}$$

An optimal solution to the above problem is $W_3^* = (0,2)$ and hence $Z_3 = (-4,6)$, and $WD_3 = WD_2 \cup \{W_3^*\} = \{(2,2),(1,2),(2,1),(0,2)\}$.

Iteration 4

To generate another member of WDSNDSs the following problem is solved.

$$\begin{array}{ll} \max & w_1 + w_2 \\ \text{s.t.} & w_1 - 2w_2 \leq 0 \\ & w_1 - 2w_2 + 100t_{11} > -2 \\ & -w_1 + 3w_2 + 100t_{21} > 4 \\ & w_1 - 2w_2 + 100t_{12} > -3 \\ & -w_1 + 3w_2 + 100t_{22} > 5 \\ & w_1 - 2w_2 + 100t_{13} > 0 \\ & -w_1 + 3w_2 + 100t_{23} > 1 \\ & w_1 - 2w_2 + 100t_{14} > -4 \\ & -w_1 + 3w_2 + 100t_{24} > 6 \\ & t_{1p} + t_{2p} \leq 1, p = 1,2,3,4 \\ & w_1, w_2 \in \{0,1,2\}, t_{1p}, t_{2p} \in \{0,1\}, \\ & p = 1,2,3,4. \end{array}$$

The vector $W_4^* = (1,1)$ is an optimal solution of the above model. Therefore, $Z_4 = (-1,2)$ and $WD_4 = WD_3 \cup \{W_4^*\} = \{(2,2),(1,2),(2,1),(0,2),(1,1)\}$.

Iteration 5

The model of this iteration is infeasible and the algorithm is terminated. Therefore, the set WDSNDSs= $\{Z_0, Z_1, \ldots, Z_4\}$ is the whole set of non-dominated solutions. This example has been solved in [11]. To find a $Z \in$WDSNDSs Sylva and Creme [11] solve two problems while our method solves only one problem in each iteration.

5 Conclusion

This paper proposed an algorithm to find a WDSNDSs of an MOMILP problem. In each iteration of the proposed algorithm, only one mixed integer linear programming problem is solved. According to the $\overline{\lambda} \in \Lambda$, the opinions of decision maker, the rank of the optimal solution of model (3.6) in the p^{th} iteration is better than its optimal solution in the $(p+1)^{th}$ iteration. Hence, the elements of the WDSNDSs of an MOMILP problem are ranked according to their distance form ideal point and the generated WDSNDSs can be used without any filtering procedures. Using suitable value for the parameter of the proposed model an appropriate WDSNDSs by less computational efforts is generated.

Similar to Sylva and Crema's method [11], corresponding to an MOMILP problem with s objective functions, in each iteration $s+1$ constraints and s variables are added to the mixed integer model which is solved. This increases the computational efforts to generate the WDSNDSs and can be studied in the future. The proposed method can be modified to solve mixed integer non-linear programming problem.

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