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Research Article

On solving ordinary differential equations of the first order by updating the Lagrange multiplier in variational iteration method

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Abstract

In this paper, we have proposed a new iterative method for finding the solution of ordinary differential equations of the first order. In this method we have extended the idea of variational iteration method by changing the general Lagrange multiplier which is defined in the context of the variational iteration method.This causes the convergent rate of the method increased compared with the variational iteration method. To prevent consuming large amount of the CPU time and computer memory and to control requires significant amounts of computations, the Taylor expansion of the iterative functions in each iteration are applied. Finally to extend the convergence region of the truncated series, also the Pade approximants are used. Error analysis and convergence of the method are studied. Some examples are given to illustrate the performance and efficiency of the proposed method. For comparison, the results obtained by the our method and the variational iteration method are presented.

Keywords : First order ordinary differential equations; Variational iteration method; Lagrange multiplier; Pade approximant.

1 Introduction

Differential equations involving partial differential equations and ordinary differential equations have significant role in most branches of applied sciences such as stochastic realization theory, optimal control, robust stabilization and financial mathematics, etc. Except for certain particular equations, most of the interesting equations have no closed form solution, so the numerical approaches are the only way to get some approximations to the solution.
In this paper, we consider the following scalar initial value problems of the first order

$$y'(t) = f(t, y(t)), \quad t \in [t_0, T], \qquad y(t_0) = Y_0, \tag{1.1}$$

where $f(t, y)$ is a sufficiently smooth function and Y_0, t_0, and T are given real numbers.
Although the solution of equation (1.1) can be obtained by applying classic numerical methods such as Runge-Kutta and multistep methods [4, 11], however by fast development of computer algebra systems and computer hardwares much attention of many scientists and engineers have drawn to analytical asymptotic techniques such as Adomian's decomposition method (ADM) [6, 8], homotopy perturbation method (HPM) [2, 3], homotopy analysis method (HAM) [12], and variational iteration method (VIM) [7, 9, 10], and so on.
In this paper, we extend the idea of variational iteration method by allowing the general Lagrange multiplier defined in the context of the variational iteration method [9] to change in each iteration. This causes the obtained sequence to have faster convergence than VIM.

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This paper is organized as follows:
In Section 2, we specify the new iterative equation. The sufficient conditions for convergence of the proposed method are proved in the Section 3. Section 4 includes some numerical examples to show efficiency of our technique and Section 5 ends this paper with a brief conclusion.

2 Description of the method

To propose the new iterative formula, let $y_0(t)$ be an initial approximate solution such that $y_0(t_0) = Y_0$ and $\|e_0\|_\infty$ be small in which $e_0(t) := y_0(t) - Y(t)$. As done in VIM, to improve the approximate solution the following estimate can be used in which the integral term plays a role of correction

$$y_{new}(t) = y_0(t) - \int_{t_0}^{t} \lambda(t,\tau)(y_0'(\tau) - f(\tau, y_0(\tau)))d\tau. \tag{2.2}$$

The function $\lambda(t,\tau)$ was introduced in VIM and called the general Lagrange multiplier [9]. A proper choice of $\lambda(t,\tau)$ can lead to a better approximate solution. Here the function $\lambda(t,\tau)$ is determined in a different manner than VIM.
Since $\frac{\partial f(t,y)}{\partial y} \in C(\overline{D})$, the mean value theorem gives

$$f(t, y_0(t)) = f(t, Y(t)) + e_0(t)\frac{\partial}{\partial y} f(t, \theta y_0(t) + (1-\theta)Y(t)), \tag{2.3}$$

for some $\theta \in (0, 1)$.
Substituting $y_0(t) = Y(t) + e_0(t)$ and (2.3) into (2.2) we get

$$y_{new}(t) = Y(t) + e_0(t) - \int_{t_0}^{t} \lambda(t,\tau)\frac{d}{d\tau}e_0(\tau)d\tau + \int_{t_0}^{t} \lambda(t,\tau)e_0(\tau) \frac{\partial}{\partial y} f(\tau, \theta y_0(\tau) + (1-\theta)Y(\tau))d\tau.$$

Using the integration by parts on the first integral leads to

$$y_{new}(t) = Y(t) + (1 - \lambda(t,t))e_0(t) + \int_{t_0}^{t} (\frac{\partial}{\partial \tau}\lambda(t,\tau) + \lambda(t,\tau)\frac{\partial}{\partial y} f(\tau, \theta y_0(\tau) + (1-\theta)Y(\tau))e_0(\tau)d\tau.$$

It is easily seen that by vanishing the coefficient of e_0 a better approximation is obtained. This gives the following conditions which can be used to determine $\lambda(t,\tau)$,

$$\begin{cases} \frac{\partial}{\partial \tau}\lambda(t,\tau) + \\ \lambda(t,\tau)\frac{\partial}{\partial y} f(\tau, \theta y_0(\tau) + (1-\theta)Y(\tau)) = 0, \\ \qquad\qquad \text{for some } \theta \in (0,1), \\ \\ \lambda(t,t) = 1. \end{cases}$$

The exact solution of the above first order linear IVP is

$$\lambda(t,\tau) = exp(-\int_t^{\tau} (\frac{\partial}{\partial y} f(\delta, \theta y_0(\delta) + (1-\theta)Y(\delta))d\delta).$$

As $Y(\delta)$ is unknown, we use $y_0(\delta)$ to have a value for $\lambda(t,\tau)$, hence this gives the following approximation

$$y_1(t) = y_0(t) - \int_{t_0}^{t} \lambda_0(t,\tau)(y_0'(\tau) - f(\tau, y_0(\tau)))d\tau,$$

where $\lambda_0(t,\tau) = exp(-\int_t^{\tau} \frac{\partial}{\partial y} f(\delta, y_0(\delta))d\delta)$.
Repeating the above scheme defines the following iterative method

$$y_{n+1}(t) = y_n(t) - \int_{t_0}^{t} \lambda_n(t,\tau)(y_n'(\tau) - f(\tau, y_n(\tau)))d\tau, \quad n \geq 0, \tag{2.4}$$

where $\lambda_n(t,\tau) = exp(-\int_t^{\tau} \frac{\partial}{\partial y} f(\delta, y_n(\delta))d\delta)$ and $y_0(t)$ is an initial approximation such that $y_0(t_0) = Y_0$. According to (2.4) $y_n(t_0) = Y_0$ for $n \geq 1$.
In the following theorem, another equivalent formula with (2.4) is introduced.

Theorem 2.1 *Let $f(t, y) \in C(D)$ and $\frac{\partial}{\partial y} f(t, y) \in C(\overline{D})$. The sequence of functions $\{y_n\}$ generated by (2.4) satisfies in the following relation*

$$y_{n+1}(t) = \sigma_n(t)(Y_0 + \int_{t_0}^{t} \sigma_n(\tau)^{-1}(f(\tau, y_n(\tau)) - \frac{\partial}{\partial y} f(\tau, y_n(\tau))y_n(\tau))d\tau), \tag{2.5}$$

where

$$\sigma_n(t) = exp(\int_{t_0}^{t} \frac{\partial}{\partial y} f(\delta, y_n(\delta))d\delta). \tag{2.6}$$

Proof. Applying integration by parts on (2.4), the following relation is inferred

$$y_{n+1}(t) = \lambda_n(t, t_0)Y_0 + \int_{t_0}^{t} (\frac{\partial}{\partial \tau}\lambda_n(t,\tau)y_n(\tau) + \lambda_n(t,\tau)f(\tau, y_n(\tau)))d\tau. \tag{2.7}$$

The outcome of derivative from the definition of $\lambda_n(t,\tau)$ in (2.4) is

$$\frac{\partial}{\partial \tau}\lambda_n(t,\tau) = -\frac{\partial}{\partial y}f(\tau, y_n(\tau))\lambda_n(t,\tau). \tag{2.8}$$

Inserting (2.8) in (2.7) leads to

$$\begin{aligned} y_{n+1}(t) &= \lambda_n(t,t_0)Y_0 + \textstyle\int_{t_0}^t \lambda_n(t,\tau)(\\ &\quad f(\tau, y_n(\tau)) - \tfrac{\partial}{\partial y}f(\tau, y_n(\tau))y_n(\tau))d\tau. \end{aligned} \tag{2.9}$$

According to (2.6), the function $\lambda_n(t,\tau)$ can be expressed as

$$\lambda_n(t,\tau) = \sigma_n(t)\sigma_n(\tau)^{-1}, \tag{2.10}$$

particularly $\lambda_n(t,t_0) = \sigma_n(t)$.
Using (2.10) in (2.9), the equation (2.5) is inferred.□

Since calculating the sequence $\{y_n\}$ via (2.4) or (2.5) requires significant amounts of computations, the Taylor expansion of y_n in each iteration can be used. For this purpose, let f be a function which its higher-order partial derivatives exist, and the exact solution of (1.1), Y, belongs to $C^\infty[t_0,T]$. Further let $\mathbb{P}_m$ be the space of polynomials of degree at most m and P_m be an operator from $C^\infty[t_0,T]$ to $\mathbb{P}_m$ which is defined as $P_m(y)(t) = \sum_{k=0}^m \frac{y^{(k)}(t_0)}{k!}(t-t_0)^k$. Thus we propose the following recursive relation in which $\{m_n\}_{n=0}^\infty$ is an increasing sequence of positive integer numbers

$$\begin{cases} \hat{y}_n(t) := \sigma_n(t)(Y_0 + \int_{t_0}^t \sigma_n(\tau)^{-1}(\\ \qquad f(\tau, y_n(\tau)) - \frac{\partial}{\partial y}f(\tau, y_n(\tau))y_n(\tau))d\tau), \\ y_{n+1} := P_{m_n}(\hat{y}_n)(t) \\ \qquad = \sum_{k=0}^{m_n} \frac{\hat{y}_n^{(k)}(t_0)}{k!}(t-t_0)^k, \ n \geq 0. \end{cases} \tag{2.11}$$

Although the iterated functions obtained by (2.11) are identical with some high order truncated Taylor series of the exact solution, but it is well known that their region of convergence are limited. Hence to extend the convergence region, here the Pade approximants are used which usually improve the convergence rate and accuracy of the truncated series [5]. Some softwares like Maple which support the symbolic computations can be easily used to evaluate the Pade approximants of any degree $[M/N]$ for the Taylor truncated series. Since the first few iterated solutions obtained by (2.11) have high order terms of Taylor series, The high degree diagonal Pade approximants can be calculated.

3 Error Analysis

Since $M = \max_{(t,y)\in\overline{D}}|\frac{\partial}{\partial y}f(t,y)|$, we have $|\frac{\partial}{\partial y}f(t,y)|\leq M$, for all $(t,y)\in D$. Clearly, a consequence of using mean value theorem is that $f(t,y)$ satisfies the Lipschitz condition with respect to its second variable

$$|f(t,y_1) - f(t,y_2)|\leq M|y_1 - y_2|, \tag{3.12}$$

for all $(t,y_1),(t,y_2)\in D$. Furthermore, for all $n \geq 1$, $\lambda_n(t,\tau)$ is a continuous function and

$$\begin{aligned} \lambda_n(t,\tau) &= exp(-\textstyle\int_t^\tau \frac{\partial}{\partial y}f(\delta, y_n(\delta))d\delta) \\ &\leq exp(M|t-\tau|) \leq exp(M(T-t_0)), \end{aligned} \tag{3.13}$$

for all $(t,\tau)\in[t_0,T]\times[t_0,T]$.

Theorem 3.1 *Let $f(t,y) \in D$, $\frac{\partial}{\partial y}f(t,y) \in C(\overline{D})$, and $\Lambda := \max_{\substack{t,\tau\in[t_0,T] \\ n\geq 0}} \lambda_n(t,\tau)$. Let y_0 be a continuous function and the sequence $\{y_n\}_{n=0}^\infty$ has been generated by relation (2.4) or (2.5). Let $m_0 := \min_{t\in[a,b]} y_0(t)$, $M_0 := \max_{t\in[a,b]} y_0(t)$, $[m_0,M_0]\subset [c,d]$ and $r > 0$ be a positive number such that $[m_0 - r, M_0 + r] \subset [c,d]$. Further assume that a neighborhood $N_r(y_0) = \{y \in C^1[a,b] : \|y-y_0\|_\infty\leq r\}$ exists such that $\|y_1 - y_0\|_\infty\leq (1-K)r$ where $K := 2\Lambda M(T-t_0)$. If $K < 1$, then the sequence $\{y_n\}_{n=0}^\infty \subset N_r(y_0)$ is uniformly bounded.*

Proof. First it is shown that $\{y_n\}_{n=0}^\infty$ is a subset of $N_r(y_0)$ by induction. Obviously this is valid for $n = 0$ and $n = 1$ by assumptions of the theorem. Now let $y_j \in N_r(y_0)$, $j = 0,1,\cdots,n$. Taking derivative from relation (2.4) yields that

$$\begin{aligned} y'_{n+1}(t) &= f(t, y_n(t)) - \textstyle\int_{t_0}^t \frac{\partial}{\partial t}\lambda_n(t,\tau)(\\ &\qquad y'_n(\tau) - f(\tau, y_n(\tau)))d\tau, \end{aligned} \tag{3.14}$$

for all $n \geq 0$. Hence inserting (2.8) in (3.14) and using (2.4) imply that

$$\begin{aligned} y'_{n+1}(t) &= f(t, y_n(t)) + \\ &\quad \tfrac{\partial}{\partial t}f(t, y_n(t))(y_{n+1}(t) - y_n(t)). \end{aligned} \tag{3.15}$$

Further putting (3.15) in (3.14) with $n-1$ leads to

$$\begin{aligned} y_{n+1}(t) &= y_n(t) + \textstyle\int_{t_0}^t \lambda_n(t,\tau)(f(\tau, y_{n-1}(\tau)) \\ &\quad - f(\tau, y_n(\tau)) + \tfrac{\partial}{\partial t}f(\tau, y_{n-1}(\tau)) \\ &\quad (y_n(\tau) - y_{n-1}(\tau)))d\tau. \end{aligned} \tag{3.16}$$

With the aid of (3.16), (3.13) and (3.12), we infer that

$$\begin{aligned}&|y_{n+1}(t)-y_n(t)|\\&\quad\le 2\Lambda M\int_{t_0}^t|y_n(\tau)-y_{n-1}(\tau)|d\tau\\&\quad\le 2\Lambda M(T-t_0)\|y_n-y_{n-1}\|_\infty,\end{aligned}$$

for all $n\ge 1$, and whereby

$$\begin{aligned}\|y_{n+1}-y_n\|_\infty&\le K\|y_n-y_{n-1}\|_\infty\\&\le K^n\|y_1-y_0\|_\infty.\end{aligned}$$

So

$$\begin{aligned}&\|y_{n+1}-y_0\|_\infty\\&\le \textstyle\sum_{i=0}^n\|y_{i+1}-y_i\|_\infty\\&\le \textstyle\sum_{i=0}^n K^i\|y_1-y_0\|_\infty\\&\le (1-K)\textstyle\sum_{i=0}^n K^i r\\&=(1-K^{n+1})r\le r,\end{aligned}$$

which prove that $y_{n+1}\in N_r(y_0)$.
Finally, for all $n\ge 1$ we get

$$\begin{aligned}\|y_n\|_\infty&\le\|y_n-y_0\|_\infty+\|y_0\|_\infty\\&\le\tfrac{1}{1-K}\|y_1-y_0\|_\infty+\|y_0\|_\infty.\end{aligned}\qquad\Box$$

The following theorem concerns the convergence of the sequence $\{y_n\}_{n=0}^\infty$ obtained by (2.4) or (2.5).

Theorem 3.2 *Let the assumptions of Theorem 3.1 satisfy, then the sequence $\{y_n\}_{n=0}^\infty$ generated by (2.4) or (2.5) is uniformly convergent to the exact solution of (1.1).*

Proof. The exact solution $Y(t)$ satisfies in a similar relation with (2.4),

$$\begin{aligned}Y(t)=Y(t)-&\\&\int_{t_0}^t\lambda_n(t,\tau)(Y'(\tau)-f(\tau,Y(\tau)))d\tau.\end{aligned}\tag{3.17}$$

Subtracting (3.17) from (2.4) leads to

$$y_{n+1}(t)-Y(t)=I_1(t)+I_2(t),$$

where

$$\begin{aligned}I_1(t):=\,&y_n(t)-Y(t)-\\&\int_{t_0}^t\lambda_n(t,\tau)(y_n'(\tau)-Y'(\tau))d\tau,\end{aligned}$$

$$\begin{aligned}I_2(t):=\,&\int_{t_0}^t\lambda_n(t,\tau)(\\&f(\tau,y_n(\tau))-f(\tau,Y(\tau)))d\tau.\end{aligned}$$

By applying the method of integration by parts on the integral in $I_1(t)$ and using (2.8) we have

$$\begin{aligned}I_1(t)&=\int_{t_0}^t\tfrac{\partial}{\partial\tau}\lambda_n(t,\tau)(y_n(\tau)-Y(\tau))d\tau\\&=\int_{t_0}^t\lambda_n(t,\tau)\tfrac{\partial}{\partial y}f(\tau,y_n(\tau))\\&\qquad(Y(\tau)-y_n(\tau))d\tau.\end{aligned}$$

Thus for all $t\in[t_0,T]$ we get

$$|I_1(t)|\le M\Lambda(T-t_0)\|Y-y_n\|_\infty.\tag{3.18}$$

With aid of (3.12) and the bound of $\lambda_n(t,\tau)$, the following inequality is obtained

$$|I_2(t)|\le M\Lambda(T-t_0)\|Y-y_n\|_\infty,\tag{3.19}$$

for all $t\in[t_0,T]$. Hence from (3.18) and (3.19), we infer

$$\|Y-y_{n+1}\|_\infty\le 2M\Lambda(T-t_0)\|Y-y_n\|_\infty.$$

The convergence is obtained as a consequence of the hypothesis of the theorem and the Banach fixed point theorem. $\Box$
The next theorem deals with the convergence of sequences obtained from (2.11).

Lemma 3.1 *Let f be a function of two variables which its higher-order partial derivatives exist. Suppose that the sequence $\{y_n\}_{n=0}^\infty$ is defined by (2.4) or (2.5) and the exact solution of (1.1) satisfies $Y\in C^\infty[t_0,T]$. If $y_n^{(i)}(t_0)=Y^{(i)}(t_0)$, $i=0,1,\cdots,k$, then $y_{n+1}^{(i)}(t_0)=Y^{(i)}(t_0)$, $i=0,1,\cdots,k+1$.*

Proof.
The proof is by induction. Obviously $y_{n+1}(t_0)=Y(t_0)$. Furthermore from the first derivative of (2.4)

$$\begin{aligned}y_{n+1}'(t)=&f(t,y_n(t))-\\&\int_{t_0}^t\tfrac{\partial}{\partial t}\lambda_n(t,\tau)(y_n'(\tau)-f(\tau,y_n(\tau)))d\tau,\end{aligned}\tag{3.20}$$

we have $y_{n+1}'(t_0)=f(t_0,y_n(t_0))=f(t_0,Y(t_0))=Y'(t_0)$. Assume that it is true for $k\ge 0$, that is $y_{n+1}^{(i)}(t_0)=Y^{(i)}(t_0)$, $i=0,1,\cdots,k$.

According to (3.20) and (2.8), the $(k+1)$th derivative of (2.4) can be expressed as

$$\begin{aligned}y_{n+1}^{(k+1)}(t)=&\tfrac{d^k}{dt^k}(f(t,y_n(t))-\tfrac{\partial}{\partial y}f(t,y_n(t))\times\\&\int_{t_0}^t\lambda_n(t,\tau)(y_n'(\tau)-f(\tau,y_n(\tau)))d\tau).\end{aligned}\tag{3.21}$$

By virtue of (2.4) we have

$$y_{n+1}^{(k+1)}(t) = \frac{d^k}{dt^k} f(t, y_n(t)) - \frac{d^k}{dt^k}\left(\frac{\partial}{\partial y} f(t, y_n(t))(y_{n+1}(t) - y_n(t))\right). \quad (3.22)$$

Putting $t = t_0$ into (3.22) and then using the Leibnitz formula and the induction hypothesis lead to

$$\begin{aligned} y_{n+1}^{(k+1)}(t_0) &= \frac{d^k}{dt^k} f(t_0, y_n(t_0)) \\ &= \frac{d^k}{dt^k} f(t_0, Y(t_0)). \end{aligned} \quad (3.23)$$

Consequently $y_{n+1}^{(k+1)}(t_0) = Y^{(k+1)}(t_0)$.

Theorem 3.3 *Let the assumptions of Lemma 3.1 be satisfied and $\|Y - P_n Y\|_\infty \to 0$ as $n \to \infty$. Then for some increasing sequence $\{m_n\}_{n=0}^{\infty}$ the sequence obtained by (2.11) converges to the exact solution of (1.1), that is $\|y_n - Y\|_\infty \to 0$ as $n \to \infty$.*

Proof.
There is an increasing sequence of nonnegative integers $\{m_n\}_{n=0}^{\infty}$ such that $y_{n+1} = P_{m_n} Y,\ n \geq 0$. In fact since $y_0(t_0) = Y(t_0)$, a nonnegative integer $l \geq 0$ exists such that $P_l y_0(t) = P_l Y(t)$, hence according to the Lemma 3.1 for some $m_0 \geq l+1$, we have $y_1(t) = P_{m_0}\hat{y}_0(t) = P_{m_0} Y(t)$.
By proceeding similarly, we obtain an $m_n \geq m_{n-1} + 1$ such that $y_{n+1}(t) = P_{m_n}\hat{y}_n(t) = P_{m_n} Y(t)$. Therefore by assumption of the theorem, $\|y_{n+1} - Y\|_\infty = \|P_{m_n} Y - Y\|_\infty \to 0$ as $n \to \infty$. □

4 Illustrative examples

In this Section, some examples are given to illustrate the performance and efficiency of the proposed method. For comparison, the results obtained by our method and VIM are presented. In all of the examples starting function is the constant function $y_0(t) = Y_0$ and furthermore the Pade approximants are used on the last iteration of the method.

Example 4.1 *[1] Consider the following quadratic Riccati differential equation*

$$\begin{cases} y'(t) = f(t, y(t)) = 2y(t) - y^2(t) - 1, \\ y(0) = 0. \end{cases} \quad (4.24)$$

The exact solution of equation (4.24) was found to be

$$Y(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1})),$$

which has the following Taylor expansion series about $t = 0$,

$$Y(t) = t + t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{3}t^4 - \tfrac{7}{15}t^5 - \tfrac{7}{45}t^6 + \tfrac{53}{315}t^7 + \tfrac{71}{315}t^8 + \tfrac{197}{2835}t^9 - \tfrac{1213}{14175}t^{10} - \tfrac{2419}{22275}t^{11} - \tfrac{2051}{66825}t^{12} + \tfrac{263701}{6081075}t^{13} + \tfrac{2223841}{42567525}t^{14} + O(t^{15}).$$

We consider $y_0(t) = 0$ as initial approximation and we take $m_1 = 2; m_n = 2m_{n-1} + 2, n \geq 2$. According to (2.11), the three first iterations are as follows:

$$\begin{aligned} y_1(t) &= t + t^2, \\ y_2(t) &= t + t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{3}t^4 - \tfrac{7}{15}t^5 - \tfrac{7}{45}t^6, \\ y_3(t) &= t + t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{3}t^4 - \tfrac{7}{15}t^5 - \tfrac{7}{45}t^6 + \tfrac{53}{315}t^7 + \tfrac{71}{315}t^8 + \tfrac{197}{2835}t^9 - \tfrac{1213}{14175}t^{10} - \tfrac{2419}{22275}t^{11} \\ &\quad - \tfrac{2051}{66825}t^{12} + \tfrac{263701}{6081075}t^{13} + \tfrac{2223841}{42567525}t^{14}. \end{aligned}$$

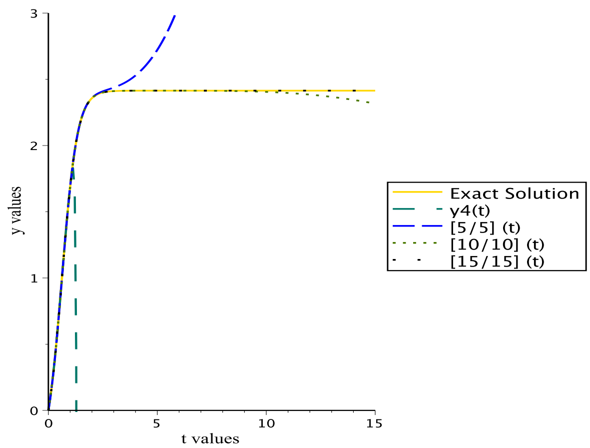

Figure 1: The plot of approximate solutions and exact solution for Example 4.1.

As can be seen, the fourteen first terms of third iteration exactly matches the Taylor expansion of $Y(t)$. Whereas the third iteration of He's variational iteration method is [1]

$$t + t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{3}t^4 - \tfrac{7}{15}t^5 - \tfrac{7}{45}t^6 + \tfrac{53}{315}t^7 + \tfrac{673}{2520}t^8 + O(t^9).$$

It should be noted that due to consuming large amount of the CPU time and computer memory,

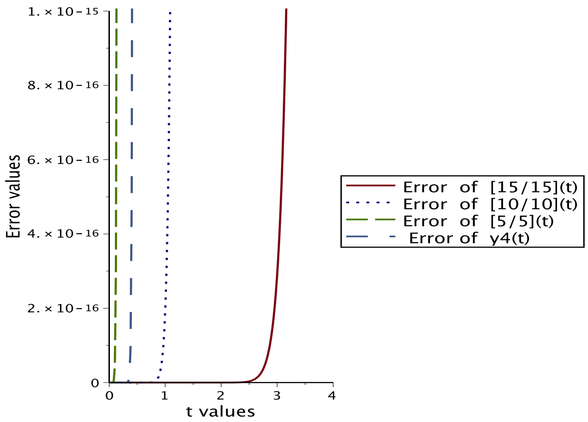

Figure 2: The plot of errors for Example 4.1.

calculating the more iterations of He's method to obtain higher accuracy approximations is almost impossible [1]. Comparing the two last equations confirms accuracy and convergence rate of the new method.
Figure 2 shows the plot of errors for the third iteration of the proposed method $y_4(t)$ and Pade approximantns $[\frac{5}{5}](t)$, $[\frac{10}{10}](t)$ and $[\frac{15}{15}](t)$ which are calculated from $y_4(t)$. To compare, the approximate solutions and the exact solution have been plotted in Figure 1.

Example 4.2 *[2] Consider the following nonlinear differential equation*

$$\begin{cases} y'(t) + (t^2-1)y(t) = t^2e^{-2t}y^3(t), \\ y(0) = 1. \end{cases} \tag{4.25}$$

The exact solution is $y(t) = e^t$. According to (2.11), with $y_0(t) = 1$ as initial approximation and $m_1 = 5, m_n = 2m_{n-1}+3, n \geq 2$, the following expansions are obtained which show the fast convergence.

$$\begin{aligned}
&y_1(t) = 1 + t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 + \tfrac{1}{24}\,t^4,\\
&y_2(t) = 1 + t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 + \tfrac{1}{24}\,t^4 + \tfrac{1}{120}\,t^5\\
&+\tfrac{1}{720}\,t^6 + \tfrac{1}{5040}\,t^7 + \tfrac{1}{40320}\,t^8 + \tfrac{1}{362880}\,t^9+\\
&+\tfrac{1}{3628800}\,t^{10} + \tfrac{1}{39916800}\,t^{11} + \tfrac{1}{479001600}\,t^{12},\\
&y_3(t) = 1 + t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 + \tfrac{1}{24}\,t^4 + \tfrac{1}{120}\,t^5\\
&+\tfrac{1}{720}\,t^6 + \tfrac{1}{5040}\,t^7 + \tfrac{1}{40320}\,t^8 + \tfrac{1}{362880}\,t^9+\\
&\tfrac{1}{3628800}\,t^{10} + \tfrac{1}{40320}\,t^8 + \tfrac{1}{39916800}\,t^{11}+\\
&\tfrac{1}{479001600}\,t^{12} + \tfrac{1}{6227020800}\,t^{13}+\\
&+\tfrac{1}{87178291200}\,t^{14} + \tfrac{1}{1307674368000}\,t^{15}+\\
&\tfrac{1}{20922789888000}\,t^{16} + \tfrac{1}{355687428096000}\,t^{17}+\\
&+\tfrac{1}{6402373705728000}\,t^{18}+\\
&\tfrac{1}{121645100408832000}\,t^{19}+\\
&\tfrac{1}{2432902008176640000}\,t^{20}+\\
&\tfrac{1}{51090942171709440000}\,t^{21}\\
&+\tfrac{1}{1124000727777607680000}\,t^{22}+\\
&\tfrac{1}{25852016738884976640000}\,t^{23}+\\
&+\tfrac{1}{620448401733239439360000}\,t^{24}+\\
&\tfrac{1}{15511210043330985984000000}\,t^{25}+\\
&\tfrac{1}{403291461126605635584000000}\,t^{26}+\\
&+\tfrac{1}{10888869450418352160768000000}\,t^{27}+\\
&\tfrac{1}{304888344611713860501504000000}\,t^{28}.
\end{aligned}$$

Taylor series of the third iteration of VIM with starting function $y_0(t) = 1$ is

$$1 + t + \tfrac{1}{2}\,t^2 + \tfrac{1}{6}\,t^3 - \tfrac{23}{24}\,t^4 + \tfrac{121}{120}\,t^5 - \tfrac{1091}{720}\,t^6 + \tfrac{1387}{720}\,t^7 + O(t^8). \tag{4.26}$$

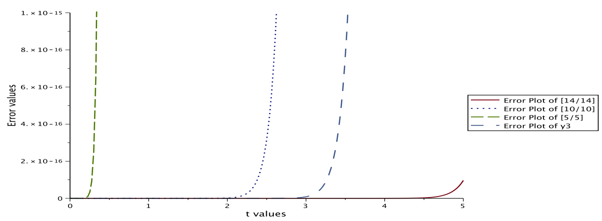

Figure 3: The plot of errors for Example 4.2.

Comparing the two last equations confirms accuracy and convergence rate of the new method. Figure 3. shows the plot of errors for the third iteration of the proposed method $y_3(t)$ and Pade approximantns $[\frac{5}{5}](t)$, $[\frac{10}{10}](t)$ and $[\frac{14}{14}](t)$ which are calculated from $y_3(t)$.

Example 4.3 *[2] Consider the following nonlinear differential equation*

$$\begin{cases} y'(t) = \frac{1}{1+y^2(t)} + y^2(t) + sin(t), \\ y(0) = 0. \end{cases} \tag{4.27}$$

The exact solution is $y(t) = tan(t)$. According to (2.11), with $y_0(t) = 0$ as initial approximation and $m_1 = 3, m_n = 2m_n + 5, n \geq 1$, the following expansion is obtained in the third iteration which shows the fast convergence.

$$y_1(t) = t + \frac{1}{3}t^3,$$
$$y_2(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11}$$
$$y_3(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} + \frac{21844}{6081075}t^{13} + \frac{929569}{638512875}t^{15} + \frac{6404582}{10854718875}t^{17} + \frac{443861162}{1856156927625}t^{19} + \frac{18888466084}{194896477400625}t^{21} + \frac{113927491862}{2900518163668125}t^{23} + \frac{58870668456604}{3698160658676859375}t^{25} + \frac{8374643517010684}{1298054391195577640625}t^{27}.$$

Taylor series of the third iteration of VIM with starting function $y_0(t) = 0$ is

$$1 + t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \frac{1382}{155925}t^{11} - \frac{144476}{6081075}t^{13} + O(t^{15}). \quad (4.28)$$

Comparing the two last equations confirms accuracy and convergence rate of the new method. Figure 4. shows the plot of errors for the third iteration of the proposed method $y_3(t)$ and Pade approximantns $[\frac{5}{5}](t)$, $[\frac{10}{10}](t)$ and $[\frac{14}{14}](t)$ which are calculated from $y_3(t)$.

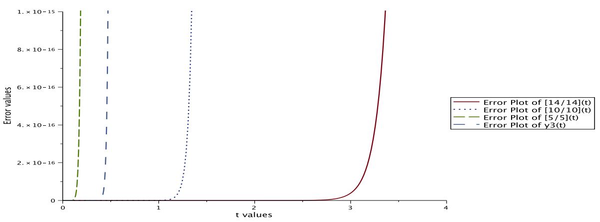

Figure 4: The plot of errors for example 4.3.

5 Conclusion

In this study, an iteration method is introduced for finding the solution of ordinary differential equations with initial condition. In the first recursive relation the idea of VIM has been extended by updating the Lagrange multipliers in each iteration. To retain proper CPU time and computer memory, the Taylor expansion of the iterative function is added in the other recursive relation. Also, at the end of the iterations the Pade approximant is used to extend the region of validity of the last iterative function. The convergence of the sequence obtained by each of the relations are discussed separately. The capability of the method is successfully shown with implementation of the method on some examples. By comparison of the numerical results obtained by the present method and the VIM, the performance and superiority of the method have been confirmed.

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