Characterization of $L_2(p^2)$ by NSE

H. Parvizi Mosaed *†, A. Tehranian †

Abstract

Let $G$ be a group and $\pi(G)$ be the set of primes $p$ such that $G$ contains an element of order $p$. Let $\text{nse}(G)$ be the set of the number of elements of the same order in $G$. In this paper, we prove that the simple group $L_2(p^2)$ is uniquely determined by $\text{nse}(L_2(p^2))$, where $p \in \{11, 13\}$.

Keywords: Element order; The set of the number of elements of the same order; Simple $K_n$-group; Projective special linear group.

1 Introduction

Let $G$ be a group and $\pi(G)$ be the set of primes $p$ such that $G$ contains an element of order $p$ and $\pi_e(G)$ be the set of element orders of $G$. If $k \in \pi_e(G)$, then we denote by $m_k$ or $m_k(G)$, the number of elements of order $k$ in $G$. Let $\text{nse}(G) = \{m_k \mid k \in \pi_e(G)\}$.

In 1987, Thompson posed a problem related to algebraic number fields as follows: (Problem 12.37 of [16])

**Thompson Problem:** Let $G$ and $H$ be two finite groups with $T(G) = T(H)$, where $T(G) = \{(k, m_k) \mid k \in \pi_e(G)\}$. If $G$ is solvable, is it true that $H$ is also necessarily solvable?

Up to now, no one can solve this problem completely even give a counterexample. It is easy to see that if $G$ and $H$ are two finite groups with $T(G) = T(H)$, then $|G| = |H|$ and $\text{nse}(G) = \text{nse}(H)$. Studies on characterizations related to $\text{nse}(G)$ started by Shao et al. In [19], they proved that if $G$ is a simple $K_4$-group, then $G$ is characterizable by $\text{nse}(G)$ and $|G|$ (The simple group $G$ is called simple $K_n$-group if $|\pi(G)| = n$). Following this result, in [4, 14], it is proved that the groups $A_{12}$ and $A_{13}$ are characterizable by $\text{nse}(G)$ and $|G|$. In [10], the authors put forward the following problem:

**Problem:** Let $G$ be a group such that $\text{nse}(G) = \text{nse}(L_2(q))$, where $q$ is a prime power. Is $G$ isomorphic to $L_2(q)$?

They proved that the groups $L_2(q)$, where $q \in \{7, 8, 11, 13\}$ are characterizable by $\text{nse}(L_2(q))$. Also in [9, 11, 12, 13, 18, 20], it is proved that the groups $L_2(q)$, where $q \in \{2, 3, 4, 9, 16, 25, 49\} \cup \{r : r < 100 \text{ is a prime}\}$ are characterizable by $\text{nse}(L_2(q))$. In this paper, we show that this problem has an affirmative answer for the case $q = p^2$, where $p \in \{11, 13\}$. In fact, we prove the following main theorem:

**Main Theorem.** Let $G$ be a group such that $\text{nse}(G) = \text{nse}(L_2(p^2))$, where $p \in \{11, 13\}$. Then $G \cong L_2(p^2)$.

2 Preliminaries

For a natural number $n$, by $\pi(n)$, we mean the set of all prime divisors of $n$, so it is obvious that if $G$ is a finite group, then $\pi(G) = \pi(|G|)$. A
Sylow $p$-subgroup of $G$ is denoted by $G_p$ and by $n_p(G)$, we mean the number of Sylow $p$-subgroups of $G$. If there is no ambiguity, then we write $n_p$ instead of $n_p(G)$. Also, the largest element of $\pi_e(G_p)$ is denoted by $exp(G_p)$. Moreover, we denote by $\varphi$ the Euler totient function and by $(a, b)$ the greatest common divisor of integers $a$ and $b$.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

**Lemma 2.1** [3] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.

**Lemma 2.2** [20] Let $G$ be a group containing more than two elements. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

**Lemma 2.3** [15] Let $G$ be a finite group and $p \in \pi(G) - \{2\}$. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^s m$, where $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 2.4** [5] Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and $h_m$ be the number of Hall $\pi$-subgroups of $G$. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, \ldots, s\}$:

- $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some $p_j$.
- The order of some chief factor of $G$ is divisible by $q_i^{\beta_i}$.

**Lemma 2.5** [20] Let $G$ be a group containing more than two elements. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

**Lemma 2.6** [15] Let $G$ be a finite group and $p \in \pi(G) - \{2\}$. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^s m$, where $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 2.7** [5] Let $G$ be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \ldots, p_r\}$ and $h_m$ be the number of Hall $\pi$-subgroups of $G$. Then $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, \ldots, s\}$:

- $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some $p_j$.
- The order of some chief factor of $G$ is divisible by $q_i^{\beta_i}$.

**Lemma 2.8** [6] Let $G$ be a solvable group and $\pi$ be any set of primes. Then

- $G$ has a Hall $\pi$-subgroup.
- If $H$ is a Hall $\pi$-subgroup of $G$ and $V$ is any $\pi$-subgroup of $G$, then $V \leq H^\pi$ for some $g \in G$. In particular, the Hall $\pi$-subgroups of $G$ form a single conjugacy class of subgroups of $G$.

**Lemma 2.9** Let $S$ be a simple $K_n$-group, where $n \in \{3, 4, 5, 6\}$. If $|S| = 2^3 3^i 5^j 7^k 13^l 17^m$, then $S$ is isomorphic to one of the following groups: $A_5$, $L_2(7)$, $L_2(13)$, $L_2(16)$, $L_2(169)$.

**Proof.** Let $S$ be a simple $K_3$-group. Then by [7], $S$ is isomorphic to one of the following groups: $A_5$, $A_6$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$. If $S \cong A_6$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$, then $3^2 \mid |S|$, which is a contradiction. So $S \cong A_5$ or $L_2(7)$.

- Let $S$ be a simple $K_4$-group. Then by [17], $S$ is isomorphic to one of the following groups:

  - $A_7$, $A_8$, $A_9$, $A_{10}$, $M_{11}$, $M_{12}$, $J_3$, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_2(243)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_5^2(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, $3D_4(2)$, $2F_4(2)$;
  - $L_2(r)$, where $r$ is a prime, $r^2 - 1 = 2^a 3^b 5^c$, $v > 3$ is a prime, $a, b, c \in \mathbb{N}$;
  - $L_2(2^m)$, where $m$, $(2^m - 1)$ and $(2^m + 1)/3$ are primes greater than 3;
  - $L_2(3^m)$, where $m$, $(3^m - 1)/2$ and $(3^m + 1)/4$ are odd primes.

If $S$ is isomorphic to one of the groups of parts $(1),(3),(4)$ except $L_2(16)$, then $2^9 \mid |S|$ or $3^2 \mid |S|$.
or $5^2 \mid |S|$, which is a contradiction. If $S \cong L_2(r)$, where $r$ is a prime, $r^2 - 1 = 2^a3^b \nu^c$, $\nu > 3$ is a prime, $a, b, c \in \mathbb{N}$, then by $|S|$, $r \in \{5, 7, 13, 17\}$ and hence $r = 13$. So we conclude that $S \cong L_2(13)$ or $L_2(16)$.

- Let $S$ be a simple $K_5$-group. Then by [8], $S$ is isomorphic to one of the following groups:

  - $L_2(q)$, where $q$ satisfies $|\pi(q^2 - 1)| = 4$;
  - $L_2(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3 - 1))| = 4$;
  - $U_3(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3 + 1))| = 4$;
  - $O_5(q)$, where $q$ satisfies $|\pi(q^2 - 1)| = 4$;
  - $S_2(2^{2m+1})$, where $|\pi((2^{2m+1} - 1)(2^{4m+2} + 1))| = 4$;

- One of the 30 other simple groups:

  - $A_{11}$, $A_{12}$, $M_{22}$, $3S$, $HS$, $He$, $M^2L$, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_6(2)$, $O_7(3)$, $S_6(3)$, $S_6(2)$, $U_4(4)$, $U_4(5)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8^+(3)$, $O_8^-(2)$, $3D_4(3)$, $G_2(4)$, $G_2(7)$, $G_2(8)$.

If $S$ is isomorphic to one of the groups of part (6), then $3^2 \mid |S|$, which is a contradiction. If $S \cong L_2(q)$, then by $|S|$, $q \in \{2, 3, 4, 5, 7, 8, 13, 16, 17, 169\}$ and since $|\pi(q^2 - 1)| = 4$, we conclude a contradiction. Similarly, we conclude that $S$ is not isomorphic to one of the groups of parts (2),(3),(4),(5).

- Let $S$ be a simple $K_6$-group. Then by [8], $S$ is isomorphic to one of the following groups:

  - $L_2(q)$, where $q$ satisfies $|\pi(q^2 - 1)| = 5$;
  - $L_3(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3 - 1))| = 5$;
  - $L_4(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3-1)(q^4-1))| = 5$;
  - $U_3(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3+1))| = 5$;
  - $U_4(q)$, where $q$ satisfies $|\pi((q^2 - 1)(q^3+1)(q^4-1))| = 5$;
  - $O_5(q)$, where $q$ satisfies $|\pi(q^2 - 1)| = 5$;
  - $G_2(q)$, where $q$ satisfies $|\pi(q^6 - 1)| = 5$;
  - $S_2(2^{2m+1})$, where $|\pi((2^{2m+1} - 1)(2^{4m+2} + 1))| = 5$;
  - $R(3^{2m+1})$, where $|\pi((3^{2m+1} - 1)(3^{6m+3} + 1))| = 5$;

- One of the 38 other simple groups:

  - $A_{13}$, $A_{14}$, $A_{15}$, $A_{16}$, $M_{23}$, $M_{24}$, $J_1$, $Suz$, $Ru$, $Co_2$, $Co_3$, $F_{422}$, $HN$, $L_5(7)$, $L_6(3)$, $L_7(2)$, $O_7(4)$, $O_7(5)$, $O_7(7)$, $O_9(3)$, $S_6(4)$, $S_6(5)$, $S_6(7)$, $S_8(3)$, $U_5(4)$, $U_5(5)$, $U_6(9)$, $U_7(2)$, $F_4(2)$, $O_{28}^+(4)$, $O_{28}^-(5)$, $O_{28}^+(7)$, $O_{10}^+(2)$, $O_{28}^-(3)$, $O_{10}^-(2)$, $3D_4(4)$, $3D_4(5)$.

If $S$ is isomorphic to one of the groups of part (10), then $3^2 \mid |S|$, which is a contradiction. If $S \cong L_2(q)$, then by $|S|$, $q \in \{2, 3, 4, 5, 7, 8, 13, 16, 17, 169\}$ and since $|\pi(q^2 - 1)| = 5$, we conclude $S \cong L_2(169)$. Similarly, we conclude that $S$ is not isomorphic to one of the groups of parts (2)-(9).

**Lemma 2.10** Let $G$ be a group such that $nse(G) = nse(L_2(p^2))$, where $p \in \{11, 13\}$. Then $G$ is finite and for every $i \in \pi_e(G)$,

$$
\left\{ \begin{array}{l}
\varphi(i) | m_i \\
| i | \sum d_i | m_d
\end{array} \right.
$$

and if $i > 2$, then $m_i$ is even.

**Proof.** The proof is straightforward according to Lemmas 2.1 and 2.5.

**3 Proof of the Main Theorem**

First, we prove the main theorem for the case $p = 13$. If $G$ is a group such that $nse(L_2(13^2)) = nse(G)$, then by [2], we have $nse(L_2(13^2)) = nse(G) = \{1, 14365, 28560, 28730, 56784, 57460, 86190, 172380, 227136, 344760, 908544\}$.

In the following lemma, we prove some basic properties of group $G$:

**Lemma 3.1** If $\{2, 3, 5, 7, 13, 17\} \subseteq \pi(G)$, then

- $m_2 = 14365$, $m_3 = 28560$, $m_5 = 56784, 908544$, $m_7 = 86190$, $m_{13} = 28560$, $m_{17} = 227136$;
- $\{17^2, 13^4, 7^2, 5^3, 3^2, 10^2, 36, 17, 7, 13, 13, 17\} \cap \pi_e(G) = \emptyset$;
- $|G_{17}| = 17$, $|G_{13}| = 13^4$, $|G_7| = 7^2$, $|G_5| = 5$, $|G_3| = 3^2$. 
Proof. According to Lemma 2.10 and $nse(G)$, the proof of parts (1) and (2) is obvious. So it is enough to prove part (3). Since $17^2 \not\in \pi_e(G)$, we conclude that $exp(G_{17}) = 17$ and hence, Lemma 2.1 implies that $|G_{17}| = 17$. Thus $G_{17}$ is cyclic and $m_{17} = 17^2/\phi(17) = 14196$.

Since $13^4 \not\in \pi_e(G)$, we conclude that $exp(G_{13}) \in \{13, 13^2, 13^3\}$. If $exp(G_{13}) = 13^3$, then Lemma 2.1 implies that $|G_{13}| = 13^3$ and hence, $G_{13}$ is cyclic and $m_{13} = 13^3/\phi(13^3) = 85$ or 448. But since every cyclic group of order 13 has only one subgroup of order 13, we conclude that $m_{13} \leq 12.448$, which is a contradiction. If $exp(G_{13}) = 13^2$, then Lemma 2.1 implies that $|G_{13}| = 13^2$ and hence, $G_{13}$ is cyclic and $n_{13} = m_{13}/\phi(13^2) \in \{364, 1105, 456, 2210, 5824\}$, which is a contradiction by Sylow’s theorem. So we conclude that $exp(G_{13}) = 13$ and hence, Lemma 2.1 implies that $|G_{13}| = 13^4$.

Since $7^2 \not\in \pi_e(G)$, Lemma 2.1 implies that $|G_7| = 7^2$.

Since $5^3 \not\in \pi_e(G)$, we conclude that $exp(G_5) \in \{5, 5^2\}$. If $exp(G_5) = 5^2$, then Lemma 2.1 implies that $|G_5| = 5^2$ and hence, $G_5$ is cyclic and $n_5 = m_5/\phi(5^2) = 8619$. But since every cyclic group of order 5 has only one subgroup of order 5, we conclude that $m_{15} \leq 4.8619$, which is a contradiction. So we conclude that $exp(G_5) = 5$ and hence, Lemma 2.1 implies that $|G_5| = 5$. Thus $G_5$ is cyclic and $n_5 = m_5/\phi(5) = 14196$ or 227136.

Since $3^2 \not\in \pi_e(G)$, we conclude that $exp(G_3) \in \{3, 3^2\}$. If $exp(G_3) = 3^2$, then Lemma 2.1 implies that $|G_3| = 3^2$. Since $3^2 \div m_3$, Lemma 2.6 implies that $G_3$ is cyclic and hence, $n_3 = m_3/\phi(3^2) = 14365$ or 57460. If $exp(G_3) = 3$, then Lemma 2.1 implies that $|G_3| = 3$ and hence, $G_3$ is cyclic and $n_3 = m_3/\phi(3) = 14365$. So $|G_3| = 3^2$. Now we are going to prove that $G \cong L_2(13^2)$. We have divided the proof into a sequence of lemmas:...

Lemma 3.2 $\pi(G) = \{2, 3, 5, 7, 13, 17\}$.

Proof. Since 14365 is the only odd number $nse(G) - \{1\}$, by Lemma 2.10, $2 \in \pi(G)$. Let $2 \not\in r \in \pi(G)$. Then by Lemma 2.10, $r \mid (1 + m_r)$ and $\varphi(r) \mid m_r$. Thus we conclude that $r \in \{3, 5, 7, 11, 13, 17\}$. If $11 \in \pi(G)$, then by Lemma 2.10, $m_{11} = 172380$. On the other hand, $22 \not\in \pi_e(G)$ because otherwise by Lemma 2.10, $\varphi(22) \mid m_{22}$ and $22 \mid (1 + m_2 + m_{11} + m_{22})$, which is a contradiction. Thus $G_{11}$ acts fixed point freely on the set of elements of order 2 by conjugation and hence $|G_{11}| \mid m_2$, which is a contradiction. Therefore $11 \not\in \pi(G)$. So we conclude that $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$.

- If $\pi(G) = \{2\}$, then by Lemma 3.1, $2^{10} \not\in \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots , 2^{9}\}$. Hence $|nse(G)| \leq 10$, which is a contradiction.

- If $\pi(G) = \{2, 7\}$, then by Lemma 3.1, $2^{10}, 7^2 \not\in \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots , 2^{9}\} \cup \{7, 7.2, \ldots , 7.2^9\}$, which implies that $|G| = 2^k \cdot 7^l = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 44760k_8 + 908544k_9$, where $l, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ and $k_9$ are non-negative integers and $l \leq 2$ and $0 \leq k_1 + \cdots + k_9 < 9$. But it is easy to check that this equation has no solution.

- If $\pi(G) = \{2, 13\}$, then by Lemma 3.1, $2^{10}, 13^2, 13.2^8 \not\in \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots , 2^{9}\} \cup \{13, 13.2, \ldots , 13.2^7\}$, which implies that $|G| = 2^k \cdot 13^l = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 44760k_8 + 908544k_9$, where $l, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ and $k_9$ are non-negative integers and $l \leq 4$ and $0 \leq k_1 + \cdots + k_9 \leq 7$. It is easy to check that this equation has no solution, which is a contradiction.

- If $\pi(G) = \{2, 7, 13\}$, then by Lemma 3.1, $7.13 \not\in \pi_e(G)$. Thus $G_7$ acts fixed point freely on the set of elements of order 13 by conjugation and hence, $|G_{13}| \mid m_{13}$. Therefore $|G_{13}| = 7$ and $n_7 = m_7/\phi(7) = 14365$. Since $n_7 \mid |G|$, we conclude that $17 \in \pi(G)$, which is a contradiction.

- If $3 \in \pi(G)$, then by Lemma 3.1, $n_3 \in \{14365, 57460\}$. Since $n_3 \mid |G|$, we conclude that $17 \in \pi(G)$.

- If $5 \in \pi(G)$, then by Lemma 3.1, $n_5 \in \{14196, 227136\}$. Since $n_5 \mid |G|$, we conclude that $3 \in \pi(G)$. Thus according to the previous case, we have $17 \in \pi(G)$.

According to the above statement, in each case, we have $17 \in \pi(G)$. By Lemma 3.1, we know that $n_{17} = 14196$ and since $n_{17} \mid |G|$, we conclude that $14196 \mid |G|$. Thus $\{2, 3, 7, 13, 17\} \subseteq \pi(G)$. On the other hand, by Lemma 3.1, $n_3 \in \{14365, 57460\}$. Since $n_3 \mid |G|$, we conclude that $5 \mid |G|$. Consequently, $\pi(G) = \{2, 3, 5, 7, 13, 17\}$.

Lemma 3.3 $|G| = 2^k \cdot 3.5.7.13^2.17$, where $k \leq 4$.

Proof. By Lemma 3.1, we have $|G_{17}| = 17$ and $|G_{31}| = 5$. Now we prove that $|G_{13}| = 13^2$, $|G_{7}| = 7$, $|G_{5}| = 5$, and...
By Lemma 3.1, we have $3.17 \not\in \pi_e(G)$. Thus $G_3$ acts fixed point freely on the set of elements of order 17 by conjugation and hence, $|G_3| = m_{17}$. So $|G_3| = 3$ and $n_3 = 14365$. According to Lemma 3.1, $\{7.13, 13.17\} \cap \pi_e(G) = \emptyset$ and hence, similar argument implies that $|G_7| = 7$, $n_7 = 14365$ and $|G_{13}| = 13^2$.

If $5.17 \not\in \pi_e(G)$, then $G_5$ acts fixed point freely on the set of elements of order 17 by conjugation and hence, $|G_5| = m_{17}$, which is a contradiction. Thus $85 = 5.17 \in \pi_e(G)$ and $m_{85} = 908544$. On the other hand, if $P$ and $Q$ are Sylow 5-subgroups of $G$, then it is obvious that $C_G(P)$ and $C_G(Q)$ are conjugate in $G$. So $m_{85} = \varphi(85)n_5k$, where $k$ is the number of cyclic subgroups of order 17 in $C_G(P)$. Hence $64n_5 = m_{85}$ and since $n_5 \in \{1, 4196, 227136\}$, we conclude that $n_5 = 4196$ and $m_5 = 56784$. Similarly, we conclude that $10 \not\in \pi_e(G)$. Thus $G_2$ acts fixed point freely on the set of elements of order 5 by conjugation and hence, $|G_2| = m_5$. So we conclude that $|G_2| = 2^4$.

**Lemma 3.4** $G$ is unsolvable.

**Proof.** If $G$ is solvable, then by Lemma 2.8, $G$ has a Hall $\pi$-subgroup $H$, where $\pi = \{3, 5, 7, 13, 17\}$ and all Hall $\pi$-subgroups of $G$ are conjugate and the number of Hall $\pi$-subgroups of $G$ is $|G : N_G(H)| = 2^4$. Since $H$ is solvable, according to Lemma 2.7, there are nonnegative integers $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t, \delta_1, \ldots, \delta_u$ such that

$$n_{17}(H) = 3^\alpha \cdot 5^\beta \cdot 7^\gamma \cdot 13^\delta \cdot 17^e,$$

where

$$3^\alpha \equiv 1 \pmod{17}, 5^\beta \equiv 1 \pmod{17}, 7^\gamma \equiv 1 \pmod{17}, 13^\delta \equiv 1 \pmod{17}.$$

Also, by Lemma 3.3, we know that $|G| = 2^k \cdot 3.5.7.13.17^2.17$, where $k \leq 4$. Thus $\sum_{j=1}^e \alpha_j \leq 1$, $\sum_{j=1}^f \beta_j \leq 1$, $\sum_{j=1}^i \gamma_j \leq 1$, $\sum_{j=1}^u \delta_j \leq 2$ which implies that $n_{17}(H) = 1$. So $16 \leq m_{17}(G) \leq (2^4 \cdot 16)$, but we have $m_{17} = 227136$, which is a contradiction.

**Lemma 3.5** $G \cong L_2(13^2)$.

**Proof.** Since $G$ is a finite unsolvable group, there is a normal series $1 \leq N \leq M \leq G$, such that $N$ is a maximal solvable normal subgroup of $G$ and $M/N$ is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups. Let $M/N \cong S_1 \times \ldots \times S_r$, where $S_1$ is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$. Since $1 \leq N \leq M \leq G$ and $|G| = 2^k \cdot 3.5.7.13.17^2.17$, where $k \leq 4$, we conclude that $r = 1$ and $M/N$ is a simple $K_n$-group, where $n \in \{3, 4, 5, 6\}$. So by Lemma 2.9, $M/N$ is isomorphic to one of the following groups: $A_5$, $L_2(7)$, $L_2(13)$, $L_2(16)$, $L_2(169)$.

- If $M/N \cong A_5$, then $(G/N)/(A/N) \cong G/A \leq \text{Aut}(M/N) \cong S_5$, where $C_{G/N}(M/N) = A/N$. Since $M/N \cong A_5$ is an unsolvable simple group, we conclude that $M/N \cap A/N = 1$ and hence, $M/N \times A/N \leq G/N$, therefore $|M/N| = |G/A|$. So we conclude that $G/A \cong A_5$ or $S_5$. Hence $7.13^2.17 = |A| = 2^2 \cdot 7 \cdot 13^2.17$. Thus by Sylow’s theorem, $n_{17}(A) = 1, 52$. Since $A \leq G$, we conclude that $n_{17}(A) = n_{17}(G)$. Therefore $m_{17}(G) \in \{16, 832\}$, which is a contradiction. Similarly, we can prove that $G \not\cong L_2(7)$, $L_2(13)$, $L_2(16)$.

- If $M/N \cong L_2(169)$, then $(G/N)/(A/N) \cong G/A \leq \text{Aut}(M/N)$, where $C_{G/N}(M/N) = A/N$. Since $M/N \cong L_2(169)$ is an unsolvable simple group, we conclude that $M/N \cap A/N = 1$, hence $M/N \times A/N \leq G/N$, therefore $|M/N| = |G/A|$. So we conclude that $2^4 \cdot 3.5.7.13^2.17 = |M/N|$.

- If $|G| = 2^3 \cdot 3.5.7.13^2.17$, then we know that $\pi_e(Au(L_2(169))) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 17, 21, 24, 26, 28, 34, 42, 56, 84, 85, 168, 170\}$. Now we have $56 \not\in \pi_e(G)$ because otherwise $m_{56} \in \{28560, 56784, 227136, 908544\}$ and similar to Lemma 3.3, $m_{56} = \varphi(56)n_7k$, thus we conclude that $n_7 \mid m_{56}$, which is a contradiction. Hence $56 \not\in \pi_e(G)$. So $168 \not\in \pi_e(G)$. Similarly, $10, 34, 170 \not\in \pi_e(G)$. So $|\pi_e(G)| \leq 19$. Thus $|G| = 2^4 \cdot 3.5.7.13^{19} = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 344760k_8 + 908544k_9$, where $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9$ are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 8$. It is easy to check that this equation has no solution, which is a contradiction. So we conclude that $|G| = 2^3 \cdot 3.5.7.13^2.17$ and since
$L_2(169) \leq G \leq \text{Aut}(L_2(169))$, we conclude that $G \cong L_2(169)$.

By the same manner, we can prove the main theorem for $p = 11$ as well. We omit the details for the sake of convenience.

References


Hosein Parvizi Mosaed received his MS degree in Pure Mathematics from Bu-Ali Sina University, Hamedan, Iran, in 2009 and his PhD degree in Algebra from Islamic Azad University, Science and Research Branch, Tehran, Iran in 2014. His research interests include finite and infinite groups.

Abolfazl Tehranian is Professor in the Department of Mathematics at Science and Research Branch, Islamic Azad University, Tehran, Iran. His primary areas of research are Algebra, Commutative Algebra, Linear Algebra, Group Theory and Graphs.