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Research Article

Characterization of $\mathbf{L_2(p^2)}$ by NSE

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Abstract

Let G be a group and $\pi(G)$ be the set of primes p such that G contains an element of order p. Let $nse(G)$ be the set of the number of elements of the same order in G. In this paper, we prove that the simple group $L_2(p^2)$ is uniquely determined by $nse(L_2(p^2))$, where $p \in \{11, 13\}$.

Keywords : Element order; The set of the number of elements of the same order; Simple K_n-group; Projective special linear group.

1 Introduction

Let G be a group and $\pi(G)$ be the set of primes p such that G contains an element of order p and $\pi_e(G)$ be the set of element orders of G. If $k \in \pi_e(G)$, then we denote by m_k or $m_k(G)$, the number of elements of order k in G. Let $nse(G) = \{m_k \mid k \in \pi_e(G)\}$.

In 1987, Thompson posed a problem related to algebraic number fields as follows: (Problem 12.37 of [16])

Thompson Problem: Let G and H be two finite groups with $T(G) = T(H)$, where $T(G) = \{(k, m_k) \mid k \in \pi_e(G)\}$. If G is solvable, is it true that H is also necessarily solvable?

Up to now, no one can solve this problem completely even give a counterexample. It is easy to see that if G and H are two finite groups with $T(G) = T(H)$, then $|G| = |H|$ and $nse(G) = nse(H)$. Studies on characterizations related to $nse(G)$ started by Shao et al. In [19], they proved that if G is a simple K_4-group, then G is characterizable by $nse(G)$ and $|G|$ (The simple group G is called simple K_n-group if $|\pi(G)| = n$). Following this result, in [4, 14], it is proved that the groups A_{12} and A_{13} are characterizable by $nse(G)$ and $|G|$. In [10], the authors put forward the following problem:

Problem: Let G be a group such that $nse(G) = nse(L_2(q))$, where q is a prime power. Is G isomorphic to $L_2(q)$?

They proved that the groups $L_2(q)$, where $q \in \{7, 8, 11, 13\}$ are characterizable by $nse(L_2(q))$. Also in [9, 11, 12, 13, 18, 20], it is proved that the groups $L_2(q)$, where $q \in \{2, 3, 4, 9, 16, 25, 49\} \cup \{r : r < 100 \text{ is a prime}\}$ are characterizable by $nse(L_2(q))$. In this paper, we show that this problem has an affirmative answer for the case $q = p^2$, where $p \in \{11, 13\}$. In fact, we prove the following main theorem:

Main Theorem. Let G be a group such that $nse(G) = nse(L_2(p^2))$, where $p \in \{11, 13\}$. Then $G \cong L_2(p^2)$.

2 Preliminaries

For a natural number n, by $\pi(n)$, we mean the set of all prime divisors of n, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A

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Sylow p-subgroup of G is denoted by G_p and by $n_p(G)$, we mean the number of Sylow p-subgroups of G. If there is no ambiguity, then we write n_p instead of $n_p(G)$. Also, the largest element of $\pi_e(G_p)$ is denoted by $exp(G_p)$. Moreover, we denote by φ the Euler totient function and by (a,b) the greatest common divisor of integers a and b.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1 *[3] Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.2 *[20] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2-1)$.*

Lemma 2.3 *[15] Let G be a finite group and $p \in \pi(G)-\{2\}$. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p,m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.4 *[5] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, \dots, s\}$:*

- $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.
- The order of some chief factor of G is divisible by $q_i^{\beta_i}$.

Lemma 2.5 *[20] Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = \sup\{m_k \mid k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2-1)$.*

Lemma 2.6 *[15] Let G be a finite group and $p \in \pi(G)-\{2\}$. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p,m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.7 *[5] Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$, $(m,n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and h_m be the number of Hall π-subgroups of G. Then $h_m = q_1^{\beta_1} \dots q_s^{\beta_s}$, satisfies the following conditions for all $i \in \{1, \dots, s\}$:*

- *$q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.*
- *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.8 *[6] Let G be a solvable group and π be any set of primes. Then*

- *G has a Hall π-subgroup.*
- *If H is a Hall π-subgroup of G and V is any π-subgroup of G, then $V \leq H^g$ for some $g \in G$. In particular, the Hall π-subgroups of G form a single conjugacy class of subgroups of G.*

Lemma 2.9 *Let S be a simple K_n-group, where $n \in \{3,4,5,6\}$. If $|S| \mid 2^4.3.5.7.13^2.17$, then S is isomorphic to one of the following groups: A_5, $L_2(7)$, $L_2(13)$, $L_2(16)$, $L_2(169)$.*

Proof. • Let S be a simple K_3-group. Then by [7], S is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$. If $S \cong A_6$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$, then $3^2 \mid |S|$, which is a contradiction. So $S \cong A_5$ or $L_2(7)$.
• Let S be a simple K_4-group. Then by [1, 17], S is isomorphic to one of the following groups:

- A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_2(243)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;
- $L_2(r)$, where r is a prime, $r^2 - 1 = 2^a 3^b v^c$, $v > 3$ is a prime, $a, b, c \in \mathbb{N}$;
- $L_2(2^m)$, where m, $(2^m - 1)$ and $(2^m + 1)/3$ are primes greater than 3;
- $L_2(3^m)$, where m, $(3^m - 1)/2$ and $(3^m + 1)/4$ are odd primes.

If S is isomorphic to one of the groups of parts (1),(3),(4) except $L_2(16)$, then $2^5 \mid |S|$ or $3^2 \mid |S|$

or $5^2 \mid |S|$, which is a contradiction. If $S \cong L_2(r)$, where r is a prime, $r^2 - 1 = 2^a 3^b v^c$, $v > 3$ is a prime, $a, b, c \in \mathbb{N}$, then by $|S|$, $r \in \{5, 7, 13, 17\}$ and hence $r = 13$. So we conclude that $S \cong L_2(13)$ or $L_2(16)$.

• Let S be a simple K_5-group. Then by [8], S is isomorphic to one of the following groups:

- $L_2(q)$, where q satisfies $|\pi(q^2 - 1)| = 4$;
- $L_3(q)$, where q satisfies $|\pi((q^2-1)(q^3-1))| = 4$;
- $U_3(q)$, where q satisfies $|\pi((q^2-1)(q^3+1))| = 4$;
- $O_5(q)$, where q satisfies $|\pi(q^4 - 1)| = 4$;
- $Sz(2^{2m+1})$, where $|\pi((2^{2m+1} - 1)(2^{4m+2} + 1))| = 4$;
- One of the 30 other simple groups: A_{11}, A_{12}, M_{22}, J_3, HS, He, M^cL, $L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, $S_6(3)$, $S_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8^+(3)$, $O_8^-(2)$, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(8)$.

If S is isomorphic to one of the groups of part (6), then $3^2 \mid |S|$, which is a contradiction. If $S \cong L_2(q)$, then by $|S|$, $q \in \{2, 3, 4, 5, 7, 8, 13, 16, 17, 169\}$ and since $|\pi(q^2 - 1)| = 4$, we conclude a contradiction. Similarly, we conclude that S is not isomorphic to one of the groups of parts (2),(3),(4),(5).

• Let S be a simple K_6-group. Then by [8], S is isomorphic to one of the following groups:

- $L_2(q)$, where q satisfies $|\pi(q^2 - 1)| = 5$;
- $L_3(q)$, where q satisfies $|\pi((q^2-1)(q^3-1))| = 5$;
- $L_4(q)$, where q satisfies $|\pi((q^2-1)(q^3-1)(q^4-1))| = 5$;
- $U_3(q)$, where q satisfies $|\pi((q^2-1)(q^3+1))| = 5$;
- $U_4(q)$, where q satisfies $|\pi((q^2-1)(q^3+1)(q^4-1))| = 5$;
- $O_5(q)$, where q satisfies $|\pi(q^4 - 1)| = 5$;
- $G_2(q)$, where q satisfies $|\pi(q^6 - 1)| = 5$;
- $Sz(2^{2m+1})$, where $|\pi((2^{2m+1} - 1)(2^{4m+2} + 1))| = 5$;
- $R(3^{2m+1})$, where $|\pi((3^{2m+1} - 1)(3^{6m+3} + 1))| = 5$;
- One of the 38 other simple groups: A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Ru, Co_2, Co_3, Fi_{22}, HN, $L_5(7)$, $L_6(3)$, $L_7(2)$, $O_7(4)$, $O_7(5)$, $O_7(7)$, $O_9(3)$, $S_6(4)$, $S_6(5)$, $S_6(7)$, $S_8(3)$, $U_5(4)$, $U_5(5)$, $U_5(9)$, $U_6(3)$, $U_7(2)$, $F_4(2)$, $O_8^+(4)$, $O_8^+(5)$, $O_8^+(7)$, $O_{10}^+(2)$, $O_8^-(3)$, $O_{10}^-(2)$, ${}^3D_4(4)$, ${}^3D_4(5)$.

If S is isomorphic to one of the groups of part (10), then $3^2 \mid |S|$, which is a contradiction. If $S \cong L_2(q)$, then by $|S|$, $q \in \{2, 3, 4, 5, 7, 8, 13, 16, 17, 169\}$ and since $|\pi(q^2 - 1)| = 5$, we conclude $S \cong L_2(169)$. Similarly, we conclude that S is not isomorphic to one of the groups of parts (2)-(9).

Lemma 2.10 *Let G be a group such that $nse(G) = nse(L_2(p^2))$, where $p \in \{11, 13\}$. Then G is finite and for every $i \in \pi_e(G)$,*

$$\begin{cases} \varphi(i) \mid m_i \\ i \mid \sum_{d|i} m_d \end{cases}$$

and if $i > 2$, then m_i is even.

Proof. The proof is straightforward according to Lemmas 2.1 and 2.5.

3 Proof of the Main Theorem

First, we prove the main theorem for the case $p = 13$. If G is a group such that $nse(L_2(13^2)) = nse(G)$, then by [2], we have $nse(L_2(13^2)) = nse(G) = \{1, 14365, 28560, 28730, 56784, 57460, 86190, 172380, 227136, 344760, 908544\}$.

In the following lemma, we prove some basic properties of group G:

Lemma 3.1 *If $\{2, 3, 5, 7, 13, 17\} \subseteq \pi(G)$, then*

- *$m_2 = 14365$, $m_3 = 28730$, $m_5 \in \{56784, 908544\}$, $m_7 = 86190$, $m_{13} = 28560$, $m_{17} = 227136$.*
- *$\{17^2, 13^4, 7^2, 5^3, 3^3, 2^{10}, 2^8.13, 3.17, 7.13, 13.17\} \cap \pi_e(G) = \emptyset$.*
- *$|G_{17}| = 17$, $|G_{13}| \mid 13^4$, $|G_7| \mid 7^2$, $|G_5| = 5$, $|G_3| \mid 3^2$.*

Proof. According to Lemma 2.10 and $nse(G)$, the proof of parts (1) and (2) is obvious. So it is enough to prove part (3). Since $17^2 \notin \pi_e(G)$, we conclude that $exp(G_{17}) = 17$ and hence, Lemma 2.1 implies that $|G_{17}|= 17$. Thus G_{17} is cyclic and $n_{17} = m_{17}/\varphi(17) = 14196$.

Since $13^4 \notin \pi_e(G)$, we conclude that $exp(G_{13}) \in \{13, 13^2, 13^3\}$. If $exp(G_{13}) = 13^3$, then Lemma 2.1 implies that $|G_{13}|| \ 13^3$ and hence, G_{13} is cyclic and $n_{13} = m_{13^3}/\varphi(13^3) =$ 85 or 448. But since every cyclic group of order 13^3 has only one subgroup of order 13, we conclude that $m_{13} \leq 12.448$, which is a contradiction. If $exp(G_{13}) = 13^2$, then Lemma 2.1 implies that $|G_{13}|| \ 13^2$ and hence, G_{13} is cyclic and $n_{13} = m_{13^2}/\varphi(13^2) \in \{364, 1105, 456, 2210, 5824\}$, which is a contradiction by Sylow's theorem. So we conclude that $exp(G_{13}) = 13$ and hence, Lemma 2.1 implies that $|G_{13}|| \ 13^4$.

Since $7^2 \notin \pi_e(G)$, Lemma 2.1 implies that $|G_7||$ 7^2.

Since $5^3 \notin \pi_e(G)$, we conclude that $exp(G_5) \in \{5, 5^2\}$. If $exp(G_5) = 5^2$, then Lemma 2.1 implies that $|G_5|| \ 5^2$ and hence, G_5 is cyclic and $n_5 = m_{5^2}/\varphi(5^2) = 8619$. But since every cyclic group of order 5^2 has only one subgroup of order 5, we conclude that $m_5 \leq 4.8619$, which is a contradiction. So we conclude that $exp(G_5) = 5$ and hence, Lemma 2.1 implies that $|G_5|= 5$. Thus G_5 is cyclic and $n_5 = m_5/\varphi(5) = 14196$ or 227136.

Since $3^3 \notin \pi_e(G)$, we conclude that $exp(G_3) \in \{3, 3^2\}$. If $exp(G_3) = 3^2$, then Lemma 2.1 implies that $|G_3|| \ 3^5$. Since $3^2 \nmid m_{3^2}$, Lemma 2.6 implies that G_3 is cyclic and hence, $n_3 = m_{3^2}/\varphi(3^2) =$ 14365 or 57460. If $exp(G_3) = 3$, then Lemma 2.1 implies that $|G_3|= 3$ and hence, G_3 is cyclic and $n_3 = m_3/\varphi(3) = 14365$. So $|G_3|| \ 3^2$. Now we are going to prove that $G \cong L_2(13^2)$. We have divided the proof into a sequence of lemmas:

Lemma 3.2 $\pi(G) = \{2, 3, 5, 7, 13, 17\}$.

Proof. Since 14365 is the only odd number $nse(G) - \{1\}$, by Lemma 2.10, $2 \in \pi(G)$. Let $2 \neq r \in \pi(G)$. Then by Lemma 2.10, $r \mid (1 + m_r)$ and $\varphi(r) \mid m_r$. Thus we conclude that $r \in \{3, 5, 7, 11, 13, 17\}$. If $11 \in \pi(G)$, then by Lemma 2.10, $m_{11} = 172380$. On the other hand, $22 \notin \pi_e(G)$ because otherwise by Lemma 2.10, $\varphi(22) \mid m_{22}$ and $22 \mid (1 + m_2 + m_{11} + m_{22})$, which is a contradiction. Thus G_{11} acts fixed point freely on the set of elements of order 2 by conjugation and hence $|G_{11}|| \ m_2$, which is a contradiction. Therefore $11 \notin \pi(G)$. So we conclude that $\{2\} \subseteq \pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$.

- If $\pi(G) = \{2\}$, then by Lemma 3.1, $2^{10} \notin \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots, 2^9\}$. Hence $|nse(G)| \leq 10$, which is a contradiction.
- If $\pi(G) = \{2, 7\}$, then by Lemma 3.1, $2^{10}, 7^2 \notin \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots, 2^9\} \cup \{7, 7.2, \ldots, 7.2^9\}$, which implies that
$|G|= 2^k.7^l = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 344760k_8 + 908544k_9$,
where $l, k, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ and k_9 are non-negative integers and $l \leq 2$ and $0 \leq k_1 + \ldots + k_9 \leq 9$. But it is easy to check that this equation has no solution.
- If $\pi(G) = \{2, 13\}$, then by Lemma 3.1, $2^{10}, 13^2, 13.2^8 \notin \pi_e(G)$. Thus $\pi_e(G) \subseteq \{1, 2, \ldots, 2^9\} \cup \{13, 13.2, \ldots, 13.2^7\}$, which implies that
$|G|= 2^k.13^l = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 344760k_8 + 908544k_9$,
where $l, k, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ and k_9 are non-negative integers and $l \leq 4$ and $0 \leq k_1 + \ldots + k_9 \leq 7$. It is easy to check that this equation has no solution, which is a contradiction.
- If $\pi(G) = \{2, 7, 13\}$, then by Lemma 3.1, $7.13 \notin \pi_e(G)$. Thus G_7 acts fixed point freely on the set of elements of order 13 by conjugation and hence, $|G_7|| \ m_{13}$. Therefore $|G_7|= 7$ and $n_7 = m_7/\varphi(7) = 14365$. Since $n_7 \mid |G|$, we conclude that $17 \in \pi(G)$, which is a contradiction.
- If $3 \in \pi(G)$, then by Lemma 3.1, $n_3 \in \{14365, 57460\}$. Since $n_3 \mid |G|$, we conclude that $17 \in \pi(G)$.
- If $5 \in \pi(G)$, then by Lemma 3.1, $n_5 \in \{14196, 227136\}$. Since $n_5 \mid |G|$, we conclude that $3 \in \pi(G)$. Thus according to the previous case, we have $17 \in \pi(G)$.

According to the above statement, in each case, we have $17 \in \pi(G)$. By Lemma 3.1, we know that $n_{17} = 14196$ and since $n_{17} \mid |G|$, we conclude that $14196 \mid |G|$. Thus $\{2, 3, 7, 13, 17\} \subseteq \pi(G)$. On the other hand, by Lemma 3.1, $n_3 \in \{14365, 57460\}$. Since $n_3 \mid |G|$, we conclude that $5 \mid |G|$. Consequently, $\pi(G) = \{2, 3, 5, 7, 13, 17\}$.

Lemma 3.3 $|G|= 2^k.3.5.7.13^2.17$, *where* $k \leq 4$.

Proof. By Lemma 3.1, we have $|G_{17}|= 17$ and $|G_5|= 5$. Now we prove that $|G_{13}|= 13^2$, $|G_7|= 7$,

$|G_3|= 3$, $|G_2|| \, 2^4$.

• By Lemma 3.1, we have $3.17 \notin \pi_e(G)$. Thus G_3 acts fixed point freely on the set of elements of order 17 by conjugation and hence, $|G_3|| \, m_{17}$. So $|G_3|= 3$ and $n_3 = 14365$. According to Lemma 3.1, $\{7.13, 13.17\} \cap \pi_e(G) = \emptyset$ and hence, similar argument implies that $|G_7|= 7$, $n_7 = 14365$ and $|G_{13}|= 13^2$.

• If $5.17 \notin \pi_e(G)$, then G_5 acts fixed point freely on the set of elements of order 17 by conjugation and hence, $|G_5|| \, m_{17}$, which is a contradiction. Thus $85 = 5.17 \in \pi_e(G)$ and $m_{85} = 908544$. On the other hand, if P and Q are Sylow 5-subgroups of G, then it is obvious that $C_G(P)$ and $C_G(Q)$ are conjugate in G. So $m_{85} = \varphi(85)n_5k$, where k is the number of cyclic subgroups of order 17 in $C_G(P)$. Hence $64n_5 \mid m_{85}$ and since $n_5 \in \{14196, 227136\}$, we conclude that $n_5 = 14196$ and $m_5 = 56784$. Similarly, we conclude that $10 \notin \pi_e(G)$. Thus G_2 acts fixed point freely on the set of elements of order 5 by conjugation and hence, $|G_2|| \, m_5$. So we conclude that $|G_2|| \, 2^4$.

Lemma 3.4 *G is unsolvable.*

Proof. If G is solvable, then by Lemma 2.8, G has a Hall π-subgroup H, where $\pi = \{3, 5, 7, 13, 17\}$ and all Hall π-subgroups of G are conjugate and the number of Hall π-subgroups of G is $|G : N_G(H)|| \, 2^4$. Since H is solvable, according to Lemma 2.7, there are nonnegative integers $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t, \delta_1, \ldots, \delta_u$ such that

$$n_{17}(H) = 3^{\sum_{i=1}^{r}\alpha_i}.5^{\sum_{j=1}^{s}\beta_j}.7^{\sum_{k=1}^{t}\gamma_k}.13^{\sum_{l=1}^{u}\delta_l},$$

where

$$3^{\alpha_i} \equiv 1 \pmod{17}, 5^{\beta_j} \equiv 1 \pmod{17}$$

$$7^{\gamma_k} \equiv 1 \pmod{17}, 13^{\delta_l} \equiv 1 \pmod{17}.$$

Also, by Lemma 3.3, we know that $|G|= 2^k.3.5.7.13^2.17$, where $k \leq 4$. Thus $\sum_{i=1}^{r}\alpha_i \leq 1$, $\sum_{j=1}^{s}\beta_j \leq 1$, $\sum_{k=1}^{t}\gamma_k \leq 1$, $\sum_{l=1}^{u}\delta_l \leq 2$ which implies that $n_{17}(H) = 1$. So $16 \leq m_{17}(G) \leq (2^4.16)$, but we have $m_{17} = 227136$, which is a contradiction.

Lemma 3.5 $G \cong L_2(13^2)$.

Proof. Since G is a finite unsolvable group, there is a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$, such that N is a maximal solvable normal subgroup of G and M/N is an unsolvable simple group or the direct product of isomorphic unsolvable simple groups. Let $M/N \cong S_1 \times \ldots \times S_r$, where S_1 is an unsolvable simple group and $S_1 \cong \ldots \cong S_r$. Since $1 \trianglelefteq N \trianglelefteq M \trianglelefteq G$ and $|G|= 2^k.3.5.7.13^2.17$, where $k \leq 4$, we conclude that $r = 1$ and M/N is a simple K_n-group, where $n \in \{3, 4, 5, 6\}$. So by Lemma 2.9, M/N is isomorphic to one of the following groups: A_5, $L_2(7)$, $L_2(13)$, $L_2(16)$, $L_2(169)$.

• If $M/N \cong A_5$, then $(G/N)/(A/N) \cong G/A \leq \mathrm{Aut}(M/N) \cong S_5$, where $C_{G/N}(M/N) = A/N$. Since $M/N \cong A_5$ is an unsolvable simple group, we conclude that $M/N \cap A/N = 1$ and hence, $M/N \times A/N \trianglelefteq G/N$, therefore $|M/N|| \, |G/A|$. So we conclude that $G/A \cong A_5$ or S_5. Hence $7.13^2.17 \mid |A|| \, 2^2.7.13^2.17$. Thus by Sylow's theorem, $n_{17}(A) \in \{1, 52\}$. Since $A \trianglelefteq G$, we conclude that $n_{17}(A) = n_{17}(G)$. Therefore $m_{17}(G) \in \{16, 832\}$, which is a contradiction. Similarly, we can prove that $G \ncong L_2(7)$, $L_2(13)$, $L_2(16)$.

• If $M/N \cong L_2(169)$, then $(G/N)/(A/N) \cong G/A \leq Aut(M/N)$, where $C_{G/N}(M/N) = A/N$. Since $M/N \cong L_2(169)$ is an unsolvable simple group, we conclude that $M/N \cap A/N = 1$, hence $M/N \times A/N \trianglelefteq G/N$, therefore $|M/N|| \, |G/A|$. So we conclude that $2^3.3.5.7.13^2.17 = |M/N|| \, |G/A|| \, |Aut(M/N)|= 2^5.3.5.7.13^2.17$. Hence $|A|| \, 2$. Let $A = \{1, x\}$ and y is element of G of order 5. Since $A \trianglelefteq G$, we conclude that $y^{-1}xy = x$, hence G have element of order 10, which is a contradiction. So $A = N = 1$ and $L_2(169) \leq G \leq Aut(L_2(169))$. Thus $|G|= 2^3.3.5.7.13^2.17$ or $2^4.3.5.7.13^2.17$. If $|G|= 2^4.3.5.7.13^2.17$, then we know that

$\pi_e(Aut(L_2(169))) = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 14, 17, 21, 24, 26, 28, 34, 42, 56, 84, 85, 168, 170\}$.

Now we have $56 \notin \pi_e(G)$ because otherwise $m_{56} \in \{28560, 56784, 227136, 908544\}$ and similar to Lemma 3.3, $m_{56} = \varphi(56)n_7k$, thus we conclude that $n_7 \mid m_{56}$, which is a contradiction. Hence $56 \notin \pi_e(G)$. So $168 \notin \pi_e(G)$. Similarly, $10, 34, 170 \notin \pi_e(G)$. So $|\pi_e(G)| \leq 19$. Thus $|G|= 2^4.3.5.7.13^2.17 = 1924910 + 28560k_1 + 28730k_2 + 56784k_3 + 57460k_4 + 86190k_5 + 172380k_6 + 227136k_7 + 344760k_8 + 908544k_9$, where $k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8$ and k_9 are non-negative integers and $0 \leq k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \leq 8$. It is easy to check that this equation has no solution, which is a contradiction. So we conclude that $|G|= 2^3.3.5.7.13^2.17$ and since

$L_2(169) \leq G \leq Aut(L_2(169))$, we conclude that $G \cong L_2(169)$.

By the same manner, we can prove the main theorem for $p = 11$ as well. We omit the details for the sake of convenience.

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