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Research Article

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New model for ranking DMUs in DDEA as a special case

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Abstract

The purpose of this paper is to offer the equitable method for ranking Decision Making Units(DMUs) based on the Dynamic Data Envelopment Analysis (DDEA) concept, where quasi-fixed inputs or intermediate products are the source of inter-temporal dependence between consecutive periods. In fact, this paper originally makes the use of an approach extending the ranking of DMUs in DEA by Khodabakhshi and Aryavash into the Dynamic DEA framework. Hence, firstly, we compute minimum and maximum efficiency values of each DMUs in dynamic state, under the assumption that the sum of efficiency values of all DMUs in dynamic state is equal to unity. Thus, with the combination of its minimum and maximum efficiency values, the rank of each DMUs is determined.

Keywords : Data Envelopment Analysis (DEA); Decision Making Units(DMU); Efficiency; Ranking; Dynamic DEA (DDEA).

1 Introduction

Data Envelopment Analysis (DEA) is a non-parametric technique developed in operation research and management science over the last four decade for measuring relative efficiency and ranking of DMUs. DEA was originated by Charnes, Cooper and Rhodes (CCR) [1]. Banker,Charnes and Cooper(BCC) introduced a variable return to scale version of the CCR model, namely the BCC model [2]. Following the CCR and BCC models, other models of DEA were introduced in the DEA literature.

The dynamic DEA model proposed by Fare and Grosskopf [3] is the first innovative scheme for dealing formally with these inter-connecting activities. Dynamic DEA was originally to cope with long time assessment point of view incorporating the concepts of quasi-fixed inputs and investment activities. DDEA enables us to measure the efficiency based upon the long time optimization in which the inter-connecting activities such as investment activities are incorporated. This feature of DDEA discriminates it from the separate time models such as Window analysis [4] and Malmquist productivity index [5]. Sofar, many DDEA models have been developed in this area, e.g. Nemoto and Goto [6], Sueyoshi and Sekitani [7], Kaoru Tone and Miki Tsutsui [8], Alireza Amirteimori [11]. In spite of developing several methods in DDEA, an equitable method for ranking decision making units has not been mentioned yet. There are many model for ranking of DMUs in DEA. Recently, Khodabakhshi and Aryavash [12] have proposed a method for ranking all units in DEA. In this work, their idea is applied to rank the DMUs in DDEA. It is assumed that the sum of efficiency values of all DMUs is equal to unity. Then, the minimum and maximum efficiency values of each DMU are computed. Finally, the rank of each DMU is determined in proportion to a combination of its minimum and maximum efficiency values.

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The rest of the work is organized as follows: In Section 2, we present a description on DDEA. The proposed ranking method in DDEA framework is presented in Section 3. Section 4 applies this method to an example involving 11 Iranian gas companies in two periods. Conclusion appears in Section 5.

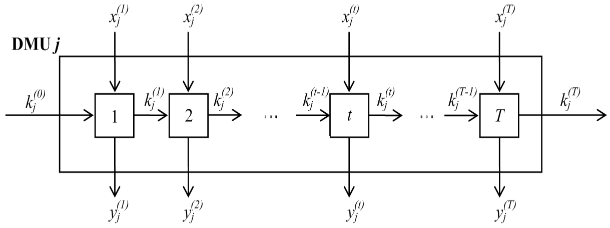

Figure 1: Dynamic Structure.

2 The DDEA

Nemoto and Goto [6] extended DEA to a dynamic framework. In the previous DEA research effort, Nemoto and Goto(1999) added an important perspective on dynamic DEA. Their research incorporates two different types of inputs (variable inputs and quasi-fixed inputs) into a framework of dynamic DEA. The introduction of quasi-fixed inputs into a DEA model can be seen as a first step toward dynamic DEA. A unique feature of the quasi-fixed inputs is that those are considered as outputs at the current period, while being treated as inputs at the next period. For example, in power generators, workers and fuels (as variable inputs) are employed to generate electricity (as an output). Most of the generated power is sold to the purchasers; though a part (as a quasi-fixed input) of the generated power is internally saved within the generator. The saved power is used to generate electricity in the next period. So, it functions as a quasi-fixed input. Accordingly, we use DDEA structure proposed by Nemoto and Goto(1999) to reach a definition of ranking in dynamic DEA.

We deal with n DMUs $(j = 1, 2, \dots, n)$ examined in T periods $(t = 1, 2, \dots, T)$. In the period t each DMU_j uses two different groups inputs $k_j^{(t-1)} \in R_+^l$ (as a vector of quasi-fixed inputs) and $x_j^{(t)} \in R_+^m$ (as a vector of variable inputs) to produce two different groups of outputs $y_j^{(t)} \in R_+^s$ (as a vector of goods) and $k_j^{(t)}$ (as a vector of quasi-fixed inputs used in the next period).

We illustrate our DDEA structure in Figure 1. Variable inputs $x_j^{(t)}$ and quasi-fixed inputs $k_j^{(t-1)}$ at the beginning of the period t are transformed by the process P_t into regular outputs $y_j^{(t)}$ and quasi-fixed inputs $k_j^{(t)}$ at the end of the period t.

A production possibility set in the period t can be defined as follows:

$$\phi_t^{CRS} = \left\{ \begin{array}{ll} (x^{(t)}, k^{(t-1)}, y^{(t)}, k^{(t)}) \in R_+^{m+l} \times R_+^{s+l} \,\Big| \\ X_t\lambda_t \leq x^{(t)}, & K_{t-1}\lambda_t \leq k^{(t-1)}, \\ Y_t\lambda_t \geq y^{(t)}, & K_t\lambda_t \geq k^{(t)}, \\ & \lambda_t \geq 0. \end{array} \right\}$$

Where $\lambda_t \in R_+^n$ is a vector of weights to connect the DMUs in the period t,

$X_t = [x_1^{(t)}, x_2^{(t)}, \dots, x_n^{(t)}]$,

$K_{t-1} = [k_1^{(t-1)}, k_2^{(t-1)}, \dots, k_n^{(t-1)}]$ and

$Y_t = [y_1^{(t)}, y_2^{(t)}, \dots, y_n^{(t)}]$

are a matrices of inputs, quasi-fixed inputs and outputs, respectively. Let DMU_o be under evaluation which uses $(x_o^{(t)}, k_o^{(t-1)})$ to produce $(y_o^{(t)}, k_o^{(t)})$ for $t = 1, 2, \dots, T$.

3 The Proposed Ranking Method in DDEA

In this section we extend the ranking method proposed by Khodabakhshi and Aryavash [12] in DDEA framework. Firstly we deal with n DMUs $(j = 1, \dots, n)$ over T terms $(t = 1, \dots, T)$, and suppose that a DMU_o is examined through an assessment window comprising T periods, i.e. $W = 1, 2, \dots, T$. To this end, assume that the DMU_o uses $(x_o^{(t)}, k_o^{(t-1)})$ to produce $(y_o^{(t)}, k_o^{(t)})$ in the period t. In this study we assume that $m = s = l = 1$. In order to achieve their goal we use the assumptions and descriptions developed by Kao and Hwang[9].

Let $x_j^{(t)}$ and $y_j^{(t)}$ denote the input and output of the jth DMU in period t. Furthermore, denote $X_j = \sum_{t=1}^{T} x_j^{(t)}$ and $Y_j = \sum_{t=1}^{T} y_j^{(t)}$

as the total quantities of the input and output, respectively, over all T periods.

We want to estimate the efficiency value of DMU_o (θ_o) under the assumption that the sum of efficiency value of all DMUs equals unity ($\sum_{j=1}^{n}\theta_j = 1$). On the other hand, the efficiency of each DMUs is defined as the weighted sum of outputs divided by the weighted sum of inputs.

In this regard, according to Kao [10], the overall efficiency of DMU_j (θ_j), and period efficiency of DMU_j ($\theta_j^{(t)}$), $t = 1,\dots,T$, $j = 1,\dots,n$, is as follow:

$$\theta_j = \frac{uY_j + wk_j^{(T)}}{vX_j + wk_j^{(0)}} \quad j = 1,\dots,n \tag{3.1}$$

$$\theta_j^{(t)} = \frac{uy_j^{(t)} + wk_j^{(t)}}{vx_j^{(t)} + wk_j^{(t-1)}} \quad j = 1,\dots,n \tag{3.2}$$

where u and v are weight vectors of output Y_j (or $y_j^{(t)}$) and input X_j (or $x_j^{(t)}$), respectively and w is weight vector of quasi-fixed input $k_j^{(t-1)}$ and output $k_j^{(t)}$.

We consider following model to determine minimum and maximum values of efficiency of DMU_o:

$$\begin{array}{lll} \min,\max & \theta_o & \\ s.t. & \theta_j \le 1 & j = 1,\dots,n \\ & \theta_j^{(t)} \le 1 & j = 1,\dots,n \\ & & t = 1,\dots,T \\ & \sum_{j=1}^{n}\theta_j = 1 & \end{array} \tag{3.3}$$

We substitute (3.1), (3.2) in (3.3), and use the Charnes-Cooper transformation with the quantity p, so that: $pu \mapsto u$, $pv \mapsto v$, $pw \mapsto w$. Also, according to definition of θ_j we have:

$$\begin{array}{l} \theta_j = \frac{uY_j + wk_j^{(T)}}{vX_j + wk_j^{(0)}}\ , \quad j = 1,\dots,n \Longrightarrow \\ uY_j + wk_j^{(T)} - [(v\theta_j)X_j + (w\theta_j)k_j^{(0)}] = 0 \end{array} \tag{3.4}$$

With substitution the transformations $v\theta_j = \overline{v}_j$ and $w\theta_j = \overline{w}_j$ in (3.4), and using the above transformations and (3.4), model (3.3) can be replaced by the following linear programming problem:

$$\begin{array}{ll} \min,\max & \theta_o = uY_o + wk_o^{(T)} \\ s.t. & vX_o + wk_o^{(0)} = 1 \\ & uY_j + wk_j^{(T)} - (vX_j + wk_j^{(0)}) \le 0 \\ & \qquad j = 1,\dots,n \\ & uy_j^{(t)} + wk_j^{(t)} - (vx_j^{(t)} + wk_j^{(t-1)}) \le 0 \\ & \qquad j = 1,\dots,n \ , \quad t = 1,\dots,T \\ & uY_j + wk_j^{(T)} - (\overline{v}_j X_j + \overline{w}_j k_j^{(0)}) = 0 \\ & \qquad j = 1,\dots,n \\ & \sum_{j=1}^{n}\overline{v}_j = v \\ & \sum_{j=1}^{n}\overline{w}_j = w \\ & u, v, w \ge 0\ , \\ & \overline{v}_j, \overline{w}_j \ge 0 \ , \quad j = 1,\dots,n. \end{array} \tag{3.5}$$

Note that the sum of the constraints in the third constraint set, i.e.,the constraints associated with the periods, is equal to the constraint in the second constraint set, i.e., the constraints associated with the system, for each DMU. Therefore, the second constraint set is redundant, and can be deleted. Since the model (3.5) always has a feasible solution, we can be calculated the minimum and the maximum objective function conclude $\theta_o^{\min}$ and $\theta_o^{\max}$, respectively using the model (3.5).

Lemma 3.1 *If $\theta_j^{\min}$ and $\theta_j^{\max}$ are optimal solutions of model (3.5) for each DMU_j, then*

$$\theta_j^{\min} \le \theta_j^{\max}$$

.

Proof. The proof is straightforward with respect to model (3.5).

Hence, we have the following interval for each θ_j:

$$\theta_j^{\min} \le \theta_j \le \theta_j^{\max}, \quad j = 1, 2, \dots, n. \tag{3.6}$$

Then, we can write θ_j as a convex combination of $\theta_j^{\min}$ and $\theta_j^{\max}$ as follow:

$$\begin{array}{l} \theta_j = \theta_j^{\min}\lambda_j + \theta_j^{\max}(1-\lambda_j), \\ 0 \le \lambda_j \le 1, \quad j = 1, 2, \dots, n. \end{array} \tag{3.7}$$

In order to compute the unique scores for $\theta_j (j = 1,\dots,n)$ in an equitable way, we use

Table 1: The normalized data used in the application in period $t = 1$

	inputs	
Companies	Budget ($x_j^{(t)}$)	Revenue of gas sold in previous period ($k_j^{(t-1)}$)
1	0.9625	0.9992
2	0.9265	0.9969
3	1	1
4	0.6009	0.8902
5	0.6617	0.6873
6	0.5464	0.4119
7	0.7287	0.5972
8	0.4038	0.1789
9	0.6186	0.3959
10	0.7309	0.3239
11	0.8250	0.9957
	outputs	
Companies	Amount of piping ($y_j^{(t)}$)	Revenue of gas sold in current period ($k_j^{(t)}$)
1	1	0.9398
2	0.569	1
3	0.357	0.9907
4	0.5915	0.8996
5	0.937	0.5277
6	0.2558	0.4064
7	0.5177	0.7782
8	0.487	0.9415
9	0.3662	0.6134
10	0.8213	0.7324
11	0.1235	0.5191

the Khodabakhshi and Aryavash [12] suggestion. Therefore, we select equal values of $\lambda_j (j = 1, 2, \ldots, n)$, i.e.

$$\lambda = \lambda_1 = \cdots = \lambda_n.$$

Accordingly, the $\theta_j (j = 1, 2, \ldots, n)$ are determined by solving the following linear equation system:

$$\begin{cases} \theta_j = \theta_j^{\min}\lambda + \theta_j^{\max}(1-\lambda), j = 1, 2, \ldots, n \\ \sum_{j=1}^{n} \theta_j = 1 \end{cases}$$

Using these equations the value of λ can be easily obtained as follow:

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \theta_j = 1 \Longrightarrow \\ &\textstyle\sum_{j=1}^{n} (\theta_j^{\min}\lambda + \theta_j^{\max}(1-\lambda)) = 1 \Longrightarrow \\ &\lambda = \frac{1-\sum_{j=1}^{n} \theta_j^{\max}}{\sum_{j=1}^{n} (\theta_j^{\min} - \theta_j^{\max})} \end{aligned} \tag{3.8}$$

With the substitution of the value of λ obtained from (3.8) in (3.7), we can determine the value of $\theta_j (j = 1, 2, \ldots, n)$. Now, with respect to their efficiency score (θ_j), all the DMUs have been fully ranked. On the other hand, a DMU with a greater efficiency score, is of a better ranking.

4 Numerical example

The proposed ranking method is applied to a numerical example consisting of eleven DMUs and two observation periods.

We apply the method to a data set consist of 11 gas companies located in 11 regions in Iran in two six-month periods during 2003 and 2004. The data set adapted from [11] and displayed in Table 1 and Table 2. Due to special case in our work we use one of each of the inputs and outputs.

Variable input($x_{t,j}$) is budget, and output($y_{t,j}$) is amount of piping. An other type of output used as input in next period, is revenue. Each gas company uses the revenue of gas sold as input in

Table 2: The normalized data used in the application in period $t = 1$

	inputs	
Companies	Budget $(x_j^{(t)})$	Revenue of gas sold in previous period$(k_j^{(t-1)})$
1	0.8973	0.9398
2	0.3884	1
3	0.7864	0.9907
4	0.6879	0.8996
5	1	0.5277
6	0.9662	0.4064
7	0.8261	0.7782
8	0.9169	0.9415
9	0.6223	0.6134
10	0.8813	0.7324
11	0.8876	0.5191
	outputs	
	Amount of piping$(y_j^{(t)})$	Revenue of gas sold in current period $(k_j^{(t)})$
1	1	0.1878
2	0.5325	0.8419
3	0.2555	1
4	0.9130	0.3372
5	0.9385	0.5516
6	0.2656	0.3555
7	0.5658	0.1811
8	0.4614	0.9852
9	0.3408	0.5262
10	0.8819	0.4786
11	0.7945	0.7394

Table 3: An equitable ranking of DMUs.

	inputs			
Companies	$\theta_j^{\min}$	$\theta_j^{\max}$	θ_j	Rank
1	0.0222	0.1287	0.0649	9
2	0.0989	0.1747	0.1293	2
3	0.0405	0.1615	0.089	7
4	0.0557	0.1407	0.0898	6
5	0.0751	0.1370	0.0999	3
6	0.0407	0.0739	0.054	10
7	0.0265	0.0839	0.0495	11
8	0.0847	0.2452	0.1492	1
9	0.0672	0.1315	0.0930	5
10	0.0705	0.1283	0.0937	4
11	0.0633	0.1238	0.0875	8

next period. At the first period, each company uses the revenue of gas sold in previous period as one input.

Using the model (3.5) we calculate $\theta_j^{\min}$ and $\theta_j^{\max}$ for all DMUs in the whole periods. Then, efficiency score (θ_j) of each DMUs is determined and finally, ranking of DMUs are performed. These results are displayed in Table 3.

5 Conclusion

In this study we provide a way of ranking DEA to a dynamic framework consist of two different type of inputs(variable inputs and quasi-fixed inputs) are incorporated into dynamic DEA.

References

[1] A. Charnes, W. W. Cooper, E. Rhodes, *Measuring the efficiency of DMUs*, European Journal of Operational Research 2 (1978) 429-444.

[2] R. D. Banker, A. Charnes, W. W. Cooper, *Some models for estimating technical and scale inefficiencies in DEA*, Management Science 30 (1984) 1078-1092.

[3] R. Fare, S. Grosskopf, *In:Intertemporal Production Frontiers: With Dynamic DEA*, Kluwer, Dordrecht (1996).

[4] G. A. Klopp, *The analysis of the efficiency of production system with multiple inputs and outputs*, Phd dissertation, University of Illinois, Industrial and system Engineering College,Chicago (1985).

[5] S. Malmquist, *Index numbers and indifference surfaces*, Trabajos de Estadistica 4 (1953) 209-242.

[6] J. Nemoto, M. Goto, *Dynamic DEA: modeling intertemporal behavior of a firm in the presence of productive inefficiencies*, Economics Letters 64 (1999) 51-56.

[7] T. Sueyoshi, K. Sekitani, *Returns to scale in dynamic DEA*, European Journal of Operational Research 161 (2005) 536-544.

[8] K.Tone, M. Tsutsui, *Network DEA: a slack-based measure approach*, European Journal of operational Research 197 (2009) 243-52.

[9] C. Kao, S. N. Hwang, *Efficiency measurment for network system:IT impact on firm performance*, Decision Support Systems 48 (2010) 437-446.

[10] C.Kao, *Dynamic data envelopment analysis: A relational analysis*, European Journal of operational Research 227 (2013) 325-330.

[11] A. Amirteimori, *Data envelopment analysis in dynamic framework*, Applied Mathematics and Computation 181 (2006) 21-28.

[12] M. Khodabakhshi, K. Aryavash, *Ranking all units in data envelopment analysis*, Applied Mathematics Letters 25 (2012) 25 2066-2070.

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