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Research Article

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Barrier options pricing of fractional version of the Black-Scholes model

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Abstract

In this paper two different methods are presented to approximate the solution of the fractional Black-Scholes equation for valuation of barrier option. Also, the two schemes need less computational work in comparison with the traditional methods. In this work, we propose a new generalization of the two-dimensional differential transform method and decomposition method that will extend the application of this methods for pricing barrier options of fractional version of the Black-Scholes model. Undoubtedly this model is the most well known model for pricing financial derivatives. This methods finds the analytical solution without any discretization or additive assumption. the approximate analytic solution is calculated in the form of convergent series with easily computable components, to solve the fractional Black-Scholes equation.

Keywords : Fractional Black-Scholes equations; Barrier option pricing problem; Analytical solution.

1 Introduction

THe pricing of options is a central problem in financial investment. It is of both theoretical and practical importance since the use of options thrives in the financial market. Undoubtedly the most well known model for pricing financial derivatives is the Black-Merton-Scholes model. This model is relatively simplistic and as such, has some drawbacks when it comes to pricing financial derivatives.

A financial derivative is an instrument whose price depends on, or is derived from, the value of another asset [?]. In 1969, Fischer Black and Myron Scholes got an idea that would change the world of financial forever. The central idea of their paper revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock.

Barrier options are one of the most popular financial contracts, and the market for these options has been expanding very rapidly, doubling every year since 1992 [?]. There has also been impressive growth in the variety of barrier options available. Barrier option is exotic option whose payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. Thus this options are so important. There is always interest to avoid specific price level or vice versa: reach a specific price level. An incomplete list of examples would include double barrier options, options with curved barriers, rainbow barriers, partial barriers, roll up and roll down options and capped options. Roll up and roll down are standard options with two barriers: when the first barrier is crossed, the options strike price is changed and it becomes a knock-

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out option. The knock-out barrier options further fall into two kinds: up-and-out and down-and-out. The academic literature on the pricing of barrier options dates back at least to Merton [?], who presented a closed-form solution for the price of a continuously monitored down-and-out European call. Rich [?] and Rubinstein and Reiner [?] also tackled the pricing of European single-barrier options, including the knock-in barrier calls and puts. More exotic variants such as partial barrier options and rainbow barrier options have been explored by Heynen and Kat [?] and Carr [?]. A detailed discussion on European and American style barrier options can be found in AtiSahlia et al. [?], Taleb [?] and Tavella [?].
The Black-Scholes model (BS) for pricing stock options has been applied to many different commodities and payoff structures. Assuming that the movement of that asset obeys the stochastic differential equation $dS(t)/S(t) = \mu dt + \sigma dz$, where the constant μ is the drift rate, σ is the volatility of the underlying asset, and dz is a Wiener process with mean zero and variance of dt , and by the Ito lemma, The Black-Scholes model for value of an option is described by the equation

$$\begin{aligned} &\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r(t)S\frac{\partial V}{\partial S} - r(t)V = 0, \\ &(S,t) \in R^+ \times (0,T), \quad V(S,T) = f(S), \end{aligned} \tag{1.1}$$

where $V(S,t)$ is the European option price at asset price S and at time t, T is the maturity, $r(t)$ is the risk free interest rate and $\sigma(S,t)$ represents the volatility function of underlying asset.
The fractional calculus is used in many fields of science and engineering. Fractional differential operators, therefore find numerous applications in the field of viscoelasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling, encompassing different branches of physics, chemistry and biological science [?]. In the area of financial markets, fractional order models have been recently used to described the probability distributions of log-price in the log-time limit, which is useful to characterise the natural variability in prices in the log term. In the last decade, several computational methods have been applied to solve fractional differential equations [?, ?, ?, ?, ?, ?, ?].
The DTM is numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. Recently, Momani and Odibat developed a semi-numerical method for solving linear partial differential equations of fractional order [?]. This method is named as generalized differential transform method (GDTM) [?] and is based on the two-dimensional differential transform method (DTM) and generalized Taylors formula [?]. In recent years, many authors have paid attention to studying the solutions of linear and nonlinear fractional differential equations by the Adomian decomposition method (ADM) [?, ?]. This method efficiently works for initial value or boundary value problems, for linear and nonlinear, ordinary or partial differential equations, and even for stochastic systems and fractional equations [?].
In the present work, for the purpose of comparison, we apply generalized differential transform and decomposition methods for solving fractional Black-Scholes equation.
The organization of this paper is as follows: in Section 2, some notation and basic definitions are presented that will be used in later section. In Section ?? and ??, the methods are discussed. Applications have been presented in Section 5 and Section 6 is the conclusion.

2 Preliminaries

In this section, we set up notation and basic definitions and main properties of fractional calculus theory which shall be used in this paper:

Definition 2.1 *The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [?]:*

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}.$$

Definition 2.2 *A real function $y(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $y(t) = t^p y_1(t)$, where $y_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m iff $y^{(m)} \in C_\mu, m \in N$.*

The Riemann-Liouville fractional integral and Caputo derivative are defined as follows.

Definition 2.3 *The Riemann-Liouville fractional derivative of y is defined as :*

$$
\begin{aligned}
&{}_{RL}D^{\alpha}y(t) = \\
&= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} y(\tau) d\tau,
\end{aligned}
$$

for $m-1 < \alpha \leq m$, $m \in N$, $t > 0$, $y \in C^m_{-1}$.

Definition 2.4 *The fractional derivative of $y(t)$ in the Caputo sense is defined as :*

$$
\begin{aligned}
D^{\alpha}y(t) &= J^{m-\alpha}D^m y(t) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau,
\end{aligned}
$$

for $m-1 < \alpha \leq m$, $m \in N$, $t > 0$, $y \in C^m_{-1}$.

Definition 2.5 *For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as*

$$
D_t^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} =
\begin{cases}
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \\
\quad m-1 < \alpha < m. \\
\frac{\partial^m u(x,t)}{\partial t^m}, \quad \alpha = m \in \mathbb{N}.
\end{cases}
$$

and the space-fractional derivative operator of order $\alpha > 0$ is defined as

$$
D_x^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial x^{\alpha}} =
\begin{cases}
\frac{1}{\Gamma(m-\alpha)} \int_0^x (x-)^{m-\alpha-1} \frac{\partial^m u(,t)}{\partial^m} d, \\
\quad m-1 < \alpha < m. \\
\frac{\partial^m u(x,t)}{\partial x^m}, \quad \alpha = m \in \mathbb{N}.
\end{cases}
$$

Definition 2.6 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $y \in C_{\mu}$, $\mu \geq -1$, is defined as:*

$$
\begin{aligned}
&J^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \\
&\alpha > 0,\ t > 0,\ J^0 y(t) = y(t).
\end{aligned}
$$

Note that the relation between Riemann-Liouville operator and Caputo fractional differential operator is given as follows

$$
\begin{aligned}
J^{\alpha}D_t^{\alpha}f(t) &= D_t^{-\alpha}D_t^{\alpha}f(t) \\
&= f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^k(0), \quad m-1 < \alpha \leq m.
\end{aligned}
\tag{2.2}
$$

Some of the most important properties of operator J^{α} for $y \in C_{\mu}$, $\mu \geq -1$, α, $\beta \geq 0$ and $\gamma > -1$, are as follows [?]:

1. $J^{\alpha}J^{\beta}y(t) = J^{(\alpha+\beta)}y(t)$,
2. $J^{\alpha}J^{\beta}y(t) = J^{\beta}J^{\alpha}y(t)$,
3. $J^{\alpha}t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

3 The analysis of the generalized differential transform method

In this section we shall derive the generalized two-dimensional differential transform method that we have developed for the numerical solution of linear partial differential equations with space- and time-fractional derivatives. The proposed method is based on a generalized Taylors formula [?].

Consider a function of two variables $v(x,t)$, and suppose that it can be represented as a product of two single-variable functions, i.e., $v(x,t) = f(x)g(t)$. Based on the properties of two-dimensional differential transform, the function $u(x,t)$ can be represented as

$$
\begin{aligned}
&v(x,t) = \\
&\sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{\alpha k} \sum_{h=0}^{\infty} G_{\beta}(h)(t-t_0)^{h\beta} \\
&= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V_{\alpha,\beta}(k,h)(x-x_0)^{\alpha k}(t-t_0)^{h\beta},
\end{aligned}
\tag{3.3}
$$

where $0 < \alpha, \beta \leq 1$. If function $v(x,t)$ is analytic and differentiated continuously with respect to time t in the domain of interest, then we define the generalized two-dimensional differential transform of the function

$$
\begin{aligned}
&V_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \\
&[(D_{x_0}^{\alpha})^k (D_{t_0}^{\beta})^h v(x,t)]_{(x_0,\ t_0)},
\end{aligned}
\tag{3.4}
$$

where $(D_{x_0}^{\alpha})^k = D_{x_0}^{\alpha} D_{x_0}^{\alpha} ... D_{x_0}^{\alpha}$, k-times. In case of $\alpha = \beta = 1$, then generalized two-dimensional differential transform (??) reduces to the classical two-dimensional differential transform. On the basis of the definitions (??) and (??), we have the following results [?]:

Theorem 3.1 *Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ are the differential transformations of the functions $u(x,t)$, $v(x,t)$ and $w(x,t)$, respectively; then*
1. If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$,

2. If $u(x,t) = av(x,t)$, $a \in R$, then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$,

3. If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\alpha,\beta}(r, h-s) W_{\alpha,\beta}(k-r, s)$,

4. If $u(x,t) = (x - x_0)^{n\alpha}(t - t_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$,

Theorem 3.2 *If $u(x,t) = D_{x_0}^{\alpha} v(x,y), 0 < \alpha \leq 1$, then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h)$.*

Proof. From relation (??) we have

$$\begin{aligned}
U_{\alpha,\beta}(k,h) &= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \\
&[(D_{x_0}^{\alpha})^k (D_{y_0}^{\beta})^h D_{x_0}^{\alpha} v(x,y)]_{(x_0,\ y_0)} \\
&= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \\
&[(D_{x_0}^{\alpha})^{k+1} (D_{y_0}^{\beta})^h v(x,y)]_{(x_0,\ y_0)} \\
&= \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\alpha(k+1)+1)} \\
&[(D_{x_0}^{\alpha})^{k+1} (D_{y_0}^{\beta})^h v(x,y)]_{(x_0,\ y_0)} \\
&= \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h).
\end{aligned}$$

Theorem 3.3 *If $u(x,t) = f(x)g(t)$ and the function $f(x) = x^{\lambda} h(x)$, where $\lambda > -1$, $h(x)$ has the generalized Taylor series expansion $h(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{\alpha n}$, and*

(a) $\beta < \lambda + 1$ and α arbitrary or
(b) $\beta \geq \lambda + 1$, α arbitrary and $a_n = 0$ for $n = 0, 1, ..., m-1$, where $m - 1 < \beta \leq m$. then (4) becomes

$$\begin{aligned}
U_{\alpha,\beta}(k,h) &= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \\
&[(D_{x_0}^{\alpha k} (D_{t_0}^{\beta})^h u(x,t)]_{(x_0,\ t_0)}.
\end{aligned} \quad (3.5)$$

Proof. The proof follows immediately from the fact that $D_{x_0}^{\alpha_1} D_{x_0}^{\alpha_2} f(x) = D_{x_0}^{\alpha_1+\alpha_2} f(x)$, under the conditions given in theorem. see details in [?].

Theorem 3.4 *If $u(x,t) = D_{x_0}^{\gamma} v(x,t), m - 1 < \gamma \leq m$ and $u(x,t) = f(x)g(t)$, where the function $f(x)$ satisfies the conditions given in Theorem 3.3, then*

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha,\beta}(k + \frac{\gamma}{\alpha}, h), \quad (3.6)$$

also if $u(x,t) = D_{t_0}^{\gamma} v(x,t)$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta h + \gamma + 1)}{\Gamma(\beta h + 1)} V_{\alpha,\beta}(k, h + \frac{\gamma}{\beta}). \quad (3.7)$$

4 The analysis of decomposition method

In this section, we illustrate the idea of the Adomian decomposition method. Let us consider the nonlinear fractional differential equation:

$$\begin{aligned}
&D_t^{\alpha} V(x,t) + \mathfrak{R}[x]V(x,t) + \mathcal{N}[x]V(x,t) = 0 \\
&t > 0,\ x \in R,\ 0 < \alpha \leq 1,
\end{aligned} \quad (4.8)$$

$$V(x,0) = g(x),$$

where $D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$, $\mathfrak{R}[x]$ is the linear operator, $\mathcal{N}[x]$ is the general nonlinear operator. We rewrite the fractional PDE in the form

$$D_t^{\alpha} V(x,t) = -\mathcal{L}[x] * V(x,t), \quad (4.9)$$

with $\mathcal{L}[x] = \mathfrak{R}[x] + \mathcal{N}[x]$.
The technique is based on the relation between Riemann-Liouville operator and Caputo fractional differential operator. On applying the operator $D_t^{-\alpha}$ to equation (??), and on taking account of (2.2) and (??), we obtain

$$V(x,t) = g(x) + D_t^{-\alpha}(-\mathcal{L}[x] * V(x,t)). \quad (4.10)$$

This begin the case, we look for a solution expandable in the form

$$V(x,t) = V_0(x,t) + \sum_{k=1}^{\infty} V_k(x,t), \quad (4.11)$$

with $V_0(x,t) = g(x)$. On substituting (??) into (??), we are led to the identification

$$V_{n+1}(x,t) = D_t^{-\alpha}(-\mathcal{L}[x] * V_n(x,t)). \quad (4.12)$$

5 Barrier option price

By changing variable to $S = e^x,\ t = T - \tau/\frac{1}{2}\sigma^2$, to a convenient form for computation as a parabolic type PDE, Black-Scholes model (1.1) can be written as:

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (m-1)\frac{\partial V}{\partial x} - mV, \qquad (5.13)$$

where $m = r/\frac{1}{2}\sigma^2$. The initial conditions becomes $V(x,0) = g(x)$.
After defining the fractional Brownian motion by Mandelbrot to model financial data, Black-Scholes model was developed to fractional Black-Scholes model. In this paper, we consider the fractional Black-Scholes model as follows [?]:

$$\frac{\partial^\alpha V}{\partial \tau^\alpha} = \frac{\partial^2 V}{\partial x^2} + (m-1)\frac{\partial V}{\partial x} - mV, \quad 0 < \alpha \le 1, \qquad (5.14)$$

with strike price of \$10 and up-and-out barrier option constraint

$$V(x,\tau) = \begin{cases} 0 & , \quad e^x \ge e^{B_u}, \quad 0 \le \tau < T, \\ e^x - 10 & , \quad 0 < e^x < e^{B_u}, \quad \tau = 0. \end{cases}$$

Therefore up-and-out barrier option is the option that the option expires worthless if the barrier S=Y, say, is reached from below before expiry. Now, we use generalized differential transform method for Barrier option pricing of fractional Black-Scholes model. Taking the two-dimensional transform of (??) by using the related properties, we have

$$\begin{aligned} &\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{1,\alpha}(k,h+1) \\ =&(k+1)(k+2)V_{1,\alpha}(k+2,h) \\ &+(m-1)(k+1)V(k+1,h) - mV_{1,\alpha}(k,h). \end{aligned} \qquad (5.15)$$

The generalized two-dimensional transform of the initial condition can be obtained as follows:

$$\begin{aligned} &V_{1,\alpha}(0,0) = -9, \\ &V_{1,\alpha}(k,0) = \frac{1}{k!} \quad k = 1,2,..., \end{aligned} \qquad (5.16)$$

by applying (??) into (??) we can obtain some value of $V(k,h)$ as follows

$$\begin{aligned} V_{1,\alpha}(0,1) &= \frac{10m}{\Gamma(\alpha+1)}, \\ V_{1,\alpha}(k,1) &= 0 \quad k = 1,2,... \\ V_{1,\alpha}(0,2) &= \frac{-10m^2}{\Gamma(2\alpha+1)}, \\ V_{1,\alpha}(k,2) &= 0 \quad k = 1,2,... \\ V_{1,\alpha}(0,3) &= \frac{10m^3}{\Gamma(3\alpha+1)}, \\ V_{1,\alpha}(k,3) &= 0 \quad k = 1,2,... \\ &... \\ &... \\ &... \\ V_{1,\alpha}(0,n) &= 10\frac{(-1)^{n+1}m^n}{\Gamma(n\alpha+1)}, \\ V_{1,\alpha}(k,n) &= 0 \quad k = 1,2,... \end{aligned}$$

Consequently, substituting all $V(k,h)$ into (??), we obtain the series form solutions of (??) as

$$\begin{aligned} &V(x,\tau) = \\ =&\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V_{1,\alpha}(k,h)(x-x_0)^k(\tau-\tau_0)^{\alpha h} \\ =&(-9 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...) \\ &+ \frac{10m\tau^\alpha}{\Gamma(\alpha+1)} - \frac{10m^2\tau^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ ... + 10\frac{(-1)^{n+1}m^n\tau^n}{\Gamma(n\alpha+1)} + ... \\ =&e^x - 10E_\alpha(-m\tau^\alpha). \end{aligned} \qquad (5.17)$$

Thus, we have

$$V(S,\tau) = S - 10E_\alpha(-m\tau^\alpha). \qquad (5.18)$$

Finally, we obtain

$$\begin{aligned} &V(S,t) = S - 10E_\alpha\left[-m\left(\sigma^2\frac{(T-t)}{2}\right)^\alpha\right], \\ &S < e^{B_u}. \end{aligned} \qquad (5.19)$$

For the special case $\alpha = 1$,

$$\begin{aligned} &V(S,t) = S - 10exp\left[-m\left(\sigma^2\frac{(T-t)}{2}\right)\right], \\ &S < e^{B_u}. \end{aligned} \qquad (5.20)$$

On the other hand, we can use decomposition procedure to solve this equation. we have

$$\frac{\partial^\alpha V}{\partial \tau^\alpha} = \mathcal{L}[x] * V(x,\tau), \tag{5.21}$$

with $\mathcal{L}[x] = (D_{xx} + (m-1)D_x - m)$.
Operating the inverse operator of D_τ^α on both sides in (**??**) and using the relation (2.2), we obtain

$$V(x,\tau) = (e^x - 10) + D_\tau^{-\alpha}(\mathcal{L}[x] * V(x,\tau)). \tag{5.22}$$

We look for a solution in the form

$$V(x,\tau) = V_0(x,\tau) + \sum_{k=1}^{\infty} V_k(x,\tau), \tag{5.23}$$

with

$$\begin{aligned} V_0(x,\tau) &= e^x - 10, \\ V_{n+1}(x,\tau) &= D_\tau^{-\alpha}(\mathcal{L}[x] * V_n(x,\tau)), \end{aligned} \tag{5.24}$$

so we have

$$\begin{aligned} V_1(x,\tau) &= D_\tau^{-\alpha}(\mathcal{L}[x] * (e^x - 10)) \\ &= \frac{\tau^\alpha}{\Gamma(1+\alpha)}[\mathcal{L}[x] * (e^x - 10)] \\ &= \frac{\tau^\alpha}{\Gamma(1+\alpha)}[10m], \end{aligned} \tag{5.25}$$

$$\begin{aligned} V_2(x,\tau) &= [\mathcal{L}^2[x] * (e^x - 10)] \\ &\quad \times D_\tau^{-\alpha}(\frac{\tau^\alpha}{\Gamma(1+\alpha)}) \\ &= \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}[-10m^2], \end{aligned} \tag{5.26}$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$\begin{aligned} V_n(x,\tau) &= [\mathcal{L}^n[x] * (e^x - 10)] \\ &\quad \times D_\tau^{-\alpha}(\frac{\tau^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}) \\ &= \frac{\tau^{n\alpha}}{\Gamma(1+n\alpha)}[10(-1)^{n+1}m^n]. \end{aligned} \tag{5.27}$$

The solution of this problem in a series form is given by

$$V(x,\tau) = \sum_{k=0}^{\infty} V_k(x,\tau) = e^x - 10E_\alpha(-m\tau^\alpha), \tag{5.28}$$

Thus, we have

$$\begin{aligned} V(S,t) =& S - 10E_\alpha\left[-m\left(\sigma^2\frac{(T-t)}{2}\right)^\alpha\right], \\ & S < e^{B_u}, \end{aligned} \tag{5.29}$$

which is consistent with the result obtained of it by the generalized differential transform method. Figure **??** shows the numerical solutions for up-and-out Barrier option price. The approximate solutions are obtained for $\alpha = \{\frac{1}{8}, \frac{1}{4}, \frac{2}{3}\}$, $\sigma = 0.2$, $T = 2$, $r = 0.05$ and $B_u = ln(83)$. Finally,

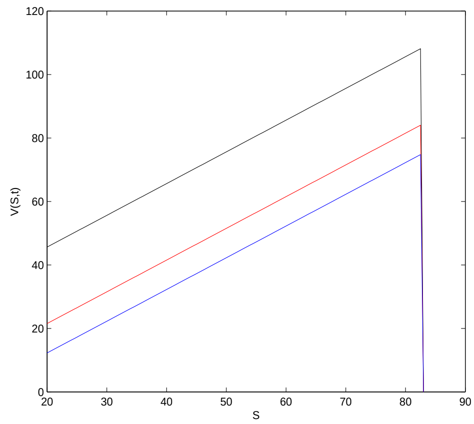

Figure 1: up-and-out barrier option values with $\alpha = \{\frac{1}{8}(black), \frac{1}{4}(red), \frac{2}{3}(blue)\}$, $t = \frac{1}{2}$, $\sigma = 0.2$, $T = 2$, $r = 0.05$ and $B_u = ln(83)$.

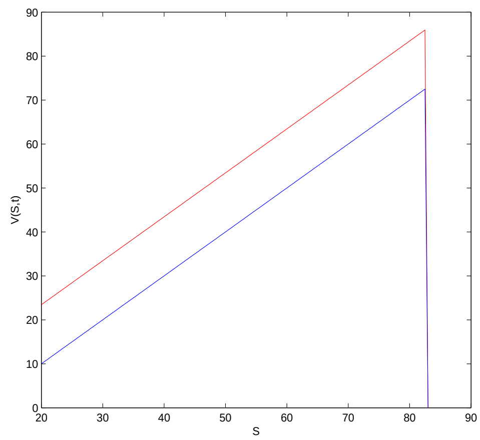

Figure 2: up-and-out barrier option values with $\alpha = \frac{1}{4}$, $t = 0(red)$, $t = T(blue)$, $\sigma = 0.2$, $T = 2$, $r = 0.05$ and $B_u = ln(83)$.

we show the difference in values for the up-and-out barrier option for $t = 0$ and $t = T = 2$ in the figure **??**.

6 Conclusion

In this paper, the application of generalized differential transform method and decomposition method was extended to explicit the analytical solutions of the fractional Black-Scholes equation. This schemes was clearly very efficient and powerful technique in finding the solutions of the proposed equations. The main advantage of this methods is to overcome the deficiency that is caused by unsatisfied conditions. We can apply this method for pricing other options in fractional black-scholes market and nonlinear black-scholes market in future.

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