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Application of CAS wavelet to construct quadrature rules for numerical integration

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Abstract

In this paper, based on CAS wavelets we present quadrature rules for numerical solution of double and triple integrals with variable limits of integration. To construct new method, first, we approximate the unknown function by CAS wavelets. Then by using suitable collocation points, we obtain the CAS wavelet coefficients that these coefficients are applied in approximating the unknown function. The major advantage of new approach is that this method can approximate the value of some improper integrals. To illustrate the efficiency and the accuracy of the method, some numerical examples are given.

Keywords : CAS Wavelets; Quadrature rules; Hybrid functions; Double and triple integrals.

1 Introduction

Construction of quadrature rules for numerical integration based on interpolating polynomials is done by many authors that these polynomials are used to find weights corresponding to nodes. To see some quadrature rules based on polynomials, one can refer to [4]-[11] and the references therein. As we know, wavelets analysis plays an important role in different areas of mathematics [12]-[16]. So, many authors applied wavelets to approximate the solution of integral equations, ordinary differential equations and partial differential equations. Recently, in [2], the authors applied Haar wavelets and hybrid functions to find numerical solution of definite integrals with constant limits. In [1], the authors extended the scope of applicability of the method presented in [2] to double and triple integrals with variable limits.

In this paper, we apply the Cos and Sin (CAS) wavelets for approximating double and triple integrals with variable limits of integration. Also, we present quadrature rules which can approximate some improper integrals. The rest of the paper is organized as follows: In Section 2, we introduce the CAS wavelets. Also, in this section, we construct quadrature rules for approximating double and triple integrals. In Section 3, we provide several examples to show the efficiency and simplicity of the presented method. Concluding remarks are given in the last section.

2 Numerical integration based on CAS wavelets

2.1 CAS wavelets and function approximation

Here first we introduce CAS wavelets from [17] and function approximation by using them.

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We know that wavelets are mathematical functions that are constructed using dilation and translation of a single function called the mother wavelet denoted by $\psi(x)$ and must satisfy certain requirements [17]. If the dilation parameter is a and translation parameter is b then we have the following family of wavelets:

$$\psi_{a,b}(x) = \mid a \mid^{-1/2} \psi(\frac{x-b}{a}), \ a, b \in \mathbb{R}, \ a \neq 0$$

By choosing a and b as $a = a_0^{-k}$ and $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and n and k as positive integers, we conclude that

$$\psi_{k,n}(x) = \mid a \mid^{k/2} \psi(a_0^k x - nb_0),$$

where $\psi_{k,n}(x)$ form a basis for $L^2(\mathbb{R})$. It is clear that, for $a_0 = 2$ and $b_0 = 1$, the set of $\psi_{k,n}(x)$ form an orthonormal basis for $L^2(\mathbb{R})$. In this paper, to construct quadrature rules, we will apply the CAS wavelets defined as the following:

$$\psi_{k,n}(x) = \begin{cases} 2^{k/2} CAS_m(2^k x - n), & \frac{n}{2^k} \leq x < \frac{n+1}{2^k}, \\ 0, & otherewise, \end{cases} \quad (2.1)$$

where

$$CAS_m(x) = cos(2m\pi x) + sin(2m\pi x),$$

and $n = 0, 1, ..., 2^k - 1$, $k \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{Z}$. Clearly, the set of CAS wavelets forms an orthonormal basis for any function $f(x)$ which is square integrable in the interval $[0, 1)$. So, we can expand $f \in L^2[0, 1]$ as the following form:

$$f(x) = \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} c_{n,m} \psi_{n,m}(x)$$

$$\cong \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m} \psi_{n,m}(x) = \mathcal{C}^T \Psi(x), \quad (2.2)$$

where $c_{n,m} =< f(x), \psi_{n,m}(x) >= \int_0^1 f(x)\psi_{n,m}(x)dx$, and $< f, g >$ is the inner product of the functions f and g. Also, $\mathcal{C}$ and $\Psi(x)$ are $2^k(2M + 1) \times 1-$ vectors given by:

$$\mathcal{C} = [C_0, C_1, ..., c_{2^k-1}]^T,$$

$$\Psi(x) = [\psi_0(x), \psi_1(x), ..., \psi_{2^k-1,M}(x)]^T,$$

$$C_i = [c_{i,-M}, c_{i,-M+1}, ..., c_{i,M}],$$

$$\psi_i(x) = [\psi_{i,-M}(x), \psi_{i,-M+1}(x), ..., \psi_{i,M}(x)],$$

$i = 0, 1, ..., 2^k - 1$.

2.2 Quadrature rules based on CAS wavelets

In this section, we present numerical integration for single, double and triple integrals by using CAS wavelets.

2.2.1 Quadrature rules for single integrals

Consider the following definite integral

$$\int_0^1 f(x)dx. \quad (2.3)$$

Using (2.2), we conclude that

$$\int_0^1 \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m}\psi_{n,m}(x)dx$$

$$= \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m} \int_0^1 \psi_{n,m}(x)dx$$

$$= \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m} \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} 2^{k/2} CAS_m(2^k x - n)dx$$

$$= \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m} 2^{-k/2} \int_0^1 CAS_m(x)dx$$

$$= \sum_{n=0}^{2^k-1} c_{n,0} 2^{-k/2}.$$

So, we have:

$$\int_0^1 f(x)dx \cong \frac{1}{2^{k/2}} \sum_{n=0}^{2^k-1} c_{n,0} \quad (2.4)$$

Now, we obtain the coefficients $c_{n,0}$ of CAS wavelets by using collocation points as

$$x_j = \frac{2j-1}{2^{k+1}(2M+1)}, \ j = 1, 2, ..., 2^k(2M + 1). \quad (2.5)$$

Therefore, by substituting these points in (2.2), we have:

$$f(x_j) = \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} c_{n,m}\psi_{n,m}(x_j), \quad (2.6)$$

$j = 1, 2, ..., 2^k(2M + 1)$. Clearly, (2.6) is a linear system of equations. Hence, by solving this system, we can obtain the CAS wavelets coefficients

Table 1: Relative errors for Example 3.1.

Proposed method	Relative errors
M=3,k=3	4.62803E-05
M=4,k=3	2.79953E-06
M=5, k=5	1.17121E-06
M=6, k=6	2.09639E-07

Table 2: Relative errors for Example 3.2.

Proposed method	Relative errors
M=3,k=2	3.96816E-07
M=4,k=3	9.60198E-06
M=5, k=4	2.51085E-09
M=6, k=5	7.19083E-09

Table 3: Relative errors for Example 3.3.

Proposed method	Relative errors	Proposed method in 2.1	Relative errors
M=3,k=2	8.14033E-03	m=3,n=5	8.9863E-03
M=4,k=3	1.58952E-03	m=4,n=8	4.6532E-03
M=5, k=4	1.300763E-03	m=5,n=12	1.8550E-03

Table 4: Relative errors for Example 3.4.

Proposed method	Relative errors	Proposed method in [1]	Relative errors
M=3,k=2	8.14033E-03	m=3,n=5	8.9863E-03
M=4,k=3	1.58952E-03	m=4,n=8	4.6532E-03
M=5, k=4	1.300763E-03	m=5,n=12	1.8550E-03

Table 5: Relative errors for Example 3.5.

Proposed method	Relative errors
M=1,k=1	3.55334E-03
M=3,k=4	1.06426E-05
M=4, k=4	1.43786E-05

$c_{n,0}$, $n = 0, 1, ..., 2^k - 1$. Here, we solve the above system for different values of M and present different quadrature rules for numerical solution of (2.3).
For $M = 1$, we have:

$$c_{n,0} = \frac{1}{3 \times 2^{k/2}} f\left(\frac{2i-1}{6 \times 2^k}\right). \tag{2.7}$$

So, numerical solution of (2.3) is as follows:

$$\int_0^1 f(x)dx \approx \frac{1}{3 \times 2^k} \sum_{i=1}^{3\times 2^k} f\left(\frac{2i-1}{6 \times 2^k}\right). \tag{2.8}$$

For $M = 2$,

$$\int_0^1 f(x)dx \approx \frac{1}{5 \times 2^k} \sum_{i=1}^{5\times 2^k} f\left(\frac{2i-1}{10 \times 2^k}\right). \tag{2.9}$$

For $M = 3$,

$$\int_0^1 f(x)dx \approx \frac{1}{7 \times 2^k} \sum_{i=1}^{7\times 2^k} f\left(\frac{2i-1}{14 \times 2^k}\right). \tag{2.10}$$

For $M = 4$,

$$\int_0^1 f(x)dx \approx \frac{1}{9 \times 2^k} \sum_{i=1}^{9\times 2^k} f\left(\frac{2i-1}{18 \times 2^k}\right). \tag{2.11}$$

For $M = 5$,

$$\int_0^1 f(x)dx \approx \frac{1}{11 \times 2^k} \sum_{i=1}^{11 \times 2^k} f\left(\frac{2i-1}{22 \times 2^k}\right). \tag{2.12}$$

It is clear that, for $M \in \mathbb{N}$, we conclude that

$$\int_0^1 f(x)dx \approx \frac{1}{r \times 2^k} \sum_{i=1}^{r \times 2^k} f\left(\frac{2i-1}{2r \times 2^k}\right), \tag{2.13}$$

where $r = 2M + 1$. Also, we can approximate

$$\int_a^b f(x)dx \tag{2.14}$$

by the following quadrature rules

$$\int_a^b f(x)dx$$

$$\approx \frac{b-a}{r \times 2^k} \sum_{i=1}^{r \times 2^k} f\left(a + \frac{(b-a)(2i-1)}{2r \times 2^k}\right). \tag{2.15}$$

2.2.2 Quadrature rules for double integrals

Here, we consider

$$\int_a^b \int_{h(y)}^{g(y)} f(x,y)dxdy. \tag{2.16}$$

To construct quadrature rules for approximating (2.16), we apply formula (2.15) for

$$\int_{h(y)}^{g(y)} f(x,y)dx. \tag{2.17}$$

Thus we obtain

$$\int_{h(y)}^{g(y)} f(x,y)dx \approx H(y), \tag{2.18}$$

where $r = 2M + 1$, and

$$H(y) = \frac{g(y)-h(y)}{r \times 2^k}$$

$$\times \sum_{i=1}^{r \times 2^k} f\left(h(y) + \frac{(g(y)-h(y))(2i-1)}{2r \times 2^k}, y\right).$$

Now, by applying formula (2.15) once again for

$$\int_a^b H(y)dy, \tag{2.19}$$

we obtain quadrature rules to approximate (2.16) as the following form

$$\int_a^b \int_{h(y)}^{g(y)} f(x,y)dxdy \approx \int_a^b H(y)dy$$

$$\approx \frac{b-a}{r \times 2^k} \sum_{i=1}^{r \times 2^k} H\left(a + \frac{(b-a)(2i-1)}{2r \times 2^k}\right). \tag{2.20}$$

2.2.3 Quadrature rules for triple integrals

Consider

$$\int_a^b \int_{h(z)}^{g(z)} \int_{e(y,z)}^{q(y,z)} f(x,y,z)dxdydz. \tag{2.21}$$

To construct quadrature rules for approximating (2.21), we apply formula (2.15) for

$$\int_{e(y,z)}^{q(y,z)} f(x,y,z)dx. \tag{2.22}$$

Thus we obtain

$$\int_{e(y,z)}^{q(y,z)} f(x,y,z)dx \approx H(y,z), \tag{2.23}$$

where $r = 2M + 1$, and

$$H(y,z) = \frac{q(y,z)-e(y,z)}{r \times 2^k} \times \sum_{i=1}^{r \times 2^k}$$

$$f\left(e(y,z) + \frac{(q(y,z)-e(y,z))(2i-1)}{2r \times 2^k}, y, z\right).$$

Now, by applying formula (2.15) once again for

$$\int_{h(z)}^{g(z)} H(y,z)dy, \tag{2.24}$$

we obtain quadrature rules to approximate (2.24) as the following form

$$\int_{h(z)}^{g(z)} H(y,z)dy \approx \frac{g(z)-h(z)}{r \times 2^k}$$

$$\times \sum_{i=1}^{r \times 2^k} H\left(h(z) + \frac{(g(z)-h(z))(2i-1)}{2r \times 2^k}, z\right). \tag{2.25}$$

Let

$$R(z) = \frac{g(z)-h(z)}{r \times 2^k}$$

$$\times \sum_{i=1}^{r \times 2^k} H\left(h(z) + \frac{(g(z)-h(z))(2i-1)}{2r \times 2^k}, z\right). \tag{2.26}$$

Now, we apply formula (2.15) once again to approximate

$$\int_a^b R(z)dz. \tag{2.27}$$

So, we obtain the following quadrature rules to approximate (2.27)

$$\int_a^b R(z)dz \approx \frac{b-a}{r \times 2^k} \sum_{i=1}^{r\times 2^k} R\left(a+\frac{(b-a)(2i-1)}{2r \times 2^k}\right), \tag{2.28}$$

and hence

$$\int_a^b \int_{h(y)}^{g(y)} \int_{e(y,z)}^{q(y,z)} f(x,y,z)dxdydz$$

$$\approx \frac{b-a}{r \times 2^k} \sum_{i=1}^{r\times 2^k} R\left(a+\frac{(b-a)(2i-1)}{2r \times 2^k}\right). \tag{2.29}$$

3 Numerical examples

To show the accuracy and efficiency of the quadrature rules defined in ([16]), ([?]) and ([?]), we present some examples.

Example 3.1 *Consider the following integral*

$$\int_0^1 sin(x^2)dx.$$

For different values of M and k, relative errors are shown in Table 1.

Example 3.2 *Consider the following integral*

$$\int_0^1 \sqrt{x^2-5x+31}dx.$$

For different values of M and n, relative errors are shown in Table 2.

Example 3.3

$$\int_0^1 \frac{e^{-1/x}}{x^2}dx.$$

Relative errors are shown in Table 3.

Example 3.4

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{x^2+y^2}}dx.$$

Relative errors are shown in Table 4.

Example 3.5

$$\int_0^{\pi} \int_0^{z} \int_0^{zy} \frac{1}{y} sin(\frac{x}{y})dx.$$

Relative errors are shown in Table 5.

4 Conclusion

In this paper, new quadrature rules to approximate double and triple integrals with variables limits are presented. For this purpose, the CAS wavelets have been used. Presented quadrature rules in this paper can approximate some improper integrals. To show the efficiency of presented methods, some test problems are considered.

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