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Duality of g-Bessel sequences and some results about RIP g-frames

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Abstract

In this paper, first we develop the duality concept for g-Bessel sequences and Bessel fusion sequences in Hilbert spaces. We obtain some results about dual, pseudo-dual and approximate dual of frames and fusion frames. We also expand every g-Bessel sequence to a frame by summing some elements. We define the restricted isometry property for g-frames and generalize some results from (B. G. Bodmann et al, Fusion frames and the restricted isometry property, Num. Func. Anal. Optim. 33 (2012) 770-790) to g-frame situation. Finally we study the stability of g-frames under erasure of operators.

Keywords : G-frames; Fusion frames; Dual frames; Pseudo-dual frames; Approximate dual frames; Bessel sequences.

1 Introduction

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{K}$, where I is a subset of $\mathbb{Z}$. For any frame $\{f_i\}_{i\in I}$ there exists at least one dual frame, i.e., a frame $\{g_i\}_{i\in I}$ for which

$$f = \sum_{i\in I} < f, g_i > f_i \qquad \forall f \in \mathcal{H}.$$

If $\{f_i\}_{i\in I}$ is a Bessel sequence with bound $B < 1$, how can we find two sequences $\{g_i\}_{i\in I}$ and $\{p_i\}_{i\in I}$ such that $\{f_i+g_i\}_{i\in I}$ and $\{p_i\}_{i\in I}$ are dual frames, i.e., such that

$$\begin{aligned} f &= \sum_{i\in I} < f, p_i > (f_i + g_i) \\ &= \sum_{i\in I} < f, f_i + g_i > p_i, \end{aligned}$$

for all $f \in \mathcal{H}$. In this paper we obtain some the more general results of the type (1). Let $\mathcal{L}(\mathcal{H}, W_i)$ be the collection of all bounded linear operators from $\mathcal{H}$ into W_i. Recall that a family of operators $\Lambda = \{\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i) : \quad i \in I\}$ is said to be a generalized frame, or simply a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants C and D are called g-frame bounds and $\sup_{i\in I} \Lambda_i$ is called the multiplicity of the g-frame. We call Λ a tight g-frame if $C = D$ and it is a Parseval g-frame if $C = D = 1$. Λ is called a ε-g-frame for $\mathcal{H}$ if $C = \frac{1}{1+\varepsilon}$ and $D = 1+\varepsilon$ for some $\varepsilon > 0$. If the right-hand side of (1.1) holds, then Λ is said a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. The representation space associated with a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is defined

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by

$$(\sum_{i\in I}\oplus W_i)_{\ell^2} = \Big\{\{g_i\}_{i\in I} | g_i \in W_i, \sum_{i\in I}\|g_i\|^2 < \infty\Big\}.$$

The synthesis operator of Λ is defined by

$$T_\Lambda : \Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2} \to \mathcal{H}$$
$$T_\Lambda(\{g_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* g_i.$$

The adjoint operator of T_Λ, which is called the analysis operator also obtain as follows

$$T_\Lambda^* : \mathcal{H} \to \Big(\sum_{i\in I}\oplus W_i\Big)_{\ell^2}$$
$$T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}.$$

By composing T_Λ with its adjoint T_Λ^*, we obtain the fusion frame operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H}$$
$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I}\Lambda_i^*\Lambda_i f,$$

which is a bounded, self-adjoint, positive and invertible operator and $CI_\mathcal{H} \le S_\Lambda \le DI_\mathcal{H}$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widetilde{\Lambda}_i\}_{i\in I}$ with $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for $\mathcal{H}$ with g-frame bounds $\frac{1}{D}$ and $\frac{1}{C}$, respectively. Also we have

$$f = \sum_{i\in I}\Lambda_i^*\widetilde{\Lambda}_i f = \sum_{i\in I}\widetilde{\Lambda}_i^*\Lambda_i f \quad \forall f \in \mathcal{H}.$$

For more details about the theory and applications of frames we refer the readers to [1, 8, 9, 10, 11] and for fusion frames to [2, 4, 5, 7], about g-frames to [3, 12, 13].

The paper is organized as follows: Section 2, contains an extension of g-Bessel sequences to dual g-frames. In this Section, we consider the dual, pseudo-dual and approximate dual frames, fusion frames and we obtain several characterizations of all this dual frames. In Section 3, we generalize the restricted isometry property to the g-frame situation. In Section 4, we study the conditions which under removing some element from a g-frame, again we obtain another g-frame.

2 Dual, approximate dual and pseudo-dual of g-frames

Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be g-Bessel sequences for $\mathcal{H}$ with synthesis operators T_Λ and T_Γ respectively. Then we say that Λ and Γ are dual g-frames for $\mathcal{H}$ if $T_\Lambda T_\Gamma^* = I_\mathcal{H}$ or $T_\Gamma T_\Lambda^* = I_\mathcal{H}$.

In the following we show that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim [9] to the situation of g-frames.

Theorem 2.1 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be two g-Bessel sequences for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then there exist g-Bessel sequences $\{\Xi_j\}_{j\in J}$ and $\{\Omega_j\}_{j\in J}$ for $\mathcal{H}$ with respect to $\{V_j\}_{j\in J}$, such that $\{\Lambda_i\}_{i\in I}\cup\{\Xi_j\}_{j\in J}$ and $\{\Gamma_i\}_{i\in I}\cup\{\Omega_j\}_{j\in J}$ form a pair of dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}\cup\{V_j\}_{j\in J}$.*

Proof. Assume that $\{\Phi_j\}_{j\in J}$ and $\{\Psi_j\}_{j\in J}$ are any pair of dual g-frames for $\mathcal{H}$ with respect to $\{V_j\}_{j\in J}$ and let $\Theta = I_\mathcal{H} - T_\Gamma T_\Lambda^*$. Then for any $f \in \mathcal{H}$ we have

$$f = \Theta f + T_\Gamma T_\Lambda^* f = \sum_{j\in J}\Psi_j^*\Phi_j\Theta f + \sum_{i\in I}\Gamma_i^*\Lambda_i f.$$

If we set $\Xi_j = \Phi_j\Theta$ and $\Omega_j = \Psi_j$ for all $j \in J$. Then $\{\Lambda_i\}_{i\in I}\cup\{\Xi_j\}_{j\in J}$ and $\{\Gamma_i\}_{i\in I}\cup\{\Omega_j\}_{j\in J}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}\cup\{V_j\}_{j\in J}$.

Theorem 2.2 *Let $\mathcal{F}$ be a Bessel sequence for $\mathcal{H}$ with Bessel bound $B < 1$ and let $\mathcal{E}$ be Parseval frame for $\mathcal{H}$. Then there exists a Bessel sequence $\mathcal{G}$ for $\mathcal{H}$ such that $\mathcal{F}+\mathcal{E}$ and $\mathcal{G}+\mathcal{E}$ are dual frames.*

Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_i\}_{i\in I}$. Since $B < 1$, $I_\mathcal{H} + T_\mathcal{F}T_\mathcal{E}^*$ is an invertible operator in $\mathcal{L}(\mathcal{H})$. If we define

$$\Theta = -(I_\mathcal{H} + T_\mathcal{F}T_\mathcal{E}^*)^{-1}T_\mathcal{F}T_\mathcal{E}^*$$

and $g_i = \Theta^* e_i$ for all $i \in I$. Then $\mathcal{G} = \{g_i\}_{i\in I}$ is a

Bessel sequence for $\mathcal{H}$ and for all $f \in \mathcal{H}$ we have

$$\begin{aligned} f &= (I_{\mathcal{H}} + T_{\mathcal{F}}T_{\mathcal{E}}^*)\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^* f + T_{\mathcal{F}}T_{\mathcal{E}}^* f \\ &= T_{\mathcal{E}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{E}}T_{\mathcal{E}}^* f + T_{\mathcal{F}}T_{\mathcal{E}}^*\Theta f + T_{\mathcal{F}}T_{\mathcal{E}}^* f \\ &= \sum_{i\in I} < \Theta f, e_i > e_i + \sum_{i\in I} < f, e_i > e_i \\ &+ \sum_{i\in I} < \Theta f, e_i > f_i + \sum_{i\in I} < f, e_i > f_i \\ &= \sum_{i\in I} < f, g_i + e_i > (f_i + e_i), \end{aligned}$$

which this finishes the proof. The following corollaries are generalizations of Theorem 2.2 to the g-frames situation. We leave the proofs to interested readers.

Corollary 2.1 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-Bessel bound $B < 1$. Then there exists g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, such that $\{\Xi_i + \Lambda_i\}_{i\in I}$ and $\{\Xi_i + \Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, where $\{\Xi_i\}_{i\in I}$ is a Parseval g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*

Corollary 2.2 *For every g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ with Bessel bound $B < 1$ and each Parseval g-frame $\Xi = \{\Xi_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, there exists g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ such that $\{\Lambda_i + \Xi_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*

Corollary 2.3 *For every g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ there exist g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ and a tight g-frame $\{\Xi_i\}_{i\in I}$ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ such that $\{\Lambda_i + \Xi_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are dual g-frames for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*

Let $\mathcal{W} = \{W_i\}_{i\in I}$ be a sequence of closed subspaces in $\mathcal{H}$, and let $\mathcal{A} = \{\alpha_i\}_{i\in I}$ be a family of weights, i.e., $\alpha_i > 0$ for all $i \in I$. A sequence $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ is a fusion frame, if there exist real numbers $0 < C \le D < \infty$ such that for all $f \in \mathcal{H}$:

$$C\|f\|^2 \le \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f)\|^2 \le D\|f\|^2, \qquad (2.2)$$

where π_{W_i} is the orthogonal projection from $\mathcal{H}$ onto W_i. The constant C, D are called the fusion frame bounds. If the right-hand inequality of (2.2) holds, then we say that $\mathcal{W}_\alpha$ is a Bessel fusion sequence with Bessel fusion bound D. Moreover if $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ is a frame for W_i for all $i \in I$. Then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i\in I}$ is called a fusion frame system for $\mathcal{H}$. The constants A, B are called the local frame bounds if they are the common frame bounds for the local frame $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ for all $i \in I$. A collection of dual frames $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$, $i \in I$ associated with the local frames is called local dual frames. By Theorem 3.2 from [7], if $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds C, D and local frame bounds A, B, then $\mathcal{F} = \{\alpha_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds AC and BD. Also if $\mathcal{F} = \{\alpha_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ with frame bounds C and D, then $\mathcal{W} = \{(W_i, \alpha_i, \mathcal{F}_i)\}_{i\in I}$ is a fusion frame system for $\mathcal{H}$ with fusion frame bounds $\frac{C}{B}$ and $\frac{D}{A}$.

Definition 2.1 *Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ be Bessel fusion sequences for $\mathcal{H}$ with synthesis operators $T_{\mathcal{W}_\alpha}$ and $T_{\mathcal{Z}_\beta}$ respectively. Then*

(i) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^* = I_{\mathcal{H}}$ or $T_{\mathcal{Z}_\alpha} T_{\mathcal{W}_\beta}^* = I_{\mathcal{H}}$.

(ii) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are approximate dual fusion frames for $\mathcal{H}$ if $\|I_{\mathcal{H}} - T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*\| < 1$ or $\|I_{\mathcal{H}} - T_{\mathcal{Z}_\alpha} T_{\mathcal{W}_\beta}^*\| < 1$.

(iii) $\mathcal{W}_\alpha, \mathcal{Z}_\beta$ are called pseudo-dual fusion frames for $\mathcal{H}$ if $T_{\mathcal{W}_\alpha} T_{\mathcal{Z}_\beta}^*$ or $T_{\mathcal{Z}_\alpha} T_{\mathcal{W}_\beta}^*$ is a bijection on $\mathcal{H}$.

Theorem 2.3 *For each $i \in I$ let $\alpha_i > 0$ and $J_i = J_{i1} \cup J_{i2}$ be a partition of J_i and let $\mathcal{W} = \{(W_i, \alpha_i, \{f_{ij}\}_{j\in J_{i1}})\}_{i\in I}$ and $\mathcal{Z} = \{(Z_i, \beta_i, \{g_{ij}\}_{j\in J_{i2}})\}_{i\in I}$ be two fusion frame system for $\mathcal{H}$. Define*

$$u_{ij} = \begin{cases} \frac{1}{\sqrt{2}} f_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{2}} \pi_{W_i} \widetilde{g}_{ij} & j \in J_{i2} \end{cases}$$

and

$$v_{ij} = \begin{cases} \frac{1}{\sqrt{2}} \pi_{Z_i} \widetilde{f}_{ij} & j \in J_{i1} \\ \frac{1}{\sqrt{2}} g_{ij} & j \in J_{i2} \end{cases}$$

for all $i \in I, j \in J_i$. Then the following conditions are equivalent:

(1) $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ and $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ are (dual, pseudo-dual, approximate dual) fusion frames.

(2) $\{\alpha_i u_{ij}\}_{i\in I, j\in J_i}$ *and* $\{\beta_i v_{ij}\}_{i\in I, j\in J_i}$ *are (dual, pseudo-dual, approximate dual) frames for* $\mathcal{H}$.

Proof. This claim follows immediately from the fact that for $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{i\in I}\sum_{j\in J_i} < f, \beta_i v_{ij} > \alpha_i u_{ij} \\
=&\sum_{i\in I}\alpha_i\beta_i \sum_{j\in J_{i1}} < f, v_{ij} > u_{ij} \\
+&\sum_{i\in I}\alpha_i\beta_i \sum_{j\in J_{i2}} < f, v_{ij} > u_{ij} \\
=&\sum_{i\in I}\alpha_i\beta_i \sum_{j\in J_{i1}} < f, \frac{1}{\sqrt{2}}\pi_{Z_i}\widetilde{f}_{ij} > \frac{1}{\sqrt{2}} f_{ij} \\
+&\sum_{i\in I}\alpha_i\beta_i \sum_{j\in J_{i2}} < f, \frac{1}{\sqrt{2}}g_{ij} > \frac{1}{\sqrt{2}}\pi_{W_i}\widetilde{g}_{ij} \\
=&\sum_{i\in I}\frac{\alpha_i\beta_i}{2} \sum_{j\in J_{i1}} < \pi_{Z_i}(f), \widetilde{f}_{ij} > f_{ij} \\
+&\sum_{i\in I}\frac{\alpha_i\beta_i}{2}\pi_{W_i}(\sum_{j\in J_{i2}} < f, g_{ij} > \widetilde{g}_{ij}) \\
=&\sum_{i\in I}\alpha_i\beta_i\pi_{W_i}\pi_{Z_i}(f)
\end{aligned}$$

Theorem 2.4 *Let* $\{(W_i, \alpha_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$ *be a fusion frame system and let* $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ *be a fusion Bessel sequence for* $\mathcal{H}$. *Put* $g_{ij} = \pi_{Z_i}(\widetilde{f}_{ij})$ *for all* $i \in I, j \in J_i$. *Then the following conditions are equivalent:*

(1) $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ *and* $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ *are (dual, pseudo-dual, approximate dual) fusion frames.*

(2) $\mathcal{F} = \{\alpha_i f_{ij}\}_{i\in I, j\in J_i}$ *and* $\mathcal{G} = \{\beta_i g_{ij}\}_{i\in I, j\in J_i}$ *are (dual, pseudo-dual, approximate dual) frames for* $\mathcal{H}$.

Proof. First we prove that $\mathcal{G}$ is a Bessel sequence for $\mathcal{H}$. Let D be the Bessel fusion bound of $\mathcal{Z}_\beta$ and A, B be the local frame bounds of $\{(W_i, \alpha_i, \{f_{ij}\}_{j\in J_i})\}_{i\in I}$, then for all $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\sum_{i\in I}\sum_{j\in J_i} |< f, \beta_i g_{ij} >|^2 \\
=&\sum_{i\in I}\sum_{j\in J_i}\beta_i^2 |< f, \pi_{Z_i}(\widetilde{f}_{ij}) >|^2 \\
=&\sum_{i\in I}\beta_i^2\sum_{j\in J_i} |< \pi_{Z_i}(f), \widetilde{f}_{ij} >|^2 \\
\leq&\sum_{i\in I}\frac{\beta_i^2}{A}\|\pi_{W_i}\pi_{Z_i}(f)\|^2 \\
\leq&\frac{1}{A}\sum_{i\in I}\beta_i^2\|\pi_{Z_i}(f)\|^2 \leq \frac{D}{A}\|f\|^2.
\end{aligned}$$

Let $T_{\mathcal{F}}$ and $T_{\mathcal{G}}$ be the synthesis operators for $\mathcal{F}$ and $\mathcal{G}$ respectively. Then for all $f \in \mathcal{H}$ we obtain

$$\begin{aligned}
T_{\mathcal{W}_\alpha}T^*_{\mathcal{Z}_\beta}(f) =&\sum_{i\in I}\alpha_i\beta_i\pi_{W_i}\pi_{Z_i}(f) \\
=&\sum_{i\in I}\alpha_i\beta_i\sum_{j\in J_i} < \pi_{Z_i}(f), \widetilde{f}_{ij} > f_{ij} \\
=&\sum_{i\in I}\sum_{j\in J_i} < f, \beta_i g_{ij} > \alpha_i f_{ij} \\
=&T_{\mathcal{F}}T^*_{\mathcal{G}}(f).
\end{aligned}$$

This finishes the proof.

Theorem 2.5 *Let* $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ *and* $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ *be Bessel fusion sequences for* $\mathcal{H}$ *and let* $T \in B(\mathcal{H})$ *be a bounded invertible operator such that* $T^*TW_i \subseteq W_i$, $T^*TZ_i \subseteq Z_i$. *Then*

(1) $\mathcal{W}_\alpha$ *and* $\mathcal{Z}_\beta$ *are (dual, pseudo-dual) fusion frames if and only if* $\mathcal{T}\mathcal{W}_\alpha = \{(TW_i, \alpha_i)\}_{i\in I}$ *and* $\mathcal{T}\mathcal{Z}_\beta = \{(TZ_i, \beta_i)\}_{i\in I}$ *are (dual, pseudo-dual) fusion frame for* $\mathcal{H}$.

(2) *If* $\mathcal{W}_\alpha$ *and* $\mathcal{Z}_\beta$ *are approximate dual fusion frames and* $\|T\|\|T^{-1}\| = 1$ *then* $\mathcal{T}\mathcal{W}_\alpha = \{(TW_i, \alpha_i)\}_{i\in I}$ *and* $\mathcal{T}\mathcal{Z}_\beta = \{(TZ_i, \beta_i)\}_{i\in I}$ *are also approximate dual fusion frames for* $\mathcal{H}$.

Proof. (1) Since T is invertible and $T^*TW_i \subseteq W_i$, $T^*TZ_i \subseteq Z_i$ hence for all $i \in I$ $\pi_{TW_i} = T\pi_{W_i}T^{-1}$, $\pi_{TZ_i} = T\pi_{Z_i}T^{-1}$. This implies that $T_{\mathcal{T}\mathcal{W}_\alpha}T^*_{\mathcal{T}\mathcal{Z}_\beta} = TT_{\mathcal{W}_\alpha}T^*_{\mathcal{Z}_\beta}T^{-1}$, that from this the claim follows immediately.
(2) We have

$$\begin{aligned}
&\|Id_{\mathcal{H}} - T_{\mathcal{T}\mathcal{W}_\alpha}T^*_{\mathcal{T}\mathcal{Z}_\beta}\| \\
&= \|TT^{-1} - TT_{\mathcal{W}_\alpha}T^*_{\mathcal{Z}_\beta}T^{-1}\| \\
&\leq \|Id_{\mathcal{H}} - T_{\mathcal{W}_\alpha}T^*_{\mathcal{Z}_\beta}\|.
\end{aligned}$$

From this the result follows at once.

Theorem 2.6 *Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$ be a fusion frame and let $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i\in I}$ be a Bessel fusion sequence for $\mathcal{H}$. Suppose that $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded invertible operator such that $TW_i \subseteq Z_i$ for all $i \in I$. Then $\mathcal{Z}_\alpha = \{(Z_i, \alpha_i)\}_{i\in I}$ and $\mathcal{TW}_\alpha = \{(TW_i, \alpha_i)\}_{i\in I}$ are pseudo-dual fusion frames for $\mathcal{H}$. Moreover if $\mathcal{TW}_\alpha$ is a Parseval fusion frame then $\mathcal{Z}_\alpha$ and $\mathcal{TW}_\alpha$ are dual fusion frames.*

Proof. Since $TW_i \subseteq Z_i$ hence $\pi_{TW_i}\pi_{Z_i} = \pi_{Z_i}\pi_{TW_i} = \pi_{TW_i}$ for all $i \in I$. It follows that $T_{\mathcal{TW}_\alpha}T^*_{\mathcal{Z}_\alpha} = T_{\mathcal{Z}_\alpha}T^*_{\mathcal{TW}_\alpha} = S_{\mathcal{TW}_\alpha}$ which finishes the proof.

Definition 2.2 *Let $\{W_i\}_{i\in I}$ and $\{\widetilde{W}_i\}_{i\in I}$ be closed subspaces in $\mathcal{H}$ and $\varepsilon > 0$. If for every $f \in \mathcal{H}$ we have*

$$\sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \leq \varepsilon \|f\|^2.$$

Then we say that $\{(\widetilde{W}_i, \alpha_i)\}_{i\in I}$ is a ε-perturbation of $\{(W_i, \alpha_i)\}_{i\in I}$.

Theorem 2.7 *Let $\mathcal{W}_\alpha = \{(W_i, \alpha_i)\}_{i\in I}$, $\mathcal{Z}_\beta = \{(Z_i, \beta_i)\}_{i\in I}$ be Bessel fusion sequences with Bessel fusion bounds D_1, D_2 respectively for $\mathcal{H}$. Let $\widetilde{\mathcal{W}}_\alpha = \{(\widetilde{W}_i, \alpha_i)\}_{i\in I}$ be a ε-perturbation of $\mathcal{W}_\alpha$ and $\varepsilon D_2 < 1$. If $\mathcal{W}_\alpha$ and $\mathcal{Z}_\beta$ are dual fusion frames, then $\widetilde{\mathcal{W}}_\alpha$ and $\mathcal{Z}_\beta$ are also approximate dual fusion frames for $\mathcal{H}$.*

Proof. By Proposition 2.4 from [4] $\widetilde{\mathcal{W}}_\alpha$ is a Bessel fusion sequence for $\mathcal{H}$. Now for all $f \in \mathcal{H}$ we have

$$\begin{aligned}
&\|f - T_{\widetilde{\mathcal{W}}_\oslash} T^*_{\mathcal{Z}_\odot}(f)\|^2 \\
=&\|T_{\mathcal{W}_\oslash} T^*_{\mathcal{Z}_\odot}(f) - T_{\widetilde{\mathcal{W}}_\oslash} T^*_{\mathcal{Z}_\odot}(f)\|^2 \\
=& \sup_{\|g\|=1} |< T_{\mathcal{W}_\oslash} T^*_{\mathcal{Z}_\odot}(f) - T_{\widetilde{\mathcal{W}}_\oslash} T^*_{\mathcal{Z}_\odot}(f), g >|^2 \\
\leq& \sup_{\|g\|=1} \Big(\sum_{i\in I} \alpha_i\beta_i \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\| \|\pi_{Z_i}(g)\| \Big)^2 \\
\leq& \sup_{\|g\|=1} \sum_{i\in I} \alpha_i^2 \|\pi_{W_i}(f) - \pi_{\widetilde{W}_i}(f)\|^2 \\
\times& \sum_{i\in I} \beta_i^2 \|\pi_{Z_i}(g)\|^2 \leq \varepsilon D_2 \|f\|^2.
\end{aligned}$$

From this the result follows at once.

3 RIP for g-frames

In this section we generalize the restricted isometry property for g-frames. We denote that $\mathcal{K}$ is a Hilbert space and $\mathcal{H}_N$ is a Hilbert space with dimension N and $\{e_j\}_{j=1}^N$ an orthonormal basis for $\mathcal{H}_N$. Moreover, the Hilbert-Schmidt norm of operator $T \in \mathcal{L}(\mathcal{H}_N, \mathcal{K})$ is defined by

$$\|T\|_{HS}^2 = \sum_{j=1}^N \|Te_j\|^2.$$

Proposition 3.1 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-frame bounds A and B and $\mathcal{H}$ be finite-dimensional. Then*

$$A \leq \frac{\sum_{i\in I} \|\Lambda_i\|_{HS}^2}{\dim \mathcal{H}} \leq B.$$

Proof. Since

$$\sum_{i\in I} \|\Lambda_i\|_{HS}^2 = \sum_{j=1}^N < S_\Lambda e_j, e_j >$$

and $AI_\mathcal{H} \leq S_\Lambda \leq BI_\mathcal{H}$, we have

$$\begin{aligned}
A \dim \mathcal{H} &= A \sum_{j=1}^N \|e_j\|^2 \\
&\leq \sum_{j=1}^N < S_\Lambda e_j, e_j > \\
&\leq B \sum_{j=1}^N \|e_j\|^2 = B \dim \mathcal{H}.
\end{aligned}$$

This yields

$$A \dim \mathcal{H} \leq \sum_{i\in I} \|\Lambda_i\|_{HS}^2 \leq B \dim \mathcal{H}.$$

From this the claim follows immediately.

Theorem 3.1 *Let $\Lambda = \{\Lambda_i\}_{i=1}^M$ be a g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^M$. Then*

(i) The optimal g-frame bounds of Λ are the smallest and biggest eigenvalues of g-frame operator S_Λ.

(ii) If $\{\lambda_i\}_{i=1}^N$ is a representation of eigenvalues of S_Λ. Then

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|\Lambda_i\|_{HS}^2$$

and

$$\lambda_j = \sum_{i=1}^{M} \|\Lambda_i e_j\|^2,$$

where $\{e_j\}_{j=1}^N$ *is the orthonormal basis consisting of eigenvectors of* S_Λ.

Proof. To prove (i), since S_Λ is a self-adjoint, $\mathcal{H}_N$ has an orthonormal basis include eigenvectors of S_Λ. Let $\{e_j\}_{j=1}^N$ be an orthogonal basis of $\mathcal{H}_N$ include of eigenvectors of S_Λ. Let $\{\lambda_j\}_{j=1}^N$ be eigenvalues of $\{e_j\}_{j=1}^N$. Then for any $f \in \mathcal{H}_N$ we have

$$\begin{aligned}\sum_{i=1}^{M} \|\Lambda_i f\|^2 &= < S_\Lambda f, f > \\ &= < \sum_{j=1}^{N} < f, e_j > S_\Lambda e_j, f > \\ &= \sum_{j=1}^{N} < f, e_j >< S_\Lambda e_j, f > \\ &= \sum_{j=1}^{N} < f, e_j >< \lambda_j e_j, f > \\ &= \sum_{j=1}^{N} \lambda_j |< f, e_j > |^2.\end{aligned}$$

Now from

$$\lambda_{\min} \leq \lambda_i \leq \lambda_{\max}, \quad (1 \leq i \leq N)$$

we obtain

$$\lambda_{\min} \|f\|^2 \leq \sum_{i=1}^{M} \|\Lambda_i f\|^2 \leq \lambda_{\max} \|f\|^2.$$

To prove (ii) we have:

$$\begin{aligned}\sum_{j=1}^{N} \lambda_j &= \sum_{j=1}^{N} < \lambda_j e_j, e_j > \\ &= \sum_{j=1}^{N} < S_\Lambda e_j, e_j > = \sum_{j=1}^{N} \sum_{i=1}^{M} \|\Lambda_i e_j\|^2 \\ &= \sum_{i=1}^{M} \sum_{j=1}^{N} \|\Lambda_i e_j\|^2 = \sum_{i=1}^{M} \|\Lambda_i\|_{HS}^2.\end{aligned}$$

We also have

$$\begin{aligned}\lambda_j &= < \lambda_j e_j, e_j > = < S_\Lambda e_j, e_j > \\ &= \sum_{i=1}^{M} \|\Lambda_i e_j\|^2.\end{aligned}$$

Corollary 3.1 *Let* $\{\Lambda_i\}_{i=1}^M$ *be an* A*-tight g-frame for* $\mathcal{H}_N$ *with respect to* $\{W_i\}_{i=1}^M$ *and* $\|\Lambda_i\|_{HS} = 1$ *for all* $1 \leq i \leq M$. *Then* $A = \frac{M}{N}$.

Proof. This is a direct result from Proposition 3.1.

Definition 3.1 *Let* $\Lambda_i \in \mathcal{L}(\mathcal{H}, W_i)$ *for all* $i \in I$. *Then*

(i) $\{\Lambda_i\}_{i\in I}$ *is called an orthonormal g-system for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$, *if* $\Lambda_i \Lambda_j^* g_j = \delta_{ij} g_j$ *for all* $i, j \in I, g_j \in W_j$.

(ii) *If* $\mathcal{H} = \{\Lambda_i^*(W_i)\}_{i\in I}$, *then we say that* $\{\Lambda_i\}_{i\in I}$ *is g-complete.*

(iii) *We say that* $\{\Lambda_i\}_{i\in I}$ *is a g-orthonormal basis for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$, *if it is a g-orthonormal g-complete system for* $\mathcal{H}$ *with respect to* $\{W_j\}_{j\in J}$.

(iv) $\{\Lambda_i\}_{i\in I}$ *is called a g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$, *if* $\{\Lambda_i\}_{i\in I}$ *is g-complete and there exist real numbers* $0 < A \leq B < \infty$ *such that:*

$$\begin{aligned}A \sum_{j\in J} \|g_j\|^2 \leq & \| \sum_{j\in J} \Lambda_j^* g_j \|^2 \\ \leq & B \sum_{j\in J} \|g_j\|^2,\end{aligned}$$

for all finite subset $J \subset I$ *and* $g_j \in W_j$. *Moreover,* $\{\Lambda_i\}_{i\in I}$ *is called an* ε*-g-Riesz basis for* $\mathcal{H}$, *if* $A = \frac{1}{1+\varepsilon}$ *and* $B = 1 + \varepsilon$ *for some* $\varepsilon > 0$. *Also* $\{\Lambda_i\}_{i\in I}$ *is an* ε*-g-Riesz sequence if* $\{\Lambda_i\}_{i\in I}$ *is an* ε*-g-Riesz basis for* $\{\Lambda_i^*(W_i)\}_{i\in I}$.

The next proposition is similar to a result of Bodmann, Cahill and Casazza [6] to the situation of g-frames.

Proposition 3.2 *Let* $\{\Lambda_i\}_{i\in I}$ *be an* ε*-g-Riesz sequence for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$ *and let* $\{I_j\}_{j=1}^L$ *be a partition of* I. *Then*

$$\begin{aligned}\frac{1}{1+\varepsilon} \sum_{j=1}^{L} \| \sum_{k\in I_j} \Lambda_k^* g_{jk} \|^2 \leq & \sum_{j=1}^{L} \sum_{k\in I_j} \|g_{jk}\|^2 \\ \leq & (1+\varepsilon) \sum_{j=1}^{L} \| \sum_{k\in I_j} \Lambda_k^* g_{jk} \|^2,\end{aligned}$$

for every $1 \leq j \leq L$ *and any sequence* $\{g_{jk}\}_{k\in I_j} \in (\sum_{k\in I_j} \oplus W_k)_{\ell^2}$. *Also*

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2 \leq \Big\|\sum_{j=1}^{L}\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2$$
$$\leq (1+\varepsilon)^2\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2.$$

Proof. Let $1 \leq j \leq L$ and $\{g_{jk}\}_{k\in I_j} \in (\sum_{k\in I_j} \oplus W_k)_{\ell^2}$

$$\frac{1}{1+\varepsilon}\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2$$
$$\leq \frac{1}{1+\varepsilon}\sum_{j=1}^{L}(1+\varepsilon)\sum_{k\in I_j}\|g_{jk}\|^2$$
$$=\sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2 \leq \sum_{j=1}^{L}(1+\varepsilon)\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2$$
$$=(1+\varepsilon)\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2.$$

This yields

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2$$
$$\leq \frac{1}{1+\varepsilon}\sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2 \leq \Big\|\sum_{j=1}^{L}\sum_{k\in I_j}\Lambda_k^* g_{jk}\Big\|^2$$
$$\leq (1+\varepsilon)\sum_{j=1}^{L}\sum_{k\in I_j}\|g_{jk}\|^2$$
$$\leq (1+\varepsilon)^2\sum_{j=1}^{L}\|\sum_{k\in I_j}\Lambda_k^* g_{jk}\|^2.$$

It is known that if $\{\Lambda_i\}_{i\in I}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-Riesz constants A and B, then $\{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with same bounds A and B. The next lemma is analogous to Lemma 3.3 in [6] to the situation of g-frames.

Lemma 3.1 *Let* $\Lambda = \{\Lambda_i\}_{i\in I}$ *be an* ε*-g-Riesz basis for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$. *Then for all* $n \in \mathbb{N}$

$$\frac{1}{(1+\varepsilon)^n}I$$

$$H \leq S_\Lambda^n \leq (1+\epsilon)^n I_{\mathcal{H}} \, and$$

$$\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^{-n} \leq (1+\epsilon)^n I_{\mathcal{H}}.$$

Proof. Since $\{\Lambda_i\}_{i\in I}$ is an ε-g-Riesz basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, so this family is a g-frame for $\mathcal{H}$ with bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectively. Hence $\frac{1}{1+\varepsilon} \leq \|S_\Lambda\| \leq (1+\varepsilon)$ and $\frac{1}{1+\varepsilon} \leq \|S_\Lambda^{-1}\| \leq (1+\varepsilon)$. On the other hand for any $f \in \mathcal{H}$ and $n \in \mathbb{N}$ we have $\|S_\Lambda^{-1}\|^{-n}\|f\| \leq \|S_\Lambda^n f\| \leq \|S_\Lambda\|^n\|f\|$. From this we have $\|S_\Lambda^{-1}\|^{-n}I_{\mathcal{H}} \leq S_\Lambda^n \leq \|S_\Lambda\|^n I_{\mathcal{H}}$. Consequently

$$\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq \|S_\Lambda^{-1}\|^{-n}I_{\mathcal{H}} \leq S_\Lambda^n$$
$$\leq \|S_\Lambda\|^n I_{\mathcal{H}} \leq (1+\varepsilon)^n I_{\mathcal{H}}.$$

This shows that $\frac{1}{(1+\varepsilon)^n}I$

H $\leq S_\Lambda^n \leq (1+\varepsilon)^n I_{\mathcal{H}}$ and so $\frac{1}{(1+\varepsilon)^n}I_{\mathcal{H}} \leq S_\Lambda^{-n} \leq (1+\varepsilon)^n I_{\mathcal{H}}$.

Proposition 3.3 *Let* $\{\Lambda_i\}_{i\in I}$ *be an* ε*-g-Riesz sequence for* $\mathcal{H}$ *with respect to* $\{W_i\}_{i\in I}$. *Then*

$$|< f, g >| \leq 2\varepsilon + \varepsilon^2,$$

for all partition $\{I_1, I_2\}$ *of* I *and* $f \in \{\Lambda_i^*(W_i)\}_{i\in I_1}, g \in \{\Lambda_i^*(W_i)\}_{i\in I_2}$ *with* $\|f\| = \|g\| = 1$.

Proof. Let $F_1 \subseteq I_1, F_2 \subseteq I_2$ be arbitrary finite subsets, $g_i \in W_i (i \in F_1 \bigcup F_2)$ and $\varphi = \sum_{i\in F_1}\Lambda_i^* g_i$ and $\psi = \sum_{i\in F_2}\Lambda_i^* g_i$ with conditions $||\varphi|| = ||\psi|| = 1$. Then for any $|\lambda| = 1$ we have

$$(< \varphi, \lambda\psi >) = \frac{2(< \varphi, \lambda\psi >) + 2}{2} - 1$$
$$= \frac{\|\varphi + \lambda\psi\|^2}{2} - 1 \leq \frac{(1+\varepsilon)}{2}\sum_{i\in F_1\cup F_2}\|g_i\|^2 - 1$$
$$= \frac{(1+\varepsilon)}{2}\Big(\sum_{i\in F_1}\|g_i\|^2 + \sum_{i\in F_2}\|g_i\|^2\Big) - 1$$
$$\leq \frac{(1+\varepsilon)^2}{2}(\|\varphi\|^2 + \|\psi\|^2) - 1 = 2\varepsilon + \varepsilon^2.$$

This yields

$$|< \varphi, \psi >| = \max_{|\lambda|=1} < \varphi, \lambda\psi > \leq 2\varepsilon + \varepsilon^2,$$

which implies that $|< f, g >| \leq 2\epsilon + \epsilon^2$.

Definition 3.2 *For every* $1 \leq i \leq M$, *let* $\Lambda_i \in \mathcal{L}(\mathcal{H}_N, W_i)$. *Then we say that the family* $\{\Lambda_i\}_{i=1}^M$ *has the restricted isometry property with constant* $0 < \varepsilon < 1$ *for sets of size* $s \leq N$, *if for every* $I \subseteq \{1, 2, ..., M\}$ *with* $|I| \leq s$, *the family* $\{\Lambda_i\}_{i\in I}$ *is an* ε*-g-Riesz sequence for* $\mathcal{H}_N$ *with respect to* $\{W_i\}_{i\in I}$.

The next theorem is a generalization of Theorem 4.2 in [6] to the g-frames situation.

Theorem 3.2 *Let $\{\Lambda_i\}_{i=1}^M$ be a tight g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^M$ with the restricted isometry property with constant $0 < \varepsilon < 1$ for sets of size $s \leq N$. Suppose that $\{I_j\}_{j=1}^L$ is an arbitrary partition of $\{1, 2, ..., M\}$ with $|I_j| \leq s$. Define $V_j = \{\Lambda_i^*(W_i)\}_{i \in I_j}$ for all $1 \leq j \leq L$, then $\{V_j\}_{j=1}^L$ is a fusion frame for $\mathcal{H}_N$ with fusion frame bounds $\frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N}$, $\frac{(1+\varepsilon)\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N}$ and*

$$\frac{1}{1+\varepsilon}\sum_{i\in I_j}\|\Lambda_i f\|^2 \leq \|\pi_{V_j} f\|^2 \leq (1+\varepsilon)\sum_{i\in Ij}\|\Lambda_i f\|^2.$$

Proof. By the hypothesis $\{\Lambda_i\}_{i\in I_j}$ is a g-frame for V_j with respect to $\{W_i\}_{i\in I_j}$ for all $1 \leq j \leq L$ with g-frame bounds $\frac{1}{1+\varepsilon}, 1+\varepsilon$ respectively. Let S_j be g-frame operator of $\{\Lambda_i\}_{i\in I_j}$ and $\{e_i\}_{i=1}^N$ be the orthonormal basis of eigenvectors of S_j with eigenvalues $\{\lambda_i\}_{i=1}^N$, then $\lambda_i = 0$ for all $|I_j| < i \leq N$ and $\frac{1}{1+\varepsilon} \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{|I_j|} \leq 1+\varepsilon$. Since $\{e_i\}_{i=1}^{|I_j|}$ is an orthonormal basis for V_j, hence $\pi_{V_j} f = \sum_{i=1}^{|I_j|} < f, e_i > e_i$, for any $f \in \mathcal{H}_N$. Now we have

$$S_j f = S_j(\sum_{i=1}^N < f, e_i > e_i) = \sum_{i=1}^N < f, e_i > S_j e_i = \sum_{i=1}^{|I_j|} < f, e_i > \lambda_i e_i$$

which implies that

$$< S_j f, f > = \sum_{i=1}^{|I_j|} \lambda_i |< f, e_i >|^2.$$

Thus we have

$$\frac{1}{1+\varepsilon}\sum_{i\in I_j}\|\Lambda_i f\|^2 = \frac{1}{1+\varepsilon} < S_j f, f > = \sum_{i\in I_j}\frac{\lambda_i}{1+\varepsilon}|< f, e_i >|^2 \leq \|\pi_{V_j}\|^2 \leq \sum_{i\in I_j}\lambda_i(1+\varepsilon)|< f, e_i >|^2 = (1+\varepsilon) < S_j f, f > = (1+\varepsilon)\sum_{i\in I_j}\|\Lambda_i f\|^2.$$

It follows that

$$\frac{1}{1+\varepsilon}\sum_{j=1}^L\sum_{i\in I_j}\|\Lambda_i f\|^2 \leq \sum_{j=1}^L\|\pi_{V_j} f\|^2 \leq (1+\varepsilon)\sum_{j=1}^L\sum_{i\in I_j}\|\Lambda_i f\|^2.$$

Now by Proposition 3.1 we have

$$\frac{\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{(1+\varepsilon)N}\|f\|^2 \leq \sum_{i=1}^L \|\pi_{V_j} f\|^2 \leq \frac{(1+\varepsilon)\sum_{i=1}^M\|\Lambda_i\|_{HS}^2}{N}\|f\|^2.$$

Corollary 3.2 *Under the assumptions of Theorem 3.2 if*

$$\{1, 2, \cdots, L\} \subseteq \{1, 2, \cdots, M\}$$

and there exists a family $\{J_j\}_{j=1}^L$ such that $\sum_{j=1}^L |J_j| \leq s$ and $J_j \subseteq I_j$ for all $1 \leq j \leq L$. Then

$$\frac{1}{(1+\varepsilon)^2}\sum_{j=1}^L\|\sum_{i\in J_j}\Lambda_i^* g_i\|^2 \leq \Big\|\sum_{j=1}^L\sum_{i\in J_j}\Lambda_i^* g_i\Big\|^2 \leq (1+\varepsilon)^2\sum_{j=1}^L\|\sum_{i\in J_j}\Lambda_i^* g_i\|^2.$$

Proof. This follows from the Proposition 3.2. The following theorem will give another method for obtaining a fusion frame from an unit norm tight frame for $\mathcal{H}_N$ without having the restricted isometry property. Another form of this result can be found in [6] Theorem 4.2.

Theorem 3.3 *Let $\{f_i\}_{i=1}^M$ be an unit norm tight frame of vectors for $\mathcal{H}_N$ and let $\{I_j\}_{j=1}^L$ be a partition of $\{1, 2, \cdots, M\}$. Define $W_j = \{f_i\}_{i\in I_j}$, then the family $\{W_j\}_{j=1}^L$ is a fusion frame for $\mathcal{H}_N$ with fusion frame bounds $\frac{AM}{N}$ and $\frac{BM}{N}$ where*

$$A = \min_{j=1}^L \min_{k=1}^{\dim W_j}\frac{1}{\lambda_{jk}}, \quad B = \max_{j=1}^L\max_{k=1}^{\dim W_j}\frac{1}{\lambda_{jk}}$$

and $\{\lambda_{jk}\}_{k=1}^{\dim W_j}$ is the family of eigenvalues of frame operator associated to $\{f_i\}_{i\in I_j}$.

Proof. Let S_j be the frame operator associated to $\{f_i\}_{i\in I_j}$ and let $\{e_{jk}\}_{k=1}^N$ be the orthonormal basis for $\mathcal{H}_N$ of eigenvectors of S_j with eigenvalues $\{\lambda_{jk}\}_{k=1}^N$. Then $\lambda_{jk}=0$ for any $\dim W_j < k \le N$ and $\{e_{jk}\}_{k=1}^{\dim W_j}$ is a orthonormal basis for W_j. Thus

$$< S_j f, f >= \sum_{k=1}^{\dim W_j} \lambda_{jk} |< f, e_k >|^2.$$

Now for any $f \in \mathcal{H}_N$ we have

$$\begin{aligned}
&\min_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i\in I_j} |< f, f_i >|^2 \\
&= \min_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} < S_j f, f > \\
&= \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\max_{1\le k\le \dim W_j} \lambda_{jk}} |< f, e_{jk} >|^2 \\
&\le \|\pi_{W_j}\|^2 \\
&\le \sum_{k=1}^{\dim W_j} \frac{\lambda_{jk}}{\min_{1\le k\le \dim W_j} \lambda_{jk}} |< f, e_{jk} >|^2 \\
&= \max_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} < S_j f, f > \\
&= \max_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} \sum_{i\in I_j} |< f, f_i >|^2.
\end{aligned}$$

This yields

$$\begin{aligned}
&\sum_{j=1}^{L} \sum_{i\in I_j} \min_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} |< f, f_i >|^2 \\
&\le \sum_{j=1}^{L} \|\pi_{W_j} f\|^2 \\
&\le \sum_{j=1}^{L} \sum_{i\in I_j} \max_{1\le k\le \dim W_j} \frac{1}{\lambda_{jk}} |< f, f_i >|^2.
\end{aligned}$$

Put

$$A = \min_{j=1}^{L} \min_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}, \ B = \max_{j=1}^{L} \max_{k=1}^{\dim W_j} \frac{1}{\lambda_{jk}}.$$

Then

$$\frac{AM}{N}\|f\|^2 \le \sum_{j=1}^{L} \|\pi_{W_j} f\|^2 \le \frac{BM}{N}\|f\|^2.$$

The next corollary generalizes Theorem 3.3 to the g-frames situation which the proof leave to interested readers.

Corollary 3.3 *Let $\{\Lambda_i\}_{i=1}^M$ be a tight g-frame for $\mathcal{H}_N$ with respect to $\{W_i\}_{i=1}^M$ and let $\{I_j\}_{j=1}^L$ be a partition of $\{1, 2, \cdots, M\}$. Define*

$$V_j = \{\Lambda_i^*(W_i)\}_{i\in I_j}.$$

Then the family $\{V_j\}_{j=1}^L$ is a fusion frame for $\mathcal{H}_N$ with fusion frame bounds

$$\frac{A\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N} \quad \textit{and} \quad \frac{B\sum_{i=1}^M \|\Lambda_i\|_{HS}^2}{N},$$

where

$$A = \min_{j=1}^{L} \min_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}, \ B = \max_{j=1}^{L} \max_{k=1}^{\dim V_j} \frac{1}{\lambda_{jk}}$$

and $\{\lambda_{jk}\}_{k=1}^{\dim V_j}$ is the family of eigenvalues of g-frame operator associated to $\{\Lambda_i\}_{i\in I_j}$.

4 Stability of g-frames

Our purpose of this section is to study the conditions which under removing some element from a g-frame, again we obtain another g-frame. The next theorem gives an erasure result of g-frames so that Theorem 4.3 obtained in [5] is a special case of it.

Theorem 4.1 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ with g-frame bounds A and B and let $J \subset I$. Then $\{\Lambda_i\}_{i\in I-J}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I-J}$ with bounds*

$$\frac{A^2}{B}\|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)^{-1}\|^{-2} \quad \textit{and} \quad B,$$

if and only if $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ be a bounded invertible operator on $\mathcal{H}$.

Proof. For any $f \in \mathcal{H}$ we have

$$\begin{aligned}
f &= \sum_{i\in I} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f \\
&= \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f + \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f.
\end{aligned}$$

Thus

$$I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i = \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i.$$

Moreover we have

$$\begin{aligned}
&\|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)f\| \\
&= \Big\| \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f \Big\| \\
&= \sup_{\|g\|=1} | < \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f, g > | \\
&= \sup_{\|g\|=1} | \sum_{i\in I-J} < \Lambda_i f, \Lambda_i S_\Lambda^{-1} g > | \\
&\le \sup_{\|g\|=1} \sum_{i\in I-J} \|\Lambda_i f\| \|\Lambda_i S_\Lambda^{-1} g\| \\
&\le \sup_{\|g\|=1} (\sum_{i\in I-J} \|\Lambda_i f\|^2)^{\frac{1}{2}} (\sum_{i\in I-J} \|\Lambda_i S_\Lambda^{-1} g\|^2)^{\frac{1}{2}} \\
&\le \sup_{\|g\|=1} \sqrt{B} \|S_\Lambda^{-1} g\| (\sum_{i\in I-J} \|\Lambda_i f\|^2)^{\frac{1}{2}} \\
&\le \frac{\sqrt{B}}{A} (\sum_{i\in I-J} \|\Lambda_i f\|^2)^{\frac{1}{2}}.
\end{aligned}$$

Now if $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ is invertible on $\mathcal{H}$. Then

$$\begin{aligned}
&\frac{A^2}{B} \|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)^{-1}\|^{-2} \|f\|^2 \\
&\le \frac{A^2}{B} \|(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i) f\|^2 \\
&\le \sum_{i\in I-J} \|\Lambda_i f\|^2.
\end{aligned}$$

On the other hand, since Λ is a g-frame hence $\{\Lambda_i\}_{i\in I-J}$ is a g-Bessel sequence. It follows that $\{\Lambda_i\}_{i\in I-J}$ is a g-frame. Conversively, suppose that $\{\Lambda_i\}_{i\in I-J}$ is a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I-J}$, with g-frame bounds A and B. We first show that $I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i$ is injective. Let

$$\begin{aligned}
&(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i) f = 0 \Rightarrow \\
&S_\Lambda^{-1}\Big(\sum_{i\in I-J} \Lambda_i^*\Lambda_i f \Big) = \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f = 0
\end{aligned}$$

hence $\sum_{i\in I-J} \Lambda_i^*\Lambda_i f = 0$. It follows that

$$\begin{aligned}
A\|f\|^2 &\le \sum_{i\in I-J} \|\Lambda_i f\|^2 \\
&= \sum_{i\in I-J} < \Lambda_i f, \Lambda_i f > \\
&=< \sum_{i\in I-J} \Lambda_i^*\Lambda_i f, f >= 0
\end{aligned}$$

which implies that $f = 0$. Also, if $(I_{\mathcal{H}} - \sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i)^* f = 0$ then $\sum_{i\in I-J} \Lambda_i^*\Lambda_i S_\Lambda^{-1} f = 0$ and therefore $S_\Lambda^{-1} f = 0$, it follows that $f = 0$. This finishes the proof.

Corollary 4.1 *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = f_0$. Then $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame for $\mathcal{H}$.*

Proof. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = f_0$, then $\sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0 = 0$, hence $\sum_{i\in I-J} \Lambda_i^*\Lambda_i f_0 = 0$. It follows that

$$\begin{aligned}
\sum_{i\in I-J} \|\Lambda_i f_0\|^2 &= \sum_{i\in I-J} < \Lambda_i f_0, \Lambda_i f_0 > \\
&=< \sum_{i\in I-J} S_\Lambda^{-1}\Lambda_i^*\Lambda_i f_0, f_0 >= 0
\end{aligned}$$

Therefore $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame.

Corollary 4.2 *Let $\{\Lambda_i\}_{i\in I}$ be a A-tight g-frame for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and let $J \subset I$. If there exists $0 \neq f_0 \in \mathcal{H}$ such that $\sum_{i\in J} \Lambda_i^*\Lambda_i f_0 = Af_0$, then $\{\Lambda_i\}_{i\in I-J}$ is not a g-frame for $\mathcal{H}$.*

5 Conclusion

In this paper, we proved that the sum of any Bessel sequence with Bessel bound less than one with a Parseval frame is a frame and computed its obtimal bounds. We also showed that a Bessel sequence is an inner summand of a frame and changed every Bessel sequence to a dual frame by summing it with any Parseval frame. Moreover, we proved that any pair of g-Bessel sequences can be extended to pair of dual g-frames. This result, generalizes a result of Christensen, Oh Kim and Young Kim in [9] to the situation of g-frames. We defined the restricted isometry property for g-frames and generalized some results from [6] to g-frames.

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