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Research Article

Robust stabilization of a class of three-dimensional uncertain fractional-order non-autonomous systems

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Abstract

This paper concerns the problem of robust stabilization of uncertain fractional-order non-autonomous systems. In this regard, a single input active control approach is proposed for control and stabilization of three-dimensional uncertain fractional-order systems. The robust controller is designed on the basis of fractional Lyapunov stability theory. Furthermore, the effects of model uncertainties are fully taken into account. Also, the robust stability and access to the equilibrium point of the control scheme are analytically proved. Moreover, fast response and easy realization in real world applications are some special features of the suggested method. Finally, as a numerical simulation, control and stabilization of three-dimensional uncertain fractional-order Chen system is provided to illustrate the usefulness and applicability of the proposed approach in practice. It is worth to notice that the proposed active control approach can be employed for robust stabilization of a large class of three-dimensional uncertain nonlinear fractional-order non autonomous dynamical systems.

Keywords : Control; Single input control; Fractional-order system.

1 Introduction

FRactional calculus, which was introduced in the early 17th century, deals with integration and derivatives of arbitrary noninteger orders. In recent years, it has been reported in many areas such as electrical circuit, population models, epidemiology models, etc [2]. Due to the existence of chaos in real fractional-order systems, control and stabilization of fractional-order systems have attracted the attention of many scholars in the past decade [3]. Therefore, studying the fractional-order chaotic systems has become an active research field. Up to now, some methods have been suggested to achieve chaos control in fractional-order chaotic systems, such as optimal control [17], active control [9], feedback control [18], PC control [12], adaptive control [19, 28], non-fragile nonlinear observer method [10], sliding mode control [4, 5, 6, 7, 8, 11], fuzzy logic control [21], etc. However, in most of the above mentioned approaches, the ideal conditions for the systems have been considered without paying attention to the unknown uncertainties which exist in reality. Moreover, in previous works the formulation of the chaos stabilization problem and the proposed controllers are complex both in design and application.

In this paper, we design a single input active controller with one driving variable to control a class of three-dimensional fractional-order systems. The effects of model uncertainties are fully taken into account. The proposed scheme is

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based on the fractional version of Lyapunov stability theory. The robust stability property and simplicity of the design are interesting capabilities of the designed method. A numerical example demonstrates the applicability and efficiency of the proposed control technique in practice. The rest of this paper is arranged as follows. Some preliminaries of fractional calculus and a lemma are given in Section 2. In Section 3, system description and problem formulation are presented. Also, the proposed control scheme is introduced in Section 3. Section 4 presents an illustrative example. Finally, in Section 5, some conclusions are included.

2 Definitions and Preliminaries

Here, some definitions about the fractional differential equations (FDEs) and an essential lemma are expressed.

Definition 2.1 *The Riemann-Liouville fractional integration of order α is presented by [25]:*

$$ {}_{t_0}I_t = {}_{t_0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{2.1} $$

where t_0 is the initial time. Also $\Gamma(.)$ is the Gamma function.

Definition 2.2 *The αth order Caputo fractional derivative of a function f(t) is defined as [25]:*

$$ \begin{aligned} {}_{t_0}D_t^{\alpha} f(t) &= {}_{t_0}D_t^{-(m-\alpha)} \frac{d^m}{dt^m} f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau. \end{aligned} \tag{2.2} $$

where where m is the smallest integer number, larger than α.

It is noted that in this paper the Caputo definition is adopted.

Definition 2.3 *Suppose that $h(t)$ is the impulse response of a linear system. The diffusive representation of $h(t)$ is called $\mu(\omega)$ with relation as follows [27]:*

$$ h(t) = \int_0^{\infty} \mu(\omega)\, e^{-\omega t}\, d\omega \tag{2.3} $$

Remark 2.1 *The fractional order integral (2.1) can rewrite as [27]:*

$$ {}_{t_0}I_t^{\alpha} f(t) = h(t) * f(t) \tag{2.4} $$

where $*$ is the convolution operator and $h(t)$ is defined as

$$ h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. $$

Also, the diffusive representation of $h(t)$ is defined as:

$$ \mu(\omega) = \frac{\sin(\alpha\pi)}{\pi} \omega^{-\alpha} \tag{2.5} $$

Definition 2.4 *Let have the nonlinear FDE [27]:*

$$ {}_{t_0}D_t^{\alpha} X = f(X) \tag{2.6} $$

According to the continuous frequency distributed model of the fractional integrator, the nonlinear system (2.6) can be rewritten as:

$$ \begin{cases} \frac{\partial z(\omega,t)}{\partial t} &= -\omega z(\omega,t) + f(x(t)) \\ x(t) &= \int_0^{\infty} \mu(\omega)\, z(\omega,t)\, d\omega \end{cases} \tag{2.7} $$

while $\mu(\omega)$ is the same as (2.5).

Lemma 2.1 *Consider $w_1 = ax^2$ and $w_2 = \int_0^{\infty} \mu(\omega)\, z(\omega,t)\, d\omega$. The quadratic form $w = w_1 + w_2$ is positive definite if and only if $a > 0$ [27].*

3 Main results

Consider the following class of uncertain three-dimensional non-autonomous fractional0order systems with a single control input.

$$ \begin{cases} D^{\alpha} X &= f_1(X,y,t) \\ D^{\alpha} y &= f_2(X,y,t) + \Delta f(X,y,t) - u(t) \end{cases} \tag{3.8} $$

where $\alpha \in (0,1)$ is the order of the system and $X(t) = [x_1, x_2]^T \in R^2$ and $y(t) \in R$ are the states of the system, $f_1(X,y,t)$ and $f_2(X,y,t)$ are the bounded nonlinear functions of X, y and t ,$\Delta f(X,y,t)$ is the system uncertainty term and $u(t) \in R$ is the single control input.

Assumption 3.1. The uncertainty terms $\Delta f(X,y,t)$ is bounded by

$$ |\Delta f(X,t)| < \rho \tag{3.9} $$

where ρ is a known positive constant.

Assumption 3.2. The function $f_1(X, y, t)$ is smooth in a neighborhood of the point $y = 0$ and the subsystem $D^\alpha X = f_1(X, y, t)$ will be asymptotically stable about the origin $X = 0$ for all X.

Remark 3.1 *The system (3.8) is very general, where it includes most of the canonical fractional-order systems, such as fractional-order Genesio-Tesi system and fractional-order Arneodo system, fractional-order unified system, fractional-order Lu system, fractional-order Lorenz system, fractional-order Chen system and fractional-order Tigan system.*

The control goal of this paper is to design a suitable robust controller for stabilization of system (3.8) around zero.

Theorem 3.1 *Consider the fractional-order system (3.8). If this system is controlled by the single active controller (3.10), then the system trajectories will converge to zero.*

$$u(t) = \lambda\zeta sign(y) + \rho + f_2(X, y, t) \tag{3.10}$$

where $\zeta = q|y|$ and λ, q are positive constants.

Proof by using Definition 2.4, $D^\alpha y$ in (3.8), can be rewritten as

$$\begin{cases} \frac{\partial z(\omega,t)}{\partial t} &= -\omega z(\omega, t) + f_2(X, y, t) \\ &+\Delta f(X, y, t) - u(t) \\ y(t) &= \int_0^\infty \mu(\omega)\, z(\omega, t)\, d\omega \end{cases} \tag{3.11}$$

We define two Lyapunov function where the first is

$$v(\omega, t) = \frac{z^2(\omega, t)}{2}.$$

For $v(\omega, t)$ one can has $\frac{\partial v(\omega,t)}{\partial z(\omega,t)} = z(\omega, t)$ and by using (3.11) we can obtain

$$\begin{aligned} \frac{\partial v(\omega, t)}{\partial t} &= \frac{\partial v(\omega, t)}{\partial z(\omega, t)} \cdot \frac{\partial z(\omega, t)}{\partial t} \\ &= z(\omega, t)[-\omega z(\omega, t) + f_2(X, y, t) \\ &+ \Delta f(X, y, t) - u(t)] \\ &\leq z(\omega, t)[-\omega z(\omega, t) + f_2(X, y, t) \\ &+ \underbrace{|\Delta f(X, y, t)|}_{<\rho} - u(t)] \\ &< z(\omega, t)[-\omega z(\omega, t) + f_2(X, y, t) \\ &+ \rho - \underbrace{(\lambda\xi sign(y) + \rho + f_2(X, y, t))}_{u(t)}] \\ &< -\omega z^2(\omega, t) - \lambda\zeta sign(y) z(\omega, t) \end{aligned}$$

Therefore,

$$\frac{\partial v(\omega, t)}{\partial t} < -\omega z^2(\omega, t) - \lambda\zeta sign(y) z(\omega, t). \tag{3.12}$$

Now we introduce the main Lyapunov function as

$$\begin{aligned} V(t) &= \int_0^\infty \mu(\omega)\, v(\omega, t)\, d\omega \\ &= \frac{1}{2}\int_0^\infty \mu(\omega)\, z^2(\omega, t)\, d\omega \end{aligned} \tag{3.13}$$

Obviously $V(t) > 0$ and based on Lyapunove stability theorem, we must demonstrate that $\frac{dV}{dt} < 0$ is holden. Therefor, by attention to (3.12), one obtains

$$\begin{aligned} \frac{dV}{dt} &= \int_0^\infty \mu(\omega) \frac{\partial v(\omega, t)}{\partial t}\, d\omega \\ &< \int_0^\infty \mu(\omega)\, [-\omega z^2(\omega, t) \\ &- \lambda\zeta sign(y) z(\omega, t)]\, d\omega \\ &< -\int_0^\infty \mu(\omega)\omega z^2(\omega, t)\, d\omega \\ &- \lambda\zeta sign(y) \underbrace{\int_0^\infty \mu(\omega) z(\omega, t)\, d\omega}_{y} \\ &< -\int_0^\infty \mu(\omega)\omega z^2(\omega, t)\, d\omega \\ &- \lambda \underbrace{q|y|}_{\zeta} sign(y) y \\ &< -(\int_0^\infty \mu(\omega)\omega z^2(\omega, t)\, d\omega + \lambda q|y|^2) \end{aligned}$$

According to Lemma 2.1, $\frac{dV}{dt} < 0$. Thus the proof is achieved completely.

4 Numerical example

Here, the robust stabilization problem of the fractional-order Chen system is splved numerically. Numerical simulation is performed using MATLAB software. The numerical approach described in [13, 14, 15, 16] with a step time of 0.001 is applied to solve the fractional-order equations.

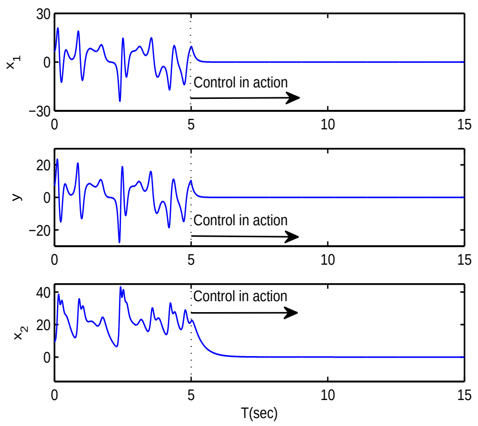

Figure 1: State trajectories of chaotic Chen system controlled with (3.10)

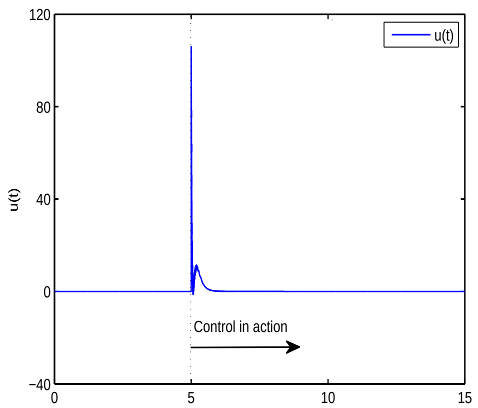

Figure 2: Time history of the single control input (3.10) applied to the Chen system.

4.1 Numerical method for solving fractional differential equations

There are several analytical and numerical methods such as the fractional difference method [25, 26], the Adomian decomposition method [22], the homotopy perturbation method [1], the variational iteration method [23, 24], the Adams-BashforthMoulton method [13, 14, 15, 16] to solve the fractional-order differential equations. In this paper, a modification of AdamsBashforthMoulton algorithm which is proposed by Diethelm et al. in [13, 14, 15, 16] is used to solve FDEs. Consider the following initial value problem (IVP) of FDEs

$$\begin{cases} D^{\alpha}y(t) &= f(y,t), \qquad 0 \le t \le T \\ y^{(k)}(0) &= y_0^{(k)},\ k = 0, 1, \dots, m-1\,(m = \lceil \alpha \rceil) \end{cases} \tag{4.14}$$

which is equivalent to the following Volterra integral equation

$$y(k) = \sum_{k=0}^{m-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(y(s),s)}{(t-s)^{1-\alpha}}\,ds \tag{4.15}$$

Setting $h = \frac{T}{N}$, $t_n = nh$, $n = 0, 1, \dots, N$ the above equation becomes

$$y_h(t_{n+1}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(y_h^{(p)}(t_{(n+1)}), t_{(n+1)}) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(y_h(t_j), t_j) \tag{4.16}$$

subject to

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, if\ j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ -2(n-j-1)^{\alpha+1},\ if\ 1 \le j \le n \\ 1,\ if\ j = n+1 \end{cases}$$

$$y_h^{(p)}(t_{(n+1)}) = \sum_{k=0}^{m-1} c_k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(y_h(t_j), t_j)$$

$$b_{j,n+1} = \frac{h^{\alpha}}{\alpha}((n+1-j)^{\alpha} - (n-j)^{\alpha}). \tag{4.17}$$

The error in this method is

$$e = \max_{j=0,1,\ldots,n} |y(t_j) - y_h(t_j)| = O(h^p) \qquad (4.18)$$

where $p = min(2, 1+\alpha)$

4.2 Robust stabilization of the fractional-order Chen system

Consider the following fractional-order Chen system [20]

$$Chen: \begin{cases} D^\alpha x_1 &= 35(x_2 - x_1), \\ D^\alpha x_2 &= 28x_2 - 7x_1 - x_1x_3, \\ D^\alpha x_3 &= x_1x_2 - 3x_3. \end{cases} \qquad (4.19)$$

It is easy to see that if $x_2(t) = 0$, the two-dimensional subsystem

$$\begin{cases} D^\alpha x_1 &= -35x_1, \\ D^\alpha x_3 &= -3x_3 \end{cases}$$

of system (4.20) will be asymptotically stable about the origin $x_1(t) = 0$ and $x_3(t) = 0$ for all x_1 and x_3. Now, let $X = [x_1, x_3]$ and $y = x_2$, then the controlled uncertain system (4.19) becomes

$$\begin{cases} D^\alpha x_1 &= 35(y - x_1), \\ D^\alpha y &= -7x_1 + 28y - x_1x_2 \\ &+\Delta f(X, y, t) - u(t), \\ D^\alpha x_2 &= x_1y - 3x_2 \end{cases} \qquad (4.20)$$

where $\Delta f(X, y, t) = 0.56\cos(3t)y$ and $\alpha = 0.98$. The initial values of the Chen system are $x_1(0) = 7, x_2(0) = 10$ and $y(0) = 7$. Besides, control parameters are selected as $\lambda = 3$ and $q = 3$. The state trajectories of the controlled fractional-order chaotic Lorenz system are displayed in Figure 1, where the controller is switched at t =5. It can be seen that the state trajectories converge to zero, which indicates that the fractional-order Chen system is indeed controlled. The time history of the control input is depicted in Figure 2. It is obvious that the control signal tends to zerobimplying that the proposed mode controller is feasible in real world applications.

5 Conclusion

This paper investigates control of three-dimensional non-autonomous fractional-order uncertain systems via a single input control technique. The analytical results of the method are proved on the basis of fractional Lyapunov stability theory. The designed active controller has several useful features such as fast response, low sensitivity to the system uncertainties and easy realization in practice. Finally, a numerical example confirm that the proposed approach can effectively stabilize the fractional-order non-autonomous systems in practice. It is worth noticing that the proposed control method can be applied to control a broad range of three-dimensional non-autonomous fractional-order dynamical systems.

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