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Research Article

Solving nonlinear Lane-Emden type equations with unsupervised combined artificial neural networks

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Abstract

In this paper we propose a method for solving some well-known classes of Lane-Emden type equations which are nonlinear ordinary differential equations on the semi-infinite domain. The proposed approach is based on an Unsupervised Combined Artificial Neural Networks (UCANN) method. Firstly, The trial solutions of the differential equations are written in the form of feed-forward neural networks containing adjustable parameters (the weights and biases); results are then optimized with the combined neural network. The proposed method is tested on series of Lane-Emden differential equations and the results are reported. Afterward, these results are compared with the solution of other methods demonstrating the efficiency and applicability of the proposed method.

Keywords : Lane-Emden type equations; Nonlinear ODE; Semi-infinite domain; Astrophysics; Artificial neural network; Combined neural network.

1 Introduction

Lane-Emden type equations are nonlinear ordinary differential equations on semi-infinite domain. They are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deals with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and has been the focus of many studies [1, 10, 65, 50, 2, 60, 25, 26, 27, 28, 29, 61, 3, 4, 38, 58, 62, 30, 31, 32, 33].

Bender et al. [10] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method. Mandelzweig and Tabakin [65] used the quasilinearization approach to solve the standard Lane-Emden equation.[33]

Shawagfeh [50] applied a nonperturbative approximate analytical solution for the Lane-Emden type equation using the Adomian Decomposition Method. His solution was in the form of a

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power series. He used Padé approximants method [39, 40] to accelerate the convergence of the power series.[33]

In [1], Wazwaz employed the Adomian Decomposition Method [41, 42] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane-Emden type. Further, the author of [2] used the Modified Decomposition Method for solving the analytical treatment of nonlinear differential equations such as the Lane-Emden type equation.[33]

Liao [60] provided an analytical algorithm for Lane-Emden type equations. This algorithm logically contains the well-known Adomian Decomposition Method. J. H. He [25] employed Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane-Emden type equation.[33]

Parand et al. [30, 31, 34, 35] presented numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Ramos [26, 27, 28, 29] solved Lane-Emden equations through different methods. Author of [27] presented the linearization method for singular initial-value problems in second-order ordinary differential equations such as Lane-Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. Later this author [29] developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. In [28],series solutions of the Lane-Emden type equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable.[33]

Yousefi [61] presented a numerical method for solving the Lane-Emden equations. Bataineh et al. [3] presented an algorithm based on Homotopy Analysis Method (HAM) [43] to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden-Fowler type equation.[33]

In [44], Chowdhury and Hashim presented an algorithm based on the Homotopy-Perturbation Method (HPM) [18, 45, 5] to solve singular IVPs of time-independent equations.[33]

Aslanov [4] introduced a further development in the Adomian Decomposition Method to overcome the difficulty at the singular point of non-homogeneous, linear and nonlinear Lane-Emden-like equations.[33]

Dehghan and Shakeri [38] applied an exponential transformation to the Lane-Emden type equations to overcome the difficulty of a singular point at $x = 0$ and solved the resulting nonsingular problem by the variational iteration method [33, 46, 47].

Yildirim and Özis [6] presented approximate solutions of a class of Lane-Emden type singular IVPs problems, by the variational iteration method.[33]

Marzban et al. [21] used a method based upon hybrid function approximations. They used the properties of hybrid of block-pulse functions and Lagrange interpolating polynomials together for solving the nonlinear second-order initial value problems and the Lane-Emden type equation.[33]

Recently, Singh et al. [55] provided an efficient analytic algorithm for Lane-Emden type equations using Modified Homotopy Analysis Method; also, they used some well-known Lane-Emden type equations as test examples.[33]

We refer the interested reader to [7, 8] for analysis of the Lane-Emden type equation based on the Lie symmetry approach.[33]

Moreover, Solving differential equations by unsupervised neural network has some advantages rather than traditional numerical methods. Firstly, in ANN, the solution is continuous over all the domain of integration; otherwise, the numerical methods provide solutions only over discrete points. In addition, in ANN, The computational complexity does not increase considerably with the number of sampling points and with the number of dimensions involved in the problem. In a sense, the unsupervised ANN learns to solve the DE analytically. Because the correct form of the DE solution is unknown beforehand, the ANN is trained in an unsupervised manner by using the Lane-Emden equation instead of error function that is derived from the DE itself and the governing boundary conditions. For designing the ANN, we use this error function in training algorithms. Ref [22] was solved ordinary differential equations with the unsupervised method.[14]

The general method for solving differential equations with unsupervised feed-forward neural networks was first introduced by Van Milligen et al. [9]. They stated the general method and applied it to a magnetohydrodynamic plasma equilibrium problem.

Other authors applied that method to other physical problems. in [23], ANN was used for solving ordinary and partial differential equations. Monterola and Saloma [11, 12] solved the non-linear Schrodinger equation. Quito et al. [48, 49] used neural networks for solving self-gravitating systems of N-bodies.

In 2003, a controlled heat problem up to three decimal digits precision was solved by using three-layered and feed-forward neural networks [51]. The solutions for KuramotoSivashinsky and Navier-Stokes equations by using neural networks were concerned in [52]. [53] contains the modeling of dynamical behavior of the KuramotoSivashinsky equation by using neural networks. P. Balasubramaniam et al. [57] was solved matrix Riccati differential equation for the linear quadratic singular system using neural networks. The Hamilton Jacobi Bellman equation was solved by the combination of the least-squares methods and the neural networks in [66].

Moreover, various ODE examples were solved by Filici [13]. Also, in [67] nonlinear Schrodinger equation was solved by feed-forward neural networks. Recently, in [24], some differential equations were solved by constructing neural networks, and in [59] systems of partial differential equations were solved by neural networks and optimization techniques.

This paper is arranged as follows: in Section 2, we describe the general formulation of the proposed approach and derive formulas for computing the gradient of the error function and some basics of combined feed forward artificial neural networks are briefly presented. In Section 3 the proposed method is applied to some types of Lane-Emden equations, and a comparison is made with the existing analytic or exact solutions that were reported in other published works in the literature. And in the final Section, conclusions are drawn and described.

2 Method Description

2.1 Feed forward artificial neural networks

A general structure of feed-forward neural networks is shown in Fig. 1. The feed-forward neural networks are the most popular architectures due to their structural flexibility, good representational capabilities and availability of a large number of training algorithm [63]. This network consists of neurons that are arranged in layers in which every neuron is connected to all neurons of the next layer (a fully connected network).

3 Main results

3.1 Feed forward artificial neural networks

A general structure of feed-forward neural networks is shown in Fig. 1. The feed-forward neural networks are the most popular architectures due to their structural flexibility, good representational capabilities and availability of a large number of training algorithm [63]. This network consists of neurons that are arranged in layers in which every neuron is connected to all neurons of the next layer (a fully connected network).

Artificial neural networks can make a nonlinear mapping from the inputs to the outputs of the corresponding system [36]. This is suitable for analyzing the systems that are described by initial-boundary value problems that have no analytical solutions or their analytical solutions are not computable easily. Any set of differential equations can be represented by the following expression:

$$D(F(z)) = 0. \tag{3.1}$$

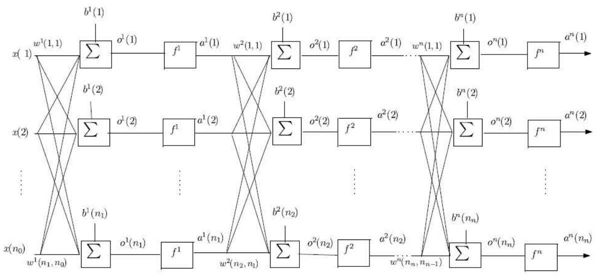

Figure 1: A multi-layer feed forward neural network structure.

Where D is any non-linear, inhomogeneous differential operator and $F(z)$ is the solution that satisfies Eq.(3.1) and the appropriate boundary conditions.
Considering that a feed-forward neural network is an approximate universal function [36, 37], the goal is finding a neural network $F^*(z)$ which approximates $F(z)$ in the finite domain $z \in [a, b]^n$. It is well known from neuron computing sciences [63]. In the case of one hidden layer, the functional form of component y of networks output F^* is given by

$$F_y^* = f^2(\sum_u w_{yu} f^1(\sum_v w_{uv} x_v + b_v) + b_u), \quad (3.2)$$

where the w's are the networks adaptive coefficients (weights), the b's are the bias and f is a sigmoid activation function. Note that F^* is a continuous and derivable function of z; therefore, the differential operator D can act on it.
For every continuous function F and given $\epsilon > 0$, there are a natural number H and real constants b_i, v_i, and $w_{ij} (i = 1, ..., H, j = 1, ..., n)$, so that [19]:

$$|F(x_1, ..., x_n) - f^2(\sum_{i=1}^{H} w_{is} f^1(\sum_{j=1}^{n} w_{ij} x_j + b_i) + b_s)| < \varepsilon. \quad (3.3)$$

In order to find an approximation of $F(z)$, it is natural to choose Eq.(3.1) plus the equations that are defining the boundary conditions as the performance function of the network.
The accuracy of the ANN solution depends on the way of convergence of Eq.(3.1) plus the equations that are defining the boundary conditions as the performance function to zero during the training. Specifically, this equation is formulated as:

$$E^q = \frac{1}{2}|D(F^*)|^2 + \frac{1}{2}\sum_{i=1}^{k}|C_i^q|^2, \quad (3.4)$$

here , $D(F^*(x_1, \cdots, x_n)) = \varepsilon$
(Eq.(3.3)) where $(x_1, x_2, ..., x_n)$ are the n independent variables of equation, and $C_k^q(x_1, \cdots, x_n)$ derived from the boundary conditions. To achieve an accurate approximate universal function, training must reduce E^q to a value that is as close as possible to zero. E^q converges to zero if and only if two terms of Eq.(3.4) identically equal to zero.

3.2 Combined Neural Network Models

The combined neural network topology was used for the computation of Lane-Emden type equations. This construction is based on a straightforward approach that has been termed stacked generalization. The stacked generalization concepts formalized by Wolpert [15] and refer to the schemes for feeding information from one set of generalizers to another before forming the final predicted value (output). The unique contribution of stacked generalization is that the information fed into the net of generalizers comes from multiple partitioning of the original learning set.[16, 17].
The network topology was the Multi-Layer Perceptron Neural Network(MLPNN) with a single or two hidden layers in the first level and two hidden layers in the second level. The network had one input neuron, equal to the number of input of ODE. We trained second level neural network to combine the predictions of the first level networks. The second level network has one input. The target for the second level network was the same as the target of the original data. Since the values of mean square errors (MSEs) converged to a small constants approximately zero in epochs, training of the neural networks was successful. In the second level analysis, the Levenberg-Marquardt training algorithms were used.

4 Application

In this Section, we apply Unsupervised Combined Neural Networks (UCNN) method for the computation of Lane-Emden type equations. In general the Lane-Emden type equations are formulated as

$$y^{''}(x) + \frac{\alpha}{x}y'(x) + f(x)g(y) = h(x), \qquad \alpha x \geq 0 \ , \tag{4.5}$$

with initial conditions

$$a) \ \ y(0) = A, \tag{4.6}$$
$$b) \ \ y^{'}(0) = B. \tag{4.7}$$

Where α, A and B are real constants and $f(x)$, $g(y)$ and $h(x)$ are some given functions. For other special forms of $g(y)$, the well-known Lane-Emden type equations were used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [33, 64, 56].
In this section we apply the Unsupervised Combined Neural Networks method to solve some well-known Lane-Emden type equations for various $f(x)$, $g(y)$, A and B, and constant form of $\alpha = 2$ and $h(x) = 0$.

4.1 Example 4.1 (The standard Lane-Emden equation)

For $f(x) = 1$, $g(y) = y^m$, $A = 1$ and $B = 0$, Eq.(4.5) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [54].

$$y^{''}(x) + \frac{2}{x}y^{'}(x) + y^m(x) = 0, \qquad x \geq 0 \ , \tag{4.8}$$

subject to the boundary conditions

$$a) \ \ y(0) = 1,$$
$$b) \ \ y^{'}(0) = 0.$$

Where $m \geq 0$ is constant. Substituting $m = 0$ into Eq.(4.8) leads to the exact solution

$$y(x) = 1 - \frac{1}{3!}x^2.$$

We apply the UCNN method to solve the standard Lane-Emden equation (4.8) for $m = 0$. For the ANN corresponding to $y(x)$, we consider the network that takes the value of x as an input and its output is one value corresponding to $y(x)$. We chose three-layer ANN consisting of one input unit, n_k hidden units with sigmoid activation functions, and one output units with linear activation functions. In terms of Eq. (3.2), the relation between the ANN and the lane-Emden equation (4.8) can be expressed as:

$$\begin{aligned} a^0 &= x, \\ a^1 &= f(w^1a^0 + b), \\ a^2 &= y(x) = w^2 f(w^1 a^0 + b), \end{aligned} \tag{4.9}$$

where

$$w^{1(T)} = (w^1_{11}, \cdots, w^1_{n_k 1}),$$

$$w^{2(T)} = (w^2_{11}, \cdots, w^2_{1 n_k}),$$

$$b^{(T)} = (b_1, \cdots, b_{n_k}),$$

$$f(x) = \frac{1}{1+e^{-x}},$$

and n_k is the number of hidden layer units. From Eq. (4.9) and using Eq. (4.8), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x}) + \frac{1}{2}c^q,$$

where c^q derived from the boundary conditions and

$$\frac{\partial y(x)}{\partial x} = w^2 w^1 f'(w^1 x + b),$$

Table 1: Comparison of y(x), between present method and series solution given by Horedt [20] for Example *4.1*.

x	Present method	Exact value	Error
0	1.000126640	1.000000000	$1.26e-04$
0.01	0.999727014	0.999983333	$2.56e-04$
0.02	0.999831415	0.999933333	$1.01e-04$
0.03	0.999930625	0.999850000	$8.06e-05$
0.04	1.000127061	0.998933333	$1.96e-04$
0.05	0.999904121	0.999583333	$3.20e-04$
0.06	0.999915186	0.999400000	$5.15e-04$
0.07	0.999989264	0.999183333	$8.05e-04$
0.1	1.000416072	0.998333333	$2.08e-03$
0.2	0.993406734	0.993333333	$7.34e-05$
0.23	0.991107044	0.991183333	$7.62e-05$
0.4	0.973300790	0.973333333	$3.25e-05$
0.8	0.893290013	0.893333333	$4.33e-05$
1	0.833508998	0.833333333	$1.75e-04$

$$\frac{\partial^2 y(x)}{\partial x^2} = w^2(w^1)^2 f''(w^1 x + b).$$

We train the network with 200 equidistance points between 0 and 1 and by trail and error found that the most trainable four-layer architecture is one with 3 hidden nodes. Tables 1 show the approximations of $y(x)$ for the standard Lane-Emden equation for $m = 0$ respectively obtained by the method proposed in this paper and those obtained by Horedt [20].

4.2 Example *4.2*

For $f(x) = 1$, $g(y) = sinh(y)$, $A = 1$ and $B = 0$, Eq.(4.5) will be one of the Lane-Emden type equations that we solve:

$$y^{''}(x) + \frac{2}{x} y^{'}(x) + sinh(y) = 0, \quad x \geq 0 \ , \quad (4.10)$$

subject to the boundary conditions

$$\begin{aligned} a)\ \ & y(0) = 1, \\ b)\ \ & y^{'}(0) = 0. \end{aligned}$$

A series solution obtained by Wazwaz [1] by using Adomian Decomposition Method (ADM) is:

$$y(x) \simeq 1 - \frac{(e^2-1)x^2}{12e} + \frac{1}{480}\frac{(e^4-1)x^4}{e^2} - \frac{1}{30240}\frac{(2e^6+3e^2-3e^4-2)x^6}{e^3} + \frac{1}{26127360}\frac{(61e^8-104e^6+104e^2-61)x^8}{e^4}.$$

We intend to apply UCNN method to solve Eq.(4.10). We choose the four-layer ANN consisting of one input unit. So the relation between the ANN and the Lane-Emden equation (4.10) can be expressed as:

$$\begin{aligned} a^0 &= x, \\ a^1 &= f(w^1 a^0 + b^1), \\ a^2 &= f(w^2 f(w^1 a^0 + b^1) + b^2). \quad (4.11) \\ a^3 &= y(x) = w^3 f(w^2 f(w^1 a^0 + b^1) + b^2). \end{aligned}$$

$$w^{3(T)} = (w^3_{11}, \cdots, w^3_{1n_k}),$$

$$b^{1(T)} = (b_1, \cdots, b_{n_k}),$$

$$b^{2(T)} = (b_1, \cdots, b_{m_k}).$$

and n_k, m_k are the number of hidden layers units. From Eq. (4.11) and using Eq. (4.10), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x} + sinh(y)) + \frac{1}{2}c^q,$$

Table 2: Comparison of y(x), between present method and series solution given by Wazwaz [1] for Example 4.2.

x	Present method	ADM[1]	Error
0	0.996822985	1.000000000	$3.17e-03$
0.4	0.968134848	0.969043758	$9.08e-04$
0.45	0.960946948	0.960947741	$7.93e-07$
0.5	0.952496238	0.951961101	$5.35e-04$
0.6	0.932103688	0.931397142	$7.06e-04$
0.7	0.907166768	0.907530823	$3.64e-04$
0.8	0.879572455	0.880565336	$9.92e-04$
0.9	0.850637959	0.850724891	$8.69e-05$
0.92	0.844425448	0.844432833	$7.38e-06$
1	0.817336078	0.818251666	$9.15e-04$
1.3	0.707622667	0.707679500	$5.68e-05$
1.5	0.625987420	0.625891607	$9.58e-05$
1.51	0.621694775	0.621688632	$6.14e-06$
2	0.430106143	0.413669103	$1.64e-02$

Table 3: Comparison of y(x), between present method and series solution given by Wazwaz [1] for Example 4.3.

x	Present method	ADM[1]	Error
0	1.000097858	1.000000000	$9.78e-05$
0.01	0.999919060	0.999985975	$6.69e-05$
0.05	0.998327374	0.999649410	$4.71e-04$
0.09	0.430106143	0.998864262	$5.36e-04$
0.1	0.998087022	0.998597927	$5.10e-04$
0.15	0.996652859	0.996846403	$1.93e-04$
0.17	0.995950198	0.995950082	$1.16e-07$
0.2	0.994737695	0.994396264	$3.41e-04$
0.5	0.974097483	0.965177780	$8.91e-03$
1	0.867326358	0.863681104	$3.64e-03$
1.5	0.708699578	0.705042495	$3.65e-03$
2	0.513320805	0.506382315	$6.93e-03$

where c^q derived from the boundary conditions and

$$\frac{\partial y(x)}{\partial x} = w^3 w^2 w^1 f'(w^2 f(w^1 x + b^1) + b^2) f'(w^1 x + b^1), \tag{4.12}$$

$$\frac{\partial^2 y(x)}{\partial x^2} = (w^3 w^2 w^1)(w^1 f''(w^1 x + b^1) f'(w^2 f(w^1 x + b^1) + b^2) + w^2 w^1 f'^2(w^1 x + b^1) f''(w^2 f(w^1 x + b^1) + b^2)). \tag{4.13}$$

We trained the network with 200 equidistance points between 0 and 2, and by trail and error, found that the most trainable four-layer architecture is two with 4 and 2 hidden nodes. Table 2 shows the comparison of the values of $y(x)$ obtained by the new method proposed in this paper, and those obtained by Wazwaz [1].

4.3 Example 4.3

For $f(x) = 1$, $g(y) = sin(y)$, $A = 1$ and $B = 0$, Eq. (4.5) will be one of the Lane-Emden type equations that we solve:

$$y''(x) + \frac{2}{x}y'(x) + sin(y) = 0, \qquad x \geq 0\ , \quad (4.14)$$

subject to the boundary conditions

$$a)\ \ y(0) = 1,$$
$$b)\ \ y'(0) = 0.$$

A series solution obtained by Wazwaz [1] by using ADM is:

$$\begin{aligned} y(x) &\simeq 1 - \frac{1}{6}k_1x^2 + \frac{1}{120}k_1k_2x^4 \\ &+k_1(\frac{1}{3024}k_1^2 - \frac{1}{5040}k_2^2)x^6 \\ &+k_1k_2(-\frac{113}{3265920}k_1^2 + \frac{1}{362880}k_2^2)x^8 \\ &+k_1(\frac{1781}{898128000}k_1^2k_2^2 - \frac{1}{399168000}k_2^4 \\ &-\frac{19}{23950080}k_1^4)x^{10}, \end{aligned}$$

where $k1 = sin(1)$ and $k2 = cos(1)$.
We intend to apply UCNN method to solve Eq. (4.14). We chose four-layer ANN consist of one input unit and From Eq. (4.11) and using Eq. (4.14), E^q is formulated as

$$E^q = \frac{1}{2}(\frac{\partial^2 y(x)}{\partial x^2} + \frac{2}{x}\frac{\partial y(x)}{\partial x} + sin(y)) + \frac{1}{2}c^q,$$

where c^q derived from the boundary conditions and $\frac{\partial^2 y(x)}{\partial x^2}, \frac{\partial y(x)}{\partial x}$ as Eq. (4.12)- (4.13).
We trained the network with 200 equidistance points between 0 and 2 and by trail and error found that the most trainable four-layer architecture is two with 3 and 6 hidden nodes. Table 3 shows the comparison of the values of y(x) obtained by the method proposed in this paper, and those obtained by Wazwaz [1].

5 Conclusion

In this study, some well-known classes of Lane-Emden type equations were investigated by using Unsupervised Combined Artificial Neural Networks. We used feed-forward neural networks containing adjustable parameters; also, the results were optimized by using combined neural network. We trained the Neural Network, and after that we could obtain the result for every point. We reported results and compared this method with the numerical methods, the results show that the solutions are so accurate in these problems.

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