Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 5, No. 4, 2013 Article ID IJIM-00336, 14 pages
Research Article

Strong convergence for variational inequalities and equilibrium problems and representations

E. Soori * †

Abstract

We introduce an implicit method for finding a common element of the set of solutions of systems of equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings and a representation of nonexpansive mappings. Then we prove the strong convergence of the proposed implicit schemes to the unique solution of a variational inequality, which is the optimality condition for a minimization problem and is also a common fixed point for a sequence of nonexpansive mappings and a representation of nonexpansive mappings.

Keywords : Representation; Equilibrium problem; Fixed point; Nonexpansive mapping; Variational inequality.

1 Introduction

Let H be a Hilbert space and let $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is

$$G(u, u) = 0 \quad \text{for every } u \in H.$$

The Equilibrium Problem is defined as follows: Find $\tilde{x} \in H$ such that

$$G(\tilde{x}, y) \geq 0 \qquad \text{for all } y \in H. \tag{1.1}$$

A solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by SEP(G). Let C be a closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Let f be an α-contraction on H (i.e. $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in H$ with $0 \leq \alpha < 1$), and A be a bounded linear operator on H. The following variational inequality problem with viscosity is of great interest [10, 11]. Find x^* in C such that

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in C), \tag{1.2}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \Big(\frac{1}{2}\langle Ax, x\rangle + h(x) \Big),$$

where γ satisfies $\|I - A\| \leq 1 - \alpha\gamma$ and h is a potential function for γf (that is $h'(x) = \gamma f(x)$). S. Takahashi and W. Takahashi [20] introduced the following viscosity approximation method for finding a common element of SEP(G) and Fix(T), where T is a nonexpansive mapping. Starting

*Corresponding author. sori.ebrahim@sci.ui.ac.ir
†Department of Mathematics, University of Isfahan, Isfahan, Iran.

with an arbitrary element $x_1 \in H$, they defined the sequences $\{u_n\}$ and $\{x_n\}$ recursively by

$$\begin{cases} G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ (y \in H), \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n)Tu_n \\ (n \in \mathbb{N}), \end{cases}$$

and Plubtieng and Punpaeng in [14] proved a strong convergence theorem for an implicit iterative sequence $\{x_n\}$ obtained from the viscosity approximation method for finding a common element in $\mathrm{SEP(G)} \cap \mathrm{Fix(T)}$ which satisfies the variational inequality (1.2):

Theorem 1.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let G be a bifunction from $H \times H$ into $\mathbb{R}$ satisfying*
(A_1) $G(x, x) = 0$ for all $x \in C$;
(A_2) G is monotone, i.e. $G(x, y) + G(y, x) \leq 0$ for all $x, y \in C$;
(A_3) For all $x, y, z \in C$,

$$\limsup_{t \to 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(A_4) For all $x \in C$, $y \mapsto G(x, y)$ is convex and lower semicontinuous.
For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be the resolvent of G i.e. $S_r(x)$ is the unique $z \in C$ for which

$$G(z, y) + \frac{1}{r}\Big\langle y - z, z - x\Big\rangle \geq 0, \qquad (y \in C).$$

Let T be a nonexpansive mapping on H such that $\mathrm{SEP(G)} \cap \mathrm{Fix(T)} \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0, 1)$ and let A be a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tu_n \\ (n \in \mathbb{N}), \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ (y \in H), \end{cases}$$

where $u_n = S_{r_n} x_n$, $\{r_n\} \subset (0, \infty)$ and $\alpha_n \subset [0, 1]$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point z in $\mathrm{Fix(T)} \cap \mathrm{SEP(G)}$ which solves the variational inequality

$$\langle (A - \gamma f)z, z - x\rangle \leq 0 \quad x \in \mathrm{Fix(T)} \cap \mathrm{SEP(G)}.$$

V. Colao, G. L. Acedo and G. Marino proved a strong convergence theorem for the following implicit sequence $\{z_n\}$ for finding a common element in $\bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \bigcap_{\mathrm{k}=1}^{\mathrm{K}} \mathrm{SEP(G_k)}$ in [4],

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)W_n S_n^K z_n,$$

where
$$S_n^K = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$$
and $n \in \mathbb{N}$. In this paper, motivated by Lau, Miyake and Takahashi [9], Atsushiba and Takahashi [2], Shimizu and Takahashi [16] and Takahashi [21], in Theorem 3.1, we use the harmonic concepts for improving the results proved in [4], in other word we use the amenability concepts and the theory of representations in our results but V. Colao, G. L. Acedo and G. Marino have not used these concepts in [4]. We introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problems $\mathrm{SEP}(\wp)$ for a family $\wp = \{G_k; k = 1, 2 \cdots, K\}$ of bifunctions and the set of fixed points of a family $\{T_i\}_{i\in\mathbb{N}}$ of nonexpansive mappings from C into itself and a representation $\varrho = \{T_t : t \in S\}$ of a semigroup S as nonexpansive mappings from C into itself, with respect to W-mappings and a left regular sequence $\{\mu_n\}$ of means defined on an appropriate subspace of bounded real-valued functions of the semigroup:

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_{\mu_n} W_n S_n^K z_n,$$

where
$$S_n^K = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$$
and $\mathrm{n} \in \mathbb{N}$.
Our goal is to prove some results of strong convergence for implicit schemes to approach a solution x^* of the problem (1.2) such that

$$x^* \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\wp).$$

2 Preliminaries

Throughout this paper H denotes a Hilbert space. Moreover we assume that A is a bounded strongly positive operator on H with constant $\overline{\gamma}$; that is there exists $\overline{\gamma} > 0$ such that

$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad (x \in H)$.

For a map $T : H \to H$ we denote by Fix(T) := $\{x \in H : x = Tx\}$ the fixed point set of T. Note that if T is a nonexpansive mapping, Fix(T) is closed and convex (see [6]).

Let S be a semigroup. We denote by $B(S)$ the Banach space of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $f \in B(S)$ we define l_s and r_s in $B(S)$ by

$(l_s f)(t) = f(st)\ , \qquad (r_s f)(t) = f(ts), \quad (t \in S)$.

Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup (see page 29 of [19]). A net $\{\mu_\alpha\}$ of means on X is said to be left regular if

$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0$,

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let X be a subspace of $B(S)$ containing all the functions $t \to \left\langle f(t), x^*\right\rangle$ with $x^* \in E^*$. We know from [7] that for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\left\langle f_\mu, x^*\right\rangle = \mu_t\left\langle f(t), x^*\right\rangle$ for all $x^* \in E^*$. We denote such f_μ by $\int f(t)\mu(t)$. Moreover, if μ is a mean on X then from [8], $\int f(t)\mu(t) \in \overline{co}\,\{f(t) : t \in S\}$.

Let C be a nonempty closed and convex subset of H. Then, a family $\varrho = \{T_s : s \in S\}$ of mappings from C into itself is said to be a representation of S as nonexpansive mapping on C into itself if satisfies the following :

(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;

(2) for every $s \in S$ the mapping $T_s : C \to C$ is nonexpansive.

We denote by Fix(ϱ) the set of common fixed points of , that is Fix(ϱ)=$\{x \in C : T_s x = x, \ (s \in S)\}$.

For an equilibrium function $G : H \times H \to \mathbb{R}$, SEP(G) := $\{x \in H : G(x, y) \geq 0\ , (y \in H)\}$ is the set of solutions of the related equilibrium problem.

Let C be a closed convex subset of a Hilbert space H. Recall that the (nearest) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following Lemma characterizes the projection P_C.

Lemma 2.1 ([19]). *Let C be a closed convex subset of a real Hilbert space H, $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if it satisfies the inequality*

$\left\langle x - y, y - z\right\rangle \geq 0 \quad (\ z \in C)$.

Lemma 2.2 ([10]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma}$ and $0 < \rho \leq \|A\|^{-1}$ Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

Theorem 2.1 ([18]). *Let S be a semigroup, C be a closed convex subset of a Hilbert space H, $\varrho = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself such that $Fix() \neq \emptyset$ and X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \left\langle T(t)x, y\right\rangle$ be an element of X for each $x \in C$ and $y \in H$, and μ be a mean on X. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$, then the following hold.*

(i) *T_μ is a nonexpansive mapping from C into C.*

(ii) *$T_\mu x = x$ for each $x \in Fix(\varrho)$.*

(iii) *$T_\mu x \in \overline{co}\,\{T_t x : t \in S\}$ for each $x \in C$.*

(iv) *If μ is left invariant, then T_μ is a nonexpansive retraction from C onto $Fix(S)$.*

Theorem 2.2 ([5]). *Let C be a nonempty closed convex subset of a Hilbert space H and $G : H \times H \to \mathbb{R}$ satisfy,*
(A_1) *$G(x,x) = 0$ for all $x \in C$;*
(A_2) *G is monotone, i.e. $G(x,y) + G(y,x) \leq 0$ for all $x, y \in C$;*
(A_3) *For all $x, y, z \in C$,*

$$\limsup_{t \to 0} G(tz+(1-t)x, y) \leq G(x,y);$$

(A_4) *For all $x \in C, y \mapsto G(x,y)$ is convex and lower semicontinuous.*
For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) := \{z \in C : G(z,y) + \frac{1}{r}\Big\langle y - z, z - x\Big\rangle \geq 0, \quad (y \in C)\},$$

then S_r is well defined and the followings are valid:
(i) *S_r is single-valued;*
(ii) *S_r is firmly nonexpansive, i.e.*

$$\|S_r x - S_r y\|^2 \leq \Big\langle S_r x - S_r y, x - y\Big\rangle,$$

for all $x, y \in H$;
(iii) $\mathrm{Fix}S_r = \mathrm{SEP}(G)$;
(iv) $\mathrm{SEP}(G)$; is closed and convex.

Theorem 2.3 ([4]). *Let $\{r_n\} \subset (0, \infty)$ be a sequence converging to $r > 0$. For a bifunction $G : H \times H \to \mathbb{R}$, satisfying conditions (A_1)- (A_4), define S_r and S_{r_n} for $n \in \mathbb{N}$ as in* Theorem 2.5, then for every $x \in H$, we have $\lim_n \|S_{r_n} - S_r\| = 0$.

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ be a mapping. Then T is said to be demiclosed at $v \in H$ if for any sequence $\{x_n\}$ in C, the following implication holds: $x_n \rightharpoonup u \in C$, $Tx_n \to v$ imply $Tu = v$, where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

Lemma 2.3 ([1]). *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Remark 2.1 *Every Hilbert space is a uniformly convex Banach space, and therefore is a strictly convex Banach space (see pages 95, 98 of [19]).*

Definition 2.1 *A vector space X is said to satisfy Opial's condition, if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad (y \in X, \ y \neq x).$$

Note that every Hilbert space satisfies the Opial's condition (see [12] and [15]).

Definition 2.2 *Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. The set of the asymptotic center of $\{x_n\}$ with respect to K, defined by*

$$A(\{x_n\}) = \Big\{x \in K : \limsup_{n\to\infty} \|x_n - x\| = \inf_{y \in K} \limsup_{n\to\infty} \|x_n - y\|\Big\}.$$

Lemma 2.4 ([1]). *Let X be a uniformly convex Banach space satisfying the Opial's condition and K be a nonempty closed convex subset of X. If a sequence $\{z_n\} \subset K$ converges weakly to a point z_0, then $\{z_0\}$ is the asymptotic center of $\{z_n\}$ with respect to K.*

Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Following [17], for any $n \geq 1$, we define a mapping W_n of C into itself as follows,

$$\begin{aligned}
U_{n,n+1} &:= I, \\
U_{n,n} &:= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\
&\vdots \\
U_{n,k} &:= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \qquad (2.3) \\
&\vdots \\
U_{n,2} &:= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I, \\
W_n &:= U_{n,1} := \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I.
\end{aligned}$$

The following results hold for the mappings W_n.

Theorem 2.4 ([17]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)} \neq \emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$. Then*
(1) W_n *is nonexpansive and* $\mathrm{Fix(W_n)} = \bigcap_{i=1}^{n} \mathrm{Fix(T_i)}$ for each $n \geq 1$,

(2) *for each $x \in C$ and for each positive integer j, the limit $\lim_{n\to\infty} U_{n,j}x$ exists.*

(3) *The mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} \quad (\mathrm{x} \in \mathrm{C}),$$

is a nonexpansive mapping satisfying $\mathrm{Fix(W)} = \bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)}$ and it is called the W-mapping generated by $\{T_i\}_{i\in\mathbb{N}}$, and $\{\lambda_i\}_{i\in\mathbb{N}}$.

Theorem 2.5 ([13]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{i=1}^{\infty} \mathrm{Fix(T_i)} \neq \emptyset$, $\{\breve{}_\mathrm{i}\}$ be a real sequence such that $0 < \lambda_i \leq b < 1$, $(i \geq 1)$. If D is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0.$$

Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. For $\epsilon > 0$ and a mapping $T : D \to H$, we let $F_\epsilon(T; D)$ be the set of ϵ-approximate fixed points of T, i.e.

$$F_\epsilon(T; D) = \{x \in D : \|x - Tx\| \leq \epsilon\}.$$

3 Main results

In this Section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and left amenable nonexpansive semigroup in a Hilbert space. These results extend the main result of [4] and many others.

Theorem 3.1 *Let S be a semigroup and let C be a closed convex subset of a Hilbert space H. Suppose that $\varrho = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself and suppose $\mathrm{Fix}(\varrho) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Let $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\varrho)) \subseteq \mathrm{Fix}(\varrho)$forevery $i \in \mathbb{N}$, and $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies* $(A_1) - (A_4)$ *of* Theorem 2.5,

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix}(\mathcal{S}) \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim_n \epsilon_n = 0$ and,

(iv) for every $k \in \{1, 2, \cdots, K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n S^K_n z_n, \tag{3.4}$$

where $S^K_n = S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}}$ for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}), \tag{3.5}$$

or, equivalently,

$$x^* = \mathrm{P}_{\mathfrak{F}}(I - (A - \gamma f))x^*,$$

or, equivalently, x^* is the unique solution of the minimization problem

$$\min_{x\in\mathfrak{F}}\Big(\frac{1}{2}\Big\langle Ax,x\Big\rangle+h(x)\Big),$$

where h is a potential function for γf.

Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \leq \min\Big\{\|A\|^{-1},\frac{1}{\bar{\gamma}}\Big\}$. We observe that if $\|p\|=1$, then

$$\begin{aligned}\langle(I-\epsilon_nA)p\ ,\ p\rangle&=1-\epsilon_n\langle Ap\ ,\ p\rangle\\&\geq 1-\epsilon_n\|A\|\geq 0.\end{aligned}$$

Hence, if $\|p\|\neq 1$ and $p\neq 0$, then we have

$$\begin{aligned}&\langle(I-\epsilon_nA)p\ ,\ p\rangle\\&\quad=\|p\|^2\langle(I-\epsilon_nA)\frac{p}{\|p\|}\ ,\ \frac{p}{\|p\|}\rangle\geq 0.\end{aligned}$$

We also have $\langle(I-\epsilon_nA)p\ ,\ p\rangle=0$, if $p=0$. Hence $\langle(I-\epsilon_nA)p\ ,\ p\rangle\geq 0$, for all $p\in C$.
By Lemma 2.2, we have

$$\|I-\epsilon_nA\|\leq 1-\epsilon_n\bar{\gamma}$$

. We shall divide the proof into eight steps.

Step 1. The existence of z_n which satisfies (3.4).
Proof. This follows immediately from the fact that for every $n\in\mathbb{N}$, the mapping N_n given by

$$N_nx:=\epsilon_n\gamma f(x)+(I-\epsilon_nA)T_{\mu_n}W_nS_n^Kx \qquad (x\in H),$$

is a contraction. To see this, put $\beta_n=1+\epsilon_n\gamma\alpha-\epsilon_n\bar{\gamma}$, then $0\leq\beta_n<1$ $(n\in\mathbb{N})$. We have,

$$\begin{aligned}&\|N_nx-N_ny\|\\&\leq\epsilon_n\gamma\|f(x)-f(y)\|\\&\quad+\Big(1-\epsilon_n\bar{\gamma}\Big)\|T_{\mu_n}W_nS_n^Kx-T_{\mu_n}W_nS_n^Ky\|\\&\leq\epsilon_n\gamma\alpha\|x-y\|+(1-\epsilon_n\bar{\gamma})\|x-y\|\\&=(1+\epsilon_n\gamma\alpha-\epsilon_n\bar{\gamma})\|x-y\|=\beta_n\|x-y\|.\end{aligned}$$

Therefore, by Banach Contraction Principle ([19], p. 4), there exist a unique point z_n such that $N_nz_n=z_n$.

Step 2. $\{z_n\}$ is bounded.
Proof. Let $p\in\mathfrak{F}$. We have

$$\begin{aligned}\|z_n-p\|^2=&\Big\langle\epsilon_n\gamma f(z_n)\\&+(I-\epsilon_nA)T_{\mu_n}W_nS_n^Kz_n-p\ ,\ z_n-p\Big\rangle\\=&\epsilon_n\gamma\Big\langle f(z_n)-f(p),z_n-p\Big\rangle\\&+\epsilon_n\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle\\&+\Big\langle(I-\epsilon_nA)(T_{\mu_n}W_nS_n^Kz_n\\&-T_{\mu_n}W_nS_n^Kp)\ ,\ z_n-p\Big\rangle\\\leq&\epsilon_n\gamma\alpha\|z_n-p\|^2+(1-\epsilon_n\bar{\gamma})\|z_n-p\|^2\\&+\epsilon_n\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle.\end{aligned}$$

Thus,

$$\|z_n-p\|^2\leq\frac{1}{\bar{\gamma}-\alpha\gamma}\Big\langle\gamma f(p)-Ap\ ,\ z_n-p\Big\rangle. \quad (3.6)$$

Hence,

$$\|z_n-p\|\leq\frac{1}{\bar{\gamma}-\alpha\gamma}\|\gamma f(p)-Ap\|.$$

That is, the sequence $\{z_n\}$ is bounded.

Step 3. For every fixed $k\in\{1,2,\cdots,K\}$, we have

$$\lim_n\|z_n-S^k_{r_{k,n}}z_n\|=0. \quad (3.7)$$

Proof. Let $k\in\{1,2,\cdots,K\}$, since by (ii) of Theorem 2.5, $S^k_{r_{k,n}}$ is firmly nonexpansive, we conclude that

$$\begin{aligned}&\|S^k_{r_{k,n}}z_n-p\|^2\\&=\|S^k_{r_{k,n}}z_n-S^k_{r_{k,n}}p\|^2\\&\leq\Big\langle S^k_{r_{k,n}}z_n-S^k_{r_{k,n}}p\ ,\ z_n-p\Big\rangle\\&=\frac{1}{2}\Big(\|S^k_{r_{k,n}}z_n-p\|^2\\&\quad+\|z_n-p\|^2-\|z_n-S^k_{r_{k,n}}z_n\|^2\Big).\end{aligned}$$

Therefore,

$$\|z_n-S^k_{r_{k,n}}z_n\|^2\leq\|z_n-p\|^2-\|S^k_{r_{k,n}}z_n-p\|^2. \quad (3.8)$$

If we put $L_n := 2\left\langle \gamma f(z_n) - AT_{\mu_n} W_n S_n^K z_n\, ,\, z_n - p \right\rangle$, then by using the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\left\langle y, x+y \right\rangle, \qquad (3.9)$$

we obtain

$$\begin{aligned} &\|z_n - p\|^2 \\ &\quad = \|\epsilon_n \gamma f(z_n) \\ &\qquad + (I - \epsilon_n A) T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\ &\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 \\ &\quad \leq \|T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\ &\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n \\ &\quad \leq \|S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n. \end{aligned}$$

So by (3.8), we have

$$\|z_n - S^k_{r_{k,n}} z_n\|^2 \leq \epsilon_n L_n.$$

That $\{L_n\}_{n\in\mathbb{N}}$ is a bounded sequence, implies

$$\lim_n \|z_n - S^k_{r_{k,n}} z_n\| = 0.$$

By induction we assume that (3.7) holds for every $k > \bar{k}$, and we prove it for $\bar{k}$.
Indeed, we have

$$\begin{aligned} &\|z_n - p\|^2 \\ &\quad \leq \|T_{\mu_n} W_n S^1_{r_{1,n}} S^2_{r_{2,n}} \\ &\qquad \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n \\ &\quad \leq \|S^{\bar{k}}_{r_{\bar{k},n}} \cdots S^K_{r_{K,n}} z_n - p\|^2 + \epsilon_n L_n. \end{aligned} \qquad (3.10)$$

Observe that

$$\begin{aligned} &\|S_{r_{\bar{k},n}}{}^{\bar{k}} \cdots S^K_{r_{K,n}} z_n - p\| \\ &\quad = \|S^{\bar{k}}_{r_{\bar{k},n}} \cdots S^K_{r_{K,n}} z_n - S^{\bar{k}}_{r_{\bar{k},n}} z_n \\ &\qquad + S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \\ &\quad \leq \|S^{\bar{k}+1}_{r_{\bar{k}+1,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| \\ &\qquad + \|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \\ &\quad \leq \|S^{\bar{k}+1}_{r_{\bar{k}+1,n}} \cdots S^K_{r_{K,n}} z_n - S^{\bar{k}+1}_{r_{\bar{k}+1,n}} z_n\| \\ &\qquad + \|S^{\bar{k}+1}_{r_{\bar{k}+1,n}} z_n - z_n\| + \|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \\ &\quad \leq \|S^{\bar{k}+2}_{r_{\bar{k}+2,n}} \cdots S^K_{r_{K,n}} z_n - z_n\| \\ &\qquad + \|S^{\bar{k}+1}_{r_{\bar{k}+1,n}} z_n - z_n\| + \|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \\ &\quad \vdots \\ &\quad \leq \|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| + \sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\|. \end{aligned}$$

Inequality (3.10) gives,

$$\begin{aligned} &\|z_n - p\|^2 \\ &\leq \Bigg(\sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \\ &\qquad + 2\|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \Bigg) \Bigg(\sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \Bigg) \\ &\qquad + \|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\|^2 + \epsilon_n L_n. \end{aligned}$$

From this inequality and (3.8), we obtain

$$\begin{aligned} &\|z_n - S^{\bar{k}}_{r_{\bar{k},n}} z_n\|^2 \\ &\leq \Bigg(\sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \\ &\qquad + 2\|S^{\bar{k}}_{r_{\bar{k},n}} z_n - p\| \Bigg) \Bigg(\sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| \Bigg) \\ &\qquad + \epsilon_n L_n. \end{aligned}$$

Since by assumption,

$$\lim_n \sum_{k=\bar{k}+1}^{K} \|S^k_{r_{k,n}} z_n - z_n\| = 0,$$

hence

$$\lim_n \| z_n - S^{\bar{k}}_{r_{\bar{k},n}} z_n \| = 0,$$

as required.

Step 4. $\lim_n \| z_n - T_{\mu_n} W_n z_n \| = 0$.
Proof. To see this, put

$$M_n := 2\Big\langle \gamma f(z_n) - A T_{\mu_n} W_n S^K_n z_n , z_n - T_{\mu_n} W_n z_n \Big\rangle.$$

It is obvious that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence. By using (3.9), we have

$$\begin{aligned}
&\| z_n - T_{\mu_n} W_n z_n \|^2 \\
&= \| \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n S^K_n z_n - T_{\mu_n} W_n z_n \|^2 \\
&\le \| S^K_n z_n - z_n \|^2 + \epsilon_n M_n,
\end{aligned}$$

and

$$\begin{aligned}
&\| S^K_n z_n - z_n \| \\
&\le \| S^1_{r_{1,n}} \cdots S^K_{r_{K,n}} z_n - S^1_{r_{1,n}} z_n \| + \| S^1_{r_{1,n}} z_n - z_n \| \\
&\le \| S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n - z_n \| + \| S^1_{r_{1,n}} z_n - z_n \| \\
&\ \ \vdots \\
&\le \sum_{k=1}^K \| S^k_{r_{k,n}} z_n - z_n \|.
\end{aligned}$$

Using (3.7) and the fact that $\{M_n\}_{n\in\mathbb{N}}$ is a bounded sequence, we can conclude that,

$$\begin{aligned}
&\lim_n \| z_n - T_{\mu_n} W_n z_n \|^2 \\
&\le \Big(\lim_n \sum_{k=1}^K \| S^k_{r_{k,n}} z_n - z_n \| \Big)^2 + \lim_n \epsilon_n M_n = 0 .
\end{aligned}$$

Step 5. $\lim_{n\to\infty} \| z_n - T_t z_n \| = 0$, for all $t \in S$.
Proof. Let $p \in \mathfrak{F}$ and put

$$M_0 = \frac{\| \gamma f(p) - Ap \|}{\overline{\gamma} - \alpha\gamma}.$$

Let $D = \{ y \in H : \| y - p \| \le M_0 \}$. It is clear that D is a bounded closed convex set, and $\{z_n : n \in \mathbb{N}\} \subseteq D$. It is also obvious that D is invariant under $\{S^k_{r_{k,n}} : k = 1, 2, \ldots K, n \in \mathbb{N}\}$, W_n for every $n \in \mathbb{N}$, and . We will show that

$$\limsup_{n\to\infty} \sup_{y\in D} \| T_{\mu_n} y - T_t T_{\mu_n} y \| = 0 \quad (\mathrm{t} \in \mathrm{S}). \tag{3.11}$$

Let $\epsilon > 0$. By Theorem 2.1 of [3], there exists $\delta > 0$ such that

$$\overline{\mathrm{co}} F_\delta(T_t; D) + B_\delta \subseteq F_\epsilon(T_t; D) \quad (\mathrm{t} \in \mathrm{S}). \tag{3.12}$$

Also by Corollary 1.1 of [3], there exists a natural number N such that

$$\Big\| \frac{1}{N+1} \sum_{i=0}^N T_{t^i s} y - T_t \Big(\frac{1}{N+1} \sum_{i=0}^N T_{t^i s} y \Big) \Big\| \le \delta, \tag{3.13}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$, since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$ such that $\| \mu_n - l^*_{t^i} \mu_n \| \le \frac{\delta}{(M_0 + \|p\|)}$ for $n \ge N_0$ and $i = 1, 2, \cdots, N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D} \Big\| T_{\mu_n} y - \int \frac{1}{N+1} \sum_{i=0}^N T_{t^i s} y \mu_n(s) \Big\| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \Big| \langle T_{\mu_n} y, z \rangle - \Big\langle \int \frac{1}{N+1} \sum_{i=0}^N T_{t^i s} y \mu_n(s), z \Big\rangle \Big| \\
&= \sup_{y\in D} \sup_{\|z\|=1} \Big| \frac{1}{N+1} \sum_{i=0}^N (\mu_n)_s \langle T_s y, z \rangle - \frac{1}{N+1} \sum_{i=0}^N (\mu_n)_s \langle T_{t^i s} y, z \rangle \Big| \\
&\le \frac{1}{N+1} \sum_{i=0}^N \sup_{y\in D} \sup_{\|z\|=1} |(\mu_n)_s \langle T_s y, z \rangle - (l^*_{t^i} \mu_n)_s \langle T_s y, z \rangle| \\
&\le \max_{i=1,2,\cdots,N} \| \mu_n - l^*_{t^i} \mu_n \| (M_0 + \|p\|) \\
&\le \delta \quad (\mathrm{n} \ge \mathrm{N_0}).
\end{aligned} \tag{3.14}$$

By Theorem 2.3 we have

$$\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s} y \mu_n(s) \in \overline{\text{co}}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S\Big\}. \quad (3.15)$$

It follows from (3.12)-(3.15) that

$$T_{\mu_n} y \in \overline{\text{co}}\Big\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s} y : s \in S\Big\} + B_\delta \subset \overline{\text{co}} F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D),$$

for all $y \in D$ and $n \geq N_0$. Therefore,

$$\limsup_{n\to\infty} \sup_{y\in D} \|T_t(T_{\mu_n} y) - T_{\mu_n} y\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get (3.11).
Let $t \in S$ and $\epsilon > 0$, then there exists $\delta > 0$, which satisfies (3.12). Take $L_0 = (\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|$. Now from (3.11) and condition (iii) there exists $N_1 \in \mathbb{N}$ such that $T_{\mu_n} y \in F_\delta(T_t; D)$ for all $y \in D$ and $\epsilon_n < \frac{\delta}{2L_0}$ for all $n \geq N_1$. We note that

$$\begin{aligned}
&\epsilon_n \|\gamma f(z_n) - AT_{\mu_n} W_n S_n^K z_n\| \\
&\leq \epsilon_n \Big(\|\gamma f(z_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\| \\
&\quad + \|Ap - AT_{\mu_n} W_n S_n^K z_n\|\Big) \\
&\leq \epsilon_n \Big(\gamma\alpha \|z_n - p\| \\
&\quad + \|\gamma f(p) - Ap\| + \|A\|\|z_n - p\|\Big) \\
&\leq \epsilon_n \Big((\gamma\alpha + \|A\|)M_0 + \|\gamma f(p) - Ap\|\Big) \\
&= \epsilon_n L_0 \leq \frac{\delta}{2},
\end{aligned}$$

for all $n \geq N_1$. Observe that

$$\begin{aligned}
z_n &= \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} W_n S_n^K z_n \\
&= T_{\mu_n} W_n S_n^K z_n + \epsilon_n \Big(\gamma f(z_n) \\
&\quad - AT_{\mu_n} W_n S_n^K z_n\Big) \\
&\in F_\delta(T_t; D) + B_{\frac{\delta}{2}} \\
&\subseteq F_\delta(T_t; D) + B_\delta \\
&\subseteq F_\epsilon(T_t; D),
\end{aligned}$$

for all $n \geq N_1$. This show that

$$\|z_n - T_t z_n\| \leq \epsilon \quad (\text{n} \geq \text{N}_1).$$

Since $\epsilon > 0$ is arbitrary, we get $\lim_{n\to\infty}\|z_n - T_t z_n\| = 0$.

Step 6. The weak ω-limit set of $\{z_n\}$ which is denoted by $\omega_\omega\{z_n\}$ is a subset of $\mathfrak{F}$.
Proof. Let $\hat{z} \in \omega_\omega\{z_n\}$ and let $\{z_{n_j}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_j} \rightharpoonup \hat{z}$. We need to show that $\hat{z} \in \mathfrak{F}$. In terms of Lemma 2.4 and Step 5, we conclude that $\hat{z} \in \text{Fix}(\mathcal{S})$. By Theorems 2.2, 2.3, the mapping $W : C \to C$, given by $Wx := \lim_n W_n x$ satisfies

$$\limsup_{n\to\infty} \|W_n \hat{z} - W\hat{z}\| = 0. \quad (3.16)$$

Putting $\lim_n r_{k,n} = \hat{r}_k$ for every $k \in \{1, 2, \cdots, K\}$, by Theorem 2.5, we have

$$S_{\hat{r}_k}^k x = \lim_n S_{r_{k,n}}^k x \qquad (x \in H). \quad (3.17)$$

Since $\hat{z} \in \text{Fix}(\mathcal{S})$, by our assumption, we have $T_i \hat{z} \in \text{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$ and then $W_n \hat{z} \in \text{Fix}(\mathcal{S})$. Hence, by (ii) of Theorem 2.3, $T_{\mu_n} W_n \hat{z} = W_n \hat{z}$.
Consider the set of the asymptotic center $A(z_{n_j})$ of $\{z_{n_j}\}$ with respect to H. Since $z_{n_j} \rightharpoonup \hat{z}$, Lemma 2.4 implies that $A(z_{n_j}) = \{\hat{z}\}$. By the definition of $A(z_{n_j})$, we have

$$\limsup_{j\to\infty} \|z_{n_j} - z\| \leq \limsup_{j\to\infty} \|z_{n_j} - T_t z_{n_j}\| \quad (t \in S),$$

for all $z \in A(z_{n_j})$. Since $A(z_{n_j}) = \{\hat{z}\}$, by Step 5, we get $z_{n_j} \to \hat{z}$. Using (3.16) and Step 4, we have

$$
\begin{aligned}
&\limsup_{j\to\infty}\|z_{n_j} - W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j} - T_{\mu_{n_j}}W_{n_j}z_{n_j}\| \\
&\qquad + \limsup_{j\to\infty}\|T_{\mu_{n_j}}W_{n_j}z_{n_j} - T_{\mu_{n_j}}W_{n_j}\hat{z}\| \\
&\qquad + \limsup_{j\to\infty}\|T_{\mu_{n_j}}W_{n_j}\hat{z} - W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j} - T_{\mu_{n_j}}W_{n_j}z_{n_j}\| \\
&\qquad + \limsup_{j\to\infty}\|z_{n_j} - \hat{z}\| \\
&\qquad + \limsup_{j\to\infty}\|W_{n_j}\hat{z} - W\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j} - \hat{z}\| = 0.
\end{aligned}
$$

This implies that $W(\hat{z}) = \hat{z}$.
Using Theorem 2.4 and (3.17) and Step 3, we have

$$
\begin{aligned}
&\limsup_{j\to\infty}\|z_{n_j} - S^k_{\hat{r}_k}\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j} - S^k_{r_k,n_j}z_{n_j}\| \\
&\qquad + \limsup_{j\to\infty}\|S^k_{r_k,n_j}z_{n_j} - S^k_{r_k,n_j}\hat{z}\| \\
&\qquad + \limsup_{j\to\infty}\|S^k_{r_k,n_j}\hat{z} - S^k_{\hat{r}_k}\hat{z}\| \\
&\quad\le \limsup_{j\to\infty}\|z_{n_j} - \hat{z}\| = 0.
\end{aligned}
\tag{3.18}
$$

This implies that $S^k_{\hat{r}_k}(\hat{z}) = \hat{z}$ for every $k \in \{1, 2, \cdots, K\}$.
Therefore, $\hat{z} \in \mathrm{Fix(W)} \cap (\bigcap_{\mathrm{k=1}}^{\mathrm{K}} \mathrm{Fix}(\mathrm{S}^{\mathrm{k}}_{\hat{\mathrm{r}}_{\mathrm{k}}}))$. In terms of Theorems 2.4 and 2.5, we conclude that $\hat{z} \in (\bigcap_{i=1}^{\infty} \mathrm{Fix}(\mathrm{T_i})) \cap \mathrm{SEP}()$. Since $\hat{z} \in \mathrm{Fix}(\mathcal{S})$, therefore, $\hat{z} \in \mathfrak{F}$.

Step 7. There exists a unique solution $x^* \in \mathfrak{F}$ of the variational inequality (3.5), such that

$$
\Gamma := \limsup_{n} \left\langle (\gamma f - A)x^*, z_n - x^* \right\rangle \le 0. \tag{3.19}
$$

Proof. Banach Contraction Mapping Principle guarantees that $P_{\mathfrak{F}}(I - (A - \gamma f))$ has a unique fixed point x^* which is, by Lemma 2.1, the unique solution of the variational inequality :

$$
\left\langle (\gamma f - A)x^*, x - x^* \right\rangle \le 0 \qquad (x \in \mathfrak{F}).
$$

Note that, from the definition of Γ and the fact that z_n is a bounded sequence, we can select a subsequence z_{n_j} of z_n with the following properties:
(i) $\lim_j \left\langle (\gamma f - A)x^*, z_{n_j} - x^* \right\rangle = \Gamma$;
(ii) z_{n_j} is weakly converge to a point $\hat{z}$;
by Step 6, we have $\hat{z} \in \mathfrak{F}$ and then

$$
\begin{aligned}
\Gamma &= \lim_j \left\langle (\gamma f - A)x^*, z_{n_j} - x^* \right\rangle \\
&= \left\langle (\gamma f - A)x^*, \hat{z} - x^* \right\rangle \le 0,
\end{aligned}
$$

as $x^* \in \mathfrak{F}$ is the unique solution of (3.5).

Step 8. $\{z_n\}$ strongly converges to x^*.

Proof. Indeed, from (3.6), (3.19) and that $x^* \in \mathfrak{F}$, we conclude

$$
\begin{aligned}
&\limsup_{n}\|z_n - x^*\|^2 \\
&\le \frac{1}{\overline{\gamma} - \alpha\gamma} \limsup_{n} \left\langle (\gamma f - A)x^*, z_n - x^* \right\rangle \le 0.
\end{aligned}
$$

That is $z_n \to x^*$.

Theorem 3.2 *Let H be a real Hilbert space, T be a nonexpansive mapping of C into itself such that $\mathrm{Fix(T)} \neq \emptyset$, $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix(T)}) \subseteq \mathrm{Fix(T)}$ for every $i \in \mathbb{N}$, and $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f be an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \le b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies $(A_1) - (A_4)$ of Theorem 2.5 ,*

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix(T)} \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim_n \epsilon_n = 0$ *and,*

(iv) *for every $k \in \{1, 2, \cdots, K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.*

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) \frac{1}{n} \sum_{k=1}^{n} T^k W_n S^K_n z_n$$

$$(n \in \mathbb{N}).$$

Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. Let $S = \{1, 2, ...\}$, $= \{T^i : i \in S\}$. For $f = (z_1, z_2, \cdots) \in B(S)$, define

$$\mu_n(f) = \frac{1}{n} \sum_{k=1}^{n} z_k \qquad (n \in \mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(S)$; for more details, see [19]. Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=1}^{n} T^k x.$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^* \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.3 *Let H be a real Hilbert space, T be a nonexpansive mapping of C into itself such that* $\mathrm{Fix(T)} \neq \emptyset$, *$\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix(T)}) \subseteq \mathrm{Fix(T)}$ for every $i \in \mathbb{N}$, $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f be an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies $(A_1) - (A_4)$ of Theorem t2.5,*

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix(T_n)} \cap \mathrm{Fix(T)} \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim\limits_{n} \epsilon_n = 0$ *and,*

(iv) *for every $k \in \{1, 2, \cdots, K\}$, $\lim\limits_{n} r_{k,n}$ exists and is a positive real number.*

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$\begin{aligned} z_n =& \epsilon_n \gamma f(z_n) \\ &+ (I - \epsilon_n A) \frac{1 - a_n}{a_n} \sum_{k=1}^{\infty} (a_n)^k T^k W_n S^K_n z_n \end{aligned}$$

$$(n \in \mathbb{N}),$$

where $\{a_n\}$ is an increasing sequence in $(0, 1)$ such that $\lim\limits_{n} a_n = 1$. Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\Big\langle (A - \gamma f)x^*, x - x^* \Big\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. Let $S = \{1, 2, ...\}$, $\varrho = \{T^i : i \in S\}$. For $f = (z_1, z_2, \cdots) \in B(S)$, define

$$\mu_n(f) = \frac{1 - a_n}{a_n} \sum_{k=1}^{\infty} (a_n)^k z_k \qquad (n \in \mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(S)$; for more details, see ([19], p. 79). Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n} x = \frac{1 - a_n}{a_n} \sum_{k=1}^{\infty} (a_n)^k T^k x.$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^* \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\left\langle (A-\gamma f)x^*, x-x^* \right\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.4 *Let H be a real Hilbert space, and C be a nonempty closed convex subset of a Hilbert space H, and $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \leq t < +\infty\}$, ϱ=$\{T\dot{}t{:}t\in \mathbb{R}^+\}$, and $\varrho = \{T_t : t \in \mathbb{R}^+\}$ be a representation of S as nonexpansive mappings of C into itself and suppose* $\mathrm{Fix}(\varrho) \neq \emptyset$. *Let X be a left invariant subspace of $B(\mathbb{R}^+)$ such that $1 \in X$ and the function $t \mapsto \left\langle T_t x, y \right\rangle$ is an element of X for each $x \in C$, $y \in H$, $\{T_i\}_{i\in\mathbb{N}}$ be a sequence of nonexpansive mappings from C into itself such that $T_i(\mathrm{Fix}(\varrho)) \subseteq \mathrm{Fix}(\varrho)$for $i \in \mathbb{N}$, $\wp = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, and f is an α-contraction on H. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. Assume that,*

(i) *for every $k \in \{1, 2, \cdots, K\}$, the function G_k satisfies $(\mathrm{A}_1)-(\mathrm{A}_4)$ of Theorem 2.5,*

(ii) $\mathfrak{F} := \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(\mathrm{T_n}) \cap \mathrm{Fix}(\varrho) \cap \mathrm{SEP}(\wp) \neq \emptyset$,

(iii) $\lim_n \epsilon_n = 0$ *and,*

(iv) *for every $k \in \{1, 2, \cdots, K\}$, $\lim_n r_{k,n}$ exists and is a positive real number.*

For every $n \in \mathbb{N}$, let W_n be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$, let $S^k_{r_{k,n}}$ be the resolvent generated by G_k and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

$$\begin{aligned} z_n =& \epsilon_n \gamma f(z_n) \\ &+ (I-\epsilon_n A)\frac{1}{a_n}\int_0^{a_n} T_t W_n S_n^K z_n t \\ (n \in& \mathbb{N}), \end{aligned}$$

where $\{a_n\}$ is an increasing sequence in $(0, \infty)$ such that $\lim_n a_n = \infty$. Then $\{z_n\}$ strongly converges to $x^* \in \mathfrak{F}$, where x^* is the unique solution of the variational inequality

$$\left\langle (A-\gamma f)x^*, x-x^* \right\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Proof. For $f \in B(\mathbb{R}^+)$, define

$$\mu_n(f) = \frac{1}{a_n}\int_0^{a_n} f(t)t \qquad (n \in \mathbb{N}).$$

Then $\{\mu_n\}$ is a regular sequence of means on $B(\mathbb{R}^+)$; for more details, see ([19], p. 80). Next for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n}x = \frac{1}{a_n}\int_0^{a_n} T_t x t \qquad (n \in \mathbb{N}).$$

Therefore, it follows from Theorem 3.1 that the sequence $\{z_n\}$ converges strongly to $x^* \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\left\langle (A-\gamma f)x^*, x-x^* \right\rangle \geq 0 \quad (x \in \mathfrak{F}).$$

Theorem 3.5 *Let $\varrho = \{T_t : t \in S\}$ be a representation of S as nonexpansive mappings of H into itself such that* $\mathrm{Fix}(\varrho) \neq \emptyset$. *Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \left\langle T_t x, y \right\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Suppose that A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$ and f is an α-contraction on H. Moreover, let $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $0 < \epsilon_n < 1$, $\lim_n \epsilon_n = 0$, $0 < \lambda_n \leq b < 1$, and γ is a real number such that $0 < \gamma < \frac{\overline{\gamma}}{\alpha}$. If $\{z_n\}$ is the sequence generated by*

$$z_n = \epsilon_n \gamma f(z_n) + (I-\epsilon_n A)T_{\mu_n} z_n \quad (n \in \mathbb{N}).$$

Then $\{z_n\}$ *strongly converges to* $x^* \in \mathrm{Fix}(\varrho)$.
Proof. Take $G_k = 0$ for every $k \in \{1, 2, \cdots K\}$, $T_i = I$ for every $i \in \mathbb{N}$ and $C = H$ in Theorem 3.1. Then we have $S^1_{r_{1,n}} S^2_{r_{2,n}} \cdots S^K_{r_{K,n}} z_n = z_n$ and $W_n = I$ for all $n \in \mathbb{N}$. So from Theorem 3.1 the sequences $\{z_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\varrho)$.

Acknowledgements

The author is grateful to the office of Graduate Studies of the University of Isfahan for their support. The author also wishes to thank The Center of Excellence for Mathematics of University of Isfahan for financial support.

References

[1] R. P. Agarwal, D. Oregan, D. R. Sahu, *Fixed point theory for Lipschitzian-type mappings with applications*, in: Topological Fixed Point Theory and its Applications, vol. 6, Springer, New York, 2009.

[2] S. Atsushiba, W. Takahashi, *Approximation common fixed points of nonexpansive semigroups by the Mann iteration process*, Ann. Univ. Mariae Curie-Sklodowska 51 (1997) 1-16.

[3] R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. 38 (1981) 304-314.

[4] V. Colao, G. L. Acedo, G. Marino, *An implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings*, Nonlinear Anal, 71 (2009) 2708-2715.

[5] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. 6 (1) (2005) 117-136.

[6] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, in: Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[7] N. Hirano, K. Kido, and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. 12 (1988), 1269-1281.

[8] K. Kido and W. Takahashi, *Mean ergodic theorems for semigroups of linear continuous in Banach spaces*, J. Math. Anal. Appl. 103 (1984), 387-394.

[9] A. T. Lau, H. Miyake, W. Takahashi, *Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Anal. 67 (2007) 1211-1225.

[10] G. Marino, H. K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. 318 (1) (2006) 43-52.

[11] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. 241 (1) (2000) 46-55.

[12] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967) 595-597.

[13] H. Piri, *Strong convergence for a minimization problem on solutions of systems of equilibrium problems and common fixed points of an infinite family and semigroup of nonexpansive mappings*, Computers and Mathematics with Applications 61 (2011) 2562-2577.

[14] S. Plubtieng, R. Punpaeng, *A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. 336 (2007) 455-469.

[15] X. Qin, M. Shang, Y. Su, *Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems*, Mathematical and Computer Modelling 48 (2008) 1033-1046.

[16] T. Shimizu, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. 211 (1997) 71-83.

[17] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math 5 (2001)387-404.

[18] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. 81 (1981) 253-256.

[19] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

[20] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. (2006) http://dx.doi.org/10.1016/j.jmaa.2006.08.036/.

[21] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces*, Nonlinear Anal. (2008) http://dx.doi.org/10.1016/j.na.2008.01.005/.

Ebrahim Soori received his B.Sc. degree in 2001 in Boali Sina University and M.Sc. degree in Kharazmi (Tarbiat Moalem Tehran) University in pure mathematics (Functional Analysis: Frame Theory) in 2003. Now he is Ph.D student in Isfahan University in Harmonic Analysis.