strong convergence for variational inequalities and equilibrium problems and representations

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Abstract

We introduce an implicit method for finding a common element of the set of solutions of systems of equilibrium problems and the set of common fixed points of a sequence of nonexpansive mappings and a representation of nonexpansive mappings. Then we prove the strong convergence of the proposed implicit schemes to the unique solution of a variational inequality, which is the optimality condition for a minimization problem and is also a common fixed point for a sequence of nonexpansive mappings and a representation of nonexpansive mappings.

Keywords: Representation; Equilibrium problem; Fixed point; Nonexpansive mapping; Variational inequality.

1 Introduction

Let $H$ be a Hilbert space and let $G : H \times H \to \mathbb{R}$ be an equilibrium function, that is

$$G(u, u) = 0 \quad \text{for every } u \in H.$$ 

The Equilibrium Problem is defined as follows: Find $\bar{x} \in H$ such that

$$G(\bar{x}, y) \geq 0 \quad \text{for all } y \in H. \quad (1.1)$$

A solution of (1.1) is said to be an equilibrium point and the set of the equilibrium points is denoted by $\text{SEP}(G)$. Let $C$ be a closed convex subset of $H$. A mapping $T$ of $C$ into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Let $f$ be an $\alpha$-contraction on $H$ (i.e. $\|f(x) - f(y)\| \leq \alpha \|x - y\|$, $x, y \in H$ with $0 \leq \alpha < 1$), and $A$ be a bounded linear operator on $H$. The following variational inequality problem with viscosity is of great interest [10, 11]. Find $x^*$ in $C$ such that

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in C), \quad (1.2)$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \left( \frac{1}{2} \langle Ax, x \rangle + h(x) \right),$$

where $\gamma$ satisfies $\|I - A\| \leq 1 - \alpha \gamma$ and $h$ is a potential function for $\gamma f$ (that is $h'(x) = \gamma f(x)$). S. Takahashi and W. Takahashi [20] introduced the following viscosity approximation method for finding a common element of $\text{SEP}(G)$ and $\text{Fix}(T)$, where $T$ is a nonexpansive mapping. Starting
with an arbitrary element \(x_1 \in H\), they defined the sequences \(\{u_n\}\) and \(\{x_n\}\) recursively by
\[
\begin{align*}
G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0 \\
(x_n + 1) = \epsilon_n \gamma f(x_n) + (I - \epsilon_n) Tu_n \\
(n \in \mathbb{N}),
\end{align*}
\]
and Plubtieng and Punpaeng in [14] proved a strong convergence theorem for an implicit iterative sequence obtained from the viscosity approximation method for finding a common element in \(\text{SEP}(G) \cap \text{Fix}(T)\) which satisfies the variational inequality (1.2):

**Theorem 1.1** Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). Let \(G\) be a bifunction from \(H \times H \to \mathbb{R}\) satisfying
\[
(A_1) \quad G(x, y) = 0 \text{ for all } x \in C;
\]
\[
(A_2) \quad G \text{ is monotone, i.e. } G(x, y) + G(y, x) \leq 0 \text{ for all } x, y \in C;
\]
\[
(A_3) \quad \text{For all } x, y, z \in C,
\]
\[
\limsup_{t \to 0} G(tx + (1 - t)x, y) \leq G(x, y);
\]
\[
(A_4) \quad \text{For all } x \in C, \quad y \mapsto G(x, y) \text{ is convex and lower semicontinuous.}
\]
For \(x \in H\) and \(r > 0\), set \(S_r : H \to C\) to be the resolvent of \(G\) i.e. \(S_r(x)\) is the unique \(z \in C\) for which
\[
G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad (y \in C).
\]

Let \(T\) be a nonexpansive mapping on \(H\) such that \(\text{SEP}(G) \cap \text{Fix}(T) \neq \emptyset\). Let \(f\) be a contraction of \(H\) into itself with \(\alpha \in (0, 1)\) and let \(A\) be a strongly positive bounded linear operator on \(H\) with coefficient \(\gamma > 0\) and \(0 < \gamma < \frac{1}{\alpha}\). Let \(\{x_n\}\) be the sequence generated by
\[
\begin{align*}
x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n A) Tu_n \\
(n \in \mathbb{N}),
\end{align*}
\]
\[
G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0 \\
(y \in H),
\]
where \(u_n = S_{r_n} x_n, \quad \{r_n\} \subset (0, \infty)\) and \(\alpha_n \in [0, 1]\) satisfying \(\lim_{n \to \infty} \alpha_n = 0\) and
\[
\liminf_{n \to \infty} r_n > 0. \quad \text{Then } \{x_n\} \text{ and } \{u_n\} \text{ converge strongly to a point } z \text{ in } \text{Fix}(T) \cap \text{SEP}(G) \text{ which solves the variational inequality}
\]
\[
\langle (A - \gamma f)z, z - x \rangle \leq 0 \quad x \in \text{Fix}(T) \cap \text{SEP}(G).
\]

V. Colao, G. L. Acedo and G. Marino proved a strong convergence theorem for the following implicit sequence \(\{z_n\}\) for finding a common element in \(\bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \bigcap_{k=1}^{n} \text{SEP}(G_k)\) in [4],
\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) W_n S_n^K z_n,
\]
where
\[
S_n^K = S_{r_1} S_{r_2} \cdots S_{r_k},
\]
and \(n \in \mathbb{N}\). In this paper, motivated by Lau, Miyake and Takahashi [9], Atsushiba and Takahashi [2], Shimizu and Takahashi [16] and Takahashi [21], in Theorem 3.1, we use the harmonic concepts for improving the results proved in [4], in other word we use the amenability concepts and the theory of representations in our results but V. Colao, G. L. Acedo and G. Marino have not used these concepts in [4]. We introduce the following general implicit algorithm for finding a common element of the set of solutions of a system of equilibrium problem \(\text{SEP}(\varphi)\) for a family \(\varphi = \{G_k; k = 1, 2, \ldots, K\}\) of bifunctions and the set of fixed points of a family \(\{T_i\}_{i \in \mathbb{N}}\) of nonexpansive mappings from \(C\) into itself and a representation \(\varphi = \{T_i : t \in S\}\) of a semigroup \(S\) as nonexpansive mappings from \(C\) into itself, with respect to \(W\)-mappings and a left regular sequence \(\{\mu_n\}\) of means defined on an appropriate subspace of bounded real-valued functions of the semigroup:
\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) W_n S_n^K z_n,
\]
where
\[
S_n^K = S_{r_1} S_{r_2} \cdots S_{r_k},
\]
and \(n \in \mathbb{N}\).

Our goal is to prove some results of strong convergence for implicit schemes to approach a solution \(x^*\) of the problem (1.2) such that
\[
x^* \in \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \text{Fix}(S) \cap \text{SEP}(\varphi).\]
2 Preliminaries

Throughout this paper \( H \) denotes a Hilbert space. Moreover we assume that \( A \) is a bounded strongly positive operator on \( H \) with constant \( \gamma \); that is there exists \( \gamma > 0 \) such that
\[
\langle Ax, x \rangle \geq \gamma \|x\|^2 \quad (x \in H).
\]

For a map \( T : H \to H \) we denote by \( \text{Fix}(T) := \{x \in H : x = Tx\} \) the fixed point set of \( T \). Note that if \( T \) is a nonexpansive mapping, \( \text{Fix}(T) \) is closed and convex (see [6]).

Let \( S \) be a semigroup. We denote by \( B(S) \) the Banach space of all bounded real-valued functions defined on \( S \) with supremum norm. For each \( s \in S \) and \( f \in B(S) \) we define \( l_s \) and \( r_s \) in \( B(S) \) by
\[
(l_s f)(t) = f(st), \quad (r_s f)(t) = f(ts), \quad (t \in S).
\]

Let \( X \) be a subspace of \( B(S) \) containing \( 1 \) and let \( X^* \) be its topological dual. An element \( \mu \) of \( X^* \) is said to be a mean on \( X \) if \( \|\mu\| = \mu(1) = 1 \). We often write \( \mu(f(t)) \) instead of \( \mu(f) \) for \( f \in X^* \) and \( t \in S \). Let \( X \) be left invariant (resp. right invariant), i.e. \( l_s(X) \subset X \) (resp. \( r_s(X) \subset X \)) for each \( s \in S \). A mean \( \mu \) on \( X \) is said to be left invariant (resp. right invariant) if \( \mu(l_s f) = \mu(f) \) (resp. \( \mu(r_s f) = \mu(f) \)) for each \( s \in S \) and \( f \in X \). \( X \) is said to be left (resp. right) amenable if \( X \) has a left (resp. right) invariant mean. \( X \) is amenable if \( X \) is both left and right amenable. As is well known, \( B(S) \) is amenable when \( S \) is a commutative semigroup (see page 29 of [19]). A net \( \{\mu_\alpha\} \) of means on \( X \) is said to be left regular if
\[
\lim_{\alpha} \|l_s^* \mu_\alpha - \mu_\alpha\| = 0,
\]
for each \( s \in S \), where \( l_s^* \) is the adjoint operator of \( l_s \).

Let \( f \) be a function of semigroup \( S \) into a reflexive Banach space \( E \) such that the weak closure of \( \{f(t) : t \in S\} \) is weakly compact and let \( X \) be a subspace of \( B(S) \) containing all the functions \( t \to \langle f(t), x^* \rangle \) with \( x^* \in E^* \). We know from [7] that for any \( \mu \in X^* \), there exists a unique element \( f_\mu \in E \) such that
\[
\langle f_\mu, x^* \rangle = \mu(\langle f(t), x^* \rangle) \quad \text{for all } x^* \in E^*.
\]
We denote such \( f_\mu \) by \( \int f(t) \mu(t) \). Moreover, if \( \mu \) is a mean on \( X \) then from [8],
\[
\int f(t) \mu(t) \in \text{co} \{f(t) : t \in S\}.
\]

Let \( C \) be a nonempty closed and convex subset of \( H \). Then, a family \( g = \{T_s : s \in S\} \) of mappings from \( C \) into itself is said to be a representation of \( S \) as nonexpansive mapping on \( C \) into itself if satisfies the following:

1. \( T_s x = T_s T_t x \) for all \( s, t \in S \) and \( x \in C \);
2. for every \( s \in S \) the mapping \( T_s : C \to C \) is nonexpansive.

We denote by \( \text{Fix}(g) \) the set of common fixed points of , that is \( \text{Fix}(g) = \{x \in C : T_s x = x, \ (s \in S)\} \).

For an equilibrium function \( G : H \times H \to \mathbb{R} \), \( \text{SEP}(G) := \{x \in H : G(x, y) \geq 0, (y \in H)\} \) is the set of solutions of the related equilibrium problem.

Let \( C \) be a closed convex subset of a Hilbert space \( H \). Recall that the (nearest) projection \( P_C \) from \( H \) onto \( C \) assigns to each \( x \in H \) the unique point \( P_C x \in C \) satisfying the property
\[
\|x - P_C x\| = \min_{y \in C} \|x - y\|.
\]
The following Lemma characterizes the projection \( P_C \).

Lemma 2.1 ([19]). Let \( C \) be a closed convex subset of a real Hilbert space \( H \), \( x \in H \) and \( y \in C \). Then \( P_C x = y \) if and only if it satisfies the inequality
\[
\langle x - y, y - z \rangle \geq 0 \quad (z \in C).
\]

Lemma 2.2 ([10]). Let \( A \) be a strongly positive linear bounded operator on a Hilbert space \( H \) with coefficient \( \gamma \) and \( 0 < \rho \leq \|A\|^{-1} \) Then \( \|I - \rho A\| \leq 1 - \rho \gamma \).

Theorem 2.1 ([18]). Let \( S \) be a semigroup, \( C \) be a closed convex subset of a Hilbert space \( H \), \( g = \{T_s : s \in S\} \) be a representation of \( S \) as nonexpansive mapping from \( C \) into itself such that \( \text{Fix}(g) \neq \emptyset \) and \( X \) be a subspace of \( B(S) \) such that \( 1 \in X \) and the mapping \( t \to \langle T(t)x, y \rangle \) be an element of \( X \) for each \( x \in C \) and \( y \in H \), and \( \mu \) be a mean on \( X \). If we write \( T_\mu x \) instead of \( \int T(t) d\mu(t) \), then the following hold.

(i) \( T_\mu \) is a nonexpansive mapping from \( C \) into \( C \).
(ii) \( T_\mu x = x \) for each \( x \in \text{Fix}(g) \).
(iii) \( T_\mu x \in \text{co} \{T_s x : s \in S\} \) for each \( x \in C \).
(iv) If \( \mu \) is left invariant, then \( T_\mu \) is a nonexpansive retraction from \( C \) onto \( \text{Fix}(S) \).
**Theorem 2.2** ([5]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $G : H \times H \to \mathbb{R}$ satisfy,

(A1) $G(x,x) = 0$ for all $x \in C$;
(A2) $G$ is monotone, i.e. $G(x,y) + G(y,x) \leq 0$ for all $x,y \in C$;
(A3) For all $x,y,z \in C$,

$$\limsup_{t \to 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(A4) For all $x \in C, y \mapsto G(x, y)$ is convex and lower semicontinuous.

For $x \in H$ and $r > 0$, set $S_r : H \to C$ to be

$$S_r(x) := \{ z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ (y \in C) \},$$

then $S_r$ is well defined and the followings are valid:

(i) $S_r$ is single-valued;
(ii) $S_r$ is firmly nonexpansive, i.e.

$$\| S_r x - S_r y \| \leq \langle S_r x - S_r y, x - y \rangle,$$

for all $x, y \in H$;
(iii) $\text{Fix} S_r = \text{SEP}(G)$;
(iv) $\text{SEP}(G)$ is closed and convex.

**Theorem 2.3** ([4]). Let $\{r_n\} \subset (0, \infty)$ be a sequence converging to $r > 0$. For a bifunction $G : H \times H \to \mathbb{R}$, satisfying conditions (A1)–(A4), define $S_r$ and $S_{r_n}$ for $n \in \mathbb{N}$ as in Theorem 2.5, then for every $x \in H$, we have

$$\lim \| S_{r_n} x - S_r x \| = 0.$$

Let $C$ be a nonempty subset of a Hilbert space $H$ and $T : C \to H$ be a mapping. Then $T$ is said to be demiclosed at $v \in H$ if for any sequence $\{x_n\}$ in $C$, the following implication holds:

$$x_n \to u \in C, \ \text{and} \ T x_n \to v \text{ imply } Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

**Lemma 2.3** ([1]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.

**Remark 2.1** Every Hilbert space is a uniformly convex Banach space, and therefore is a strictly convex Banach space (see pages 95, 98 of [19]).

**Definition 2.1** A vector space $X$ is said to satisfy Opial’s condition, if for each sequence $\{x_n\}$ in $X$ which converges weakly to point $x \in X$,

$$\liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \| \quad (y \in X, \ y \neq x).$$

Note that every Hilbert space satisfies the Opial’s condition (see [12] and [15]).

**Definition 2.2** Let $K$ be a nonempty subset of a Banach space $X$ and $\{x_n\}$ be a sequence in $K$. The set of the asymptotic center of $\{x_n\}$ with respect to $K$, defined by

$$A(\{x_n\}) = \left\{ x \in K : \limsup_{n \to \infty} \| x_n - x \| = \inf_{y \in K} \limsup_{n \to \infty} \| x_n - y \| \right\}.$$ 

**Lemma 2.4** ([1]). Let $X$ be a uniformly convex Banach space satisfying the Opial’s condition and $K$ be a nonempty closed convex subset of $X$. If a sequence $\{z_n\} \subset K$ converges weakly to a point $z_0$, then $\{z_0\}$ is the asymptotic center of $\{z_n\}$ with respect to $K$.

Let $C$ be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings of $C$ into itself and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Following [17], for any $n \geq 1$, we define a mapping $W_n$ of $C$ into itself as follows,

$$U_{n,n+1} := I,$$

$$U_{n,n} := \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I,$$

$$U_{n,k} := \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I,$$

$$U_{n,2} := \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I,$$

$$W_n := U_{n,1} := \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I.$$ 

The following results hold for the mappings $W_n$. 

Theorem 2.4 ([17]). Let $C$ be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings of $C$ into itself such that
\[
\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \neq \emptyset
\]
and let $\{\lambda_i\}$ be a real sequence such that $0 \leq \lambda_i \leq b < 1$ for every $i \in \mathbb{N}$. Then

1. $W_n$ is nonexpansive and $\text{Fix}(W_n) = \bigcap_{i=1}^{n} \text{Fix}(T_i)$ for each $n \geq 1$,

2. for each $x \in C$ and for each positive integer $j$, the limit $\lim_{n \to \infty} U_{n,j}x$ exists.

3. The mapping $W : C \to C$ defined by

\[
Wx := \lim_{n \to \infty} W_nx = \lim_{n \to \infty} U_{n,1}x \quad (x \in C),
\]
is a nonexpansive mapping satisfying $\text{Fix}(W) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$ and it is called the $W$-mapping generated by $\{T_i\}_{i \in \mathbb{N}}$, and $\{\lambda_i\}_{i \in \mathbb{N}}$.

Theorem 2.5 ([13]). Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of $C$ into itself such that $\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \neq \emptyset$, $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \leq b < 1$, $(i \geq 1)$. If $D$ is any bounded subset of $C$, then

\[
\lim_{n \to \infty} \sup_{x \in D} \|Wx - W_nx\| = 0.
\]

Throughout the rest of this paper, the open ball of radius $r$ centered at $0$ is denoted by $B_r$. For $\epsilon > 0$ and a mapping $T : D \to H$, we let $F_\epsilon(T; D)$ be the set of $\epsilon$-approximate fixed points of $T$, i.e.

\[
F_\epsilon(T; D) = \{x \in D : \|x - Tx\| \leq \epsilon\}.
\]

3 Main results

In this Section, we deal with the strong convergence approximation scheme for finding a common element of the set of solutions of a system of an equilibrium problem and the set of common fixed points of a sequence of nonexpansive mappings and left amenable nonexpansive semigroup in a Hilbert space. These results extend the main result of [4] and many others.

Theorem 3.1 Let $S$ be a semigroup and let $C$ be a closed convex subset of a Hilbert space $H$. Suppose that $\varrho = \{T_s : s \in S\}$ be a representation of $S$ as nonexpansive mapping from $C$ into itself and suppose $\text{Fix}(\varrho) \neq \emptyset$. Let $X$ be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto (T_tx, y)$ is an element of $X$ for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on $X$. Let $\{T_i\}_{i \in \mathbb{N}}$ be a sequence of nonexpansive mappings from $C$ into itself such that $T_i(\text{Fix}(\varrho)) \subseteq \text{Fix}(\varrho)$ for every $i \in \mathbb{N}$, and $\varrho = \{G_k : k = 1, 2, \cdots K\}$ be a finite family of bifunctions from $H \times H$ into $\mathbb{R}$. Suppose that $A$ is a strongly positive bounded linear operator with coefficient $\gamma$ and $f$ is an $\alpha$-contraction on $H$. Moreover, let $\{r_{k,n}\}$, $\{\epsilon_n\}$ and $\{\lambda_n\}$ be real sequences such that $r_{k,n} > 0$, $0 < \epsilon_n < 1$ and $0 < \lambda_n \leq b < 1$, and $\gamma$ is a real number such that $0 < \gamma < \frac{\alpha}{\gamma}$. Assume that,

1. for every $k \in \{1, 2, \cdots, K\}$, the function $G_k$ satisfies $(A_1) - (A_4)$ of Theorem 2.5,

2. $\mathcal{F} := \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \text{Fix}(S) \cap \text{SEP}(\varrho) \neq \emptyset$,

3. $\lim_{n} \epsilon_n = 0$ and,

4. for every $k \in \{1, 2, \cdots, K\}$, $\lim_{n} r_{k,n}$ exists and is a positive real number.

For every $n \in \mathbb{N}$, let $W_n$ be the mapping generated by $\{T_i\}$ and $\{\lambda_n\}$ as in (2.3), for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Let $S^K_{r_{k,n}}$ be the resolvent generated by $G_k$ and $r_{k,n}$ as in Theorem 2.5. If $\{z_n\}$ is the sequence generated by

\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_{\mu_n}W_n S^K_{r_{k,n}} z_n,
\]

where $S^K_{r_{k,n}} = S^1_{r_{k,n}} S^2_{r_{k,n}} \cdots S^K_{r_{k,n}}$ for every $k \in \{1, 2, \cdots, K\}$ and $n \in \mathbb{N}$. Then $\{z_n\}$ strongly converges to $x^* \in \mathcal{F}$, where $x^*$ is the unique solution of the variational inequality

\[
\langle (A - \gamma f)x^* - x - x^*, x - x^* \rangle \geq 0 \quad (x \in \mathcal{F}),
\]
or, equivalently,

\[
x^* = P_\mathcal{F}(I - (A - \gamma f))x^*,
\]
or, equivalently, $x^*$ is the unique solution of the minimization problem

$$\min_{x \in \mathbb{S}} \left\{ \frac{1}{2} \langle Ax, x \rangle + h(x) \right\},$$

where $h$ is a potential function for $\gamma f$.

Since $\epsilon_n \to 0$, we may assume that $\epsilon_n \leq \min \left\{ \|A\|^{-1}, \frac{1}{\gamma} \right\}$. We observe that if $\|p\| = 1$, then

$$\langle (I - \epsilon_n A)p, p \rangle = 1 - \epsilon_n \langle Ap, p \rangle \geq 1 - \epsilon_n \|A\| \geq 0.$$

Hence, if $\|p\| \neq 1$ and $p \neq 0$, then we have

$$\langle (I - \epsilon_n A)p, p \rangle = \|p\|^2 \left( \frac{(I - \epsilon_n A)p}{\|p\|}, \frac{p}{\|p\|} \right) \geq 0.$$

We also have $\langle (I - \epsilon_n A)p, p \rangle = 0$, if $p = 0$. Hence $\langle (I - \epsilon_n A)p, p \rangle \geq 0$, for all $p \in C$.

By Lemma 2.2, we have

$$\|I - \epsilon_n A\| \leq 1 - \epsilon_n \gamma.$$

We shall divide the proof into eight steps.

Step 1. The existence of $z_1$ which satisfies (3.4).

Proof. This follows immediately from the fact that for every $n \in \mathbb{N}$, the mapping $N_n$ given by

$$N_n x := \epsilon_n \gamma f(x) + (I - \epsilon_n A)T_{\mu_n} W_n S_n^K x, \quad (x \in H),$$

is a contraction. To see this, put $\beta_n = 1 + \epsilon_n \gamma \alpha - \epsilon_n \gamma \gamma$, then $0 \leq \beta_n < 1$ ($n \in \mathbb{N}$). We have,

$$\|N_n x - N_n y\|$$

$$\leq \epsilon_n \gamma \|f(x) - f(y)\|$$

$$+ \left(1 - \epsilon_n \gamma\right) \|T_{\mu_n} W_n S_n^K x - T_{\mu_n} W_n S_n^K y\|$$

$$\leq \epsilon_n \gamma \alpha \|x - y\| + (1 - \epsilon_n \gamma) \|x - y\|$$

$$= (1 + \epsilon_n \gamma \alpha - \epsilon_n \gamma \gamma) \|x - y\| = \beta_n \|x - y\|.$$

Therefore, by Banach Contraction Principle ([19], p. 4), there exist a unique point $z_n$ such that $N_n z_n = z_n$.

Step 2. $\{z_n\}$ is bounded.

Proof. Let $p \in \mathbb{S}$. We have

$$\|z_n - p\|^2 = \left\langle \epsilon_n \gamma f(z_n) + (I - \epsilon_n A)T_{\mu_n} W_n S_n^K z_n - p, z_n - p \right\rangle$$

$$= \epsilon_n \gamma \left\langle f(z_n) - f(p), z_n - p \right\rangle$$

$$+ \epsilon_n \left\langle \gamma f(p) - Ap, z_n - p \right\rangle$$

$$+ \left\langle (I - \epsilon_n A)T_{\mu_n} W_n S_n^K z_n - T_{\mu_n} W_n S_n^K p, z_n - p \right\rangle$$

$$\leq \epsilon_n \gamma \alpha \|z_n - p\|^2 + (1 - \epsilon_n \gamma) \|z_n - p\|^2$$

$$+ \epsilon_n \left\langle \gamma f(p) - Ap, z_n - p \right\rangle.$$

Thus,

$$\|z_n - p\|^2 \leq \frac{1}{\gamma - \gamma \alpha} \left\langle \gamma f(p) - Ap, z_n - p \right\rangle. \quad (3.6)$$

Hence,

$$\|z_n - p\| \leq \frac{1}{\gamma - \gamma \alpha} \|\gamma f(p) - Ap\|.$$

That is, the sequence $\{z_n\}$ is bounded.

Step 3. For every fixed $k \in \{1, 2, \cdots, K\}$, we have

$$\lim_n \|z_n - S_{r_k,n}^k z_n\| = 0. \quad (3.7)$$

Proof. Let $k \in \{1, 2, \cdots, K\}$, since by (ii) of Theorem 2.5, $S_{r_k,n}^k$ is firmly nonexpansive, we conclude that

$$\|S_{r_k,n}^k z_n - p\|^2$$

$$= \|S_{r_k,n}^k z_n - S_{r_k,n}^k p\|^2$$

$$\leq \left\langle S_{r_k,n}^k z_n - S_{r_k,n}^k p, z_n - p \right\rangle$$

$$= \frac{1}{2} \left( \|S_{r_k,n}^k z_n - p\|^2 + \|z_n - p\|^2 - \|z_n - S_{r_k,n}^k z_n\|^2 \right).$$

Therefore,

$$\|z_n - S_{r_k,n}^k z_n\|^2 \leq \|z_n - p\|^2 - \|S_{r_k,n}^k z_n - p\|^2. \quad (3.8)$$
If we put
\( L_n := 2\left\langle \gamma f(z_n) - AT_n W_n S_n^K z_n, z_n - p \right\rangle \), then
by using the inequality
\[ \|x + y\|^2 \leq \|x\|^2 + 2\left\langle y, x + y \right\rangle, \]   \hspace{1cm} (3.9)
we obtain
\[
\|z_n - p\|^2
= \|\epsilon_n f(z_n)
+ (I - \epsilon_n A) T_n W_n S_{r_1,n}^1 S_{r_2,n}^2 
\cdots S_{r_k,n}^K z_n - p\|^2
\leq \|T_n W_n S_{r_1,n}^1 S_{r_2,n}^2 
\cdots S_{r_k,n}^K z_n - p\|^2 + \epsilon_n L_n
\leq \|S_{r_k,n}^K z_n - p\|^2 + \epsilon_n L_n.
\]

So by (3.8), we have
\[
\|z_n - S_{r_k,n}^k z_n\|^2 \leq \epsilon_n L_n.
\]
That \( \{L_n\}_{n \in \mathbb{N}} \) is a bounded sequence, implies
\[
\lim_{n} \|z_n - S_{r_k,n}^k z_n\| = 0.
\]

By induction we assume that (3.7) holds for every \( k > \bar{k} \), and we prove it for \( \bar{k} \).

Indeed, we have
\[
\|z_n - p\|^2
\leq \|T_n W_n S_{r_1,n}^1 S_{r_2,n}^2 
\cdots S_{r_k,n}^K z_n - p\|^2 + \epsilon_n L_n
\leq \|S_{r_k,n}^\bar{k} \cdots S_{r_k,n}^K z_n - p\|^2 + \epsilon_n L_n. \]   \hspace{1cm} (3.10)

Observe that
\[
\|S_{r_k,n}^\bar{k} \cdots S_{r_k,n}^K z_n - p\|
= \|S_{r_k,n}^\bar{k} \cdots S_{r_k,n}^K z_n - S_{r_k,n}^\bar{k} z_n
+ S_{r_k,n}^\bar{k} z_n - p\|
\leq \|S_{r_k,n}^{\bar{k}+1} \cdots S_{r_k,n}^K z_n - z_n\|
+ \|S_{r_k,n}^{\bar{k}} z_n - p\|
\leq \|S_{r_k,n}^{\bar{k}+1} \cdots S_{r_k,n}^K z_n - S_{r_k,n}^{\bar{k}+1} z_n\|
+ \|S_{r_k,n}^{\bar{k}+1} z_n - z_n\| + \|S_{r_k,n}^{\bar{k}} z_n - p\|
\leq \|S_{r_k,n}^{\bar{k}+2} \cdots S_{r_k,n}^K z_n - z_n\|
+ \|S_{r_k,n}^{\bar{k}+1} z_n - z_n\| + \|S_{r_k,n}^{\bar{k}} z_n - p\|
\vdots
\leq \|S_{r_k,n}^{\bar{k}} z_n - p\| + \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\|.
\]

Inequality (3.10) gives,
\[
\|z_n - p\|^2
\leq \left( \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\| + 2\|S_{r_k,n}^\bar{k} z_n - p\| \left( \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\| \right) \right)
+ \|S_{r_k,n}^{\bar{k}} z_n - p\|^2 + \epsilon_n L_n.
\]

From this inequality and (3.8), we obtain
\[
\|z_n - S_{r_k,n}^k z_n\|^2
\leq \left( \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\| + 2\|S_{r_k,n}^\bar{k} z_n - p\| \left( \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\| \right) \right)
+ \epsilon_n L_n.
\]

Since by assumption,
\[
\lim_{n} \sum_{k=\bar{k}+1}^{K} \|S_{r_k,n}^k z_n - z_n\| = 0,
\]
hence
\[ \lim_n \| z_n - S_{r_{\alpha},n} F z_n \| = 0, \]
as required.

Step 4. \( \lim_n \| z_n - T_{\mu_n} W_n z_n \| = 0. \)

Proof. To see this, put
\[
M_n := 2\left( \gamma f(z_n) - AT_{\mu_n} W_n S^K z_n , z_n - T_{\mu_n} W_n z_n \right).
\]

It is obvious that \( \{M_n\}_{n \in \mathbb{N}} \) is a bounded sequence. By using (3.9), we have
\[
\| z_n - T_{\mu_n} W_n z_n \|^2
= \| \epsilon_n f(z_n)
+ (I - \epsilon_n A)T_{\mu_n} W_n S^K z_n - T_{\mu_n} W_n z_n \|^2
\leq \| S^K z_n - z_n \|^2 + \epsilon_n M_n,
\]
and
\[
\| S^K z_n - z_n \|^2
\leq \| S^1_{r_{\alpha},1} \cdots S^K_{r_{\alpha},n} z_n - S^1_{r_{\alpha},1} z_n \|
+ \| S^1_{r_{\alpha},1} z_n - z_n \|
\leq \| S^2_{r_{\alpha},1} \cdots S^K_{r_{\alpha},n} z_n - z_n \|
+ \| S^1_{r_{\alpha},1} z_n - z_n \|
\vdots
\leq \sum_{k=1}^{K} \| S^k_{r_{\alpha},n} z_n - z_n \|.
\]

Using (3.7) and the fact that \( \{M_n\}_{n \in \mathbb{N}} \) is a bounded sequence, we can conclude that,
\[
\lim_n \| z_n - T_{\mu_n} W_n z_n \|^2
\leq \left( \lim_{n \to \infty} \sum_{k=1}^{K} \| S^k_{r_{\alpha},n} z_n - z_n \|^2 \right)^2 + \lim \epsilon_n M_n = 0 .
\]

Step 5. \( \lim_{n \to \infty} \| z_n - T_i z_n \| = 0, \) for all \( t \in S. \)

Proof. Let \( p \in \mathfrak{F} \) and put
\[
M_0 = \frac{\| \gamma f(p) - Ap \|}{\gamma - \alpha \gamma}.
\]

Let \( D = \{ y \in H : \| y - p \| \leq M_0 \} \). It is clear that \( D \) is a bounded closed convex set, and \( \{ z_n : n \in \mathbb{N} \} \subseteq D \). It is also obvious that \( D \) is invariant under \( \{ S^k_{r_{\alpha},n} : k = 1, 2, \ldots, K, n \in \mathbb{N} \} \), \( W_n \) for every \( n \in \mathbb{N} \), and \( \mu_n \). We will show that
\[
\limsup_{n \to \infty} \sup_{y \in D} \| T_{\mu_n} y - T_i T_{\mu_n} y \| = 0 \quad (t \in S).
\]

(3.11)

Let \( \epsilon > 0 \). By Theorem 2.1 of [3], there exists \( \delta > 0 \) such that
\[
\| \epsilon \mathfrak{F}(T_i; D) + B_\delta \subseteq \mathfrak{F}(T_i; D) \quad (t \in S).
\]

(3.12)

Also by Corollary 1.1 of [3], there exists a natural number \( N \) such that
\[
\left\| \frac{1}{N + 1} \sum_{i=0}^{N} T_{i,s} y - T_i \left( \frac{1}{N + 1} \sum_{i=0}^{N} T_{i,s} y \right) \right\|
\leq \delta,
\]

(3.13)

for all \( t, s \in S \) and \( y \in D \). Let \( t \in S \), since \( \{\mu_n\} \) is strongly left regular, there exists \( N_0 \in \mathbb{N} \) such that \( \| \mu_n - l_i^* \mu_n \| \leq \frac{\delta}{\| M_0 + \|p\|} \) for \( n \geq N_0 \) and \( i = 1, 2, \ldots, N \). Then, we have
\[
\sup_{y \in D} \left\| T_{\mu_n} y - \int \frac{1}{N + 1} \sum_{i=0}^{N} T_{i,s} y \mu_n(s) \right\|
\leq \sup_{y \in D} \sup_{\|z\| = 1} \left| \langle T_{\mu_n} y, z \rangle \right|
\leq \left\| \frac{1}{N + 1} \sum_{i=0}^{N} T_{i,s} y \mu_n(s), z \right\|
\leq \frac{1}{N + 1} \sum_{i=0}^{N} \mu_n(s) \langle T_{i,s} y, z \rangle
\leq \frac{1}{N + 1} \sum_{i=0}^{N} \sup_{y \in D} \sup_{\|z\| = 1} \left| \langle (l_i^* \mu_n)_s (T_{i,s} y, z) \rangle \right|
\leq \frac{1}{N + 1} \sum_{i=0}^{N} \sup_{j=1,2,\ldots,N} \left| \langle (l_i^* \mu_n)_s (T_{j,s} y, z) \rangle \right|
\leq \max_{i=1,2,\ldots,N} \| l_i^* \mu_n \| (M_0 + \|p\|)
\leq \delta \quad (n \geq N_0).
\]

(3.14)
By Theorem 2.3 we have
\[
\frac{1}{N+1} \sum_{i=0}^{N} T_{\psi_i} y \mu_n(s) \in \mathbb{C} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{\psi_i} y : s \in S \right\}.
\] (3.15)

It follows from (3.12)-(3.15) that
\[
T_{\mu_n} y \in \mathbb{C} \left\{ \frac{1}{N+1} \sum_{i=0}^{N} T_{\psi_i} y : s \in S \right\} + B_\delta \\
\subseteq \mathbb{C} T\delta(T_i; D) + B_\delta \subseteq F_i(T_i; D),
\]
for all \( y \in D \) and \( n \geq N_0 \). Therefore,
\[
\limsup_{n \to \infty} \sup_{y \in D} \| T_i(T_{\mu_n} y) - T_{\mu_n} y \| \leq \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, we get (3.11).

Let \( t \in S \) and \( \epsilon > 0 \), then there exists \( \delta > 0 \), which satisfies (3.12). Take \( L_0 = (\gamma \alpha + \| A \|) M_0 + \| f(p) - A p \| \). Now from (3.11) and condition (iii) there exists \( N_1 \in \mathbb{N} \) such that \( T_{\mu_n} y \in F_\delta(T_i; D) \) for all \( y \in D \) and \( \epsilon_n < \frac{\delta}{2 M_0} \) for all \( n \geq N_1 \). We note that
\[
\epsilon_n \| f(z_n) - A T_{\mu_n} W_n S^K_n z_n \| \\
\leq \epsilon_n \left( \| f(z_n) - f(p) \| + \| f(p) - A p \| \right) \\
+ \| A p - A T_{\mu_n} W_n S^K_n z_n \| \\
\leq \epsilon_n \left( \gamma \alpha \| z_n - p \| \\
+ \| f(p) - A p \| \right) \\
\leq \epsilon_n \left( \gamma \alpha + \| A \| \right) M_0 + \| f(p) - A p \| \\
= \epsilon_n L_0 \leq \frac{\delta}{2},
\]
for all \( n \geq N_1 \). Observe that
\[
z_n = \epsilon_n f(z_n) + (1 - \epsilon_n \alpha) T_{\mu_n} W_n S^K_n z_n \\
= T_{\mu_n} W_n S^K_n z_n + \epsilon_n \left( f(z_n) - A T_{\mu_n} W_n S^K_n z_n \right) \\
\subseteq F_\delta(T_i; D) + B_\delta \\
= F_\delta(T_i; D) + B_\delta \\
\subseteq F_i(T_i; D),
\]
for all \( n \geq N_1 \). This show that
\[
\| z_n - T_i z_n \| \leq \epsilon \quad (n \geq N_1).
\]

Since \( \epsilon > 0 \) is arbitrary, we get \( \lim_{n \to \infty} \| z_n - T_i z_n \| = 0 \).

Step 6. The weak \( \omega \)-limit set of \( \{ z_n \} \) which is denoted by \( \omega \{ z_n \} \) is a subset of \( \mathfrak{F} \).

Proof. Let \( \hat{z} \in \omega \{ z_n \} \) and let \( \{ z_{n_j} \} \) be a subsequence of \( \{ z_n \} \) such that \( z_{n_j} \to \hat{z} \). We need to show that \( \hat{z} \in \mathfrak{F} \). In terms of Lemma 2.4 and Step 5, we conclude that \( \hat{z} \in \text{Fix}(S) \). By Theorems 2.2, 2.3, the mapping \( W : C \to C \), given by \( W x := \lim_{n \to \infty} W_n x \) satisfies
\[
\limsup_{n \to \infty} \| W_n \hat{z} - W \hat{z} \| = 0. 
\] (3.16)

Putting \( \lim r_{k,n} = \hat{r}_k \) for every \( k \in \{ 1, 2, \ldots, K \} \), by Theorem 2.5, we have
\[
S^K_{k,n} x = \lim_{n \to \infty} S^K_{r_{k,n}} x \quad (x \in H). 
\] (3.17)

Since \( \hat{z} \in \text{Fix}(S) \), by our assumption, we have \( T_i \hat{z} \in \text{Fix}(S) \) for all \( i \in \mathbb{N} \) and then \( W_n \hat{z} \in \text{Fix}(S) \). Hence, by (ii) of Theorem 2.3, \( T_{\mu_n} W_n \hat{z} = W_n \hat{z} \).

Consider the set of the asymptotic center \( A(z_{n_j}) \) of \( \{ z_{n_j} \} \) with respect to \( H \). Since \( z_{n_j} \to \hat{z} \), Lemma 2.4 implies that \( A(z_{n_j}) = \{ \hat{z} \} \). By the definition of \( A(z_{n_j}) \), we have
\[
\limsup_{j \to \infty} \| z_{n_j} - \hat{z} \| \leq \limsup_{j \to \infty} \| z_{n_j} - T_i z_{n_j} \| \\
\quad (t \in S),
\]
for all \( z \in A(z_{n_j}) \). Since \( A(z_{n_j}) = \{ \hat{z} \} \), by Step 5, we get \( z_{n_j} \to \hat{z} \). Using (3.16) and Step 4, we
have
\[ \limsup_{j \to \infty} \|z_{nj} - W \hat{z}\| \]
\[ \leq \limsup_{j \to \infty} \|z_{nj} - T_{\mu_{nj}} W_{nj} z_{nj}\| \]
\[ + \limsup_{j \to \infty} \|T_{\mu_{nj}} W_{nj} z_{nj} - T_{\mu_{nj}} W_{nj} \hat{z}\| \]
\[ + \limsup_{j \to \infty} \|T_{\mu_{nj}} W_{nj} \hat{z} - W \hat{z}\| \]
\[ \leq \limsup_{j \to \infty} \|z_{nj} - T_{\mu_{nj}} W_{nj} z_{nj}\| \]
\[ + \limsup_{j \to \infty} \|z_{nj} - \hat{z}\| \]
\[ + \limsup_{j \to \infty} \|W_{nj} \hat{z} - W \hat{z}\| \]
\[ \leq \limsup_{j \to \infty} \|z_{nj} - \hat{z}\| = 0. \] (3.18)

This implies that \( W(\hat{z}) = \hat{z} \).

Using Theorem 2.4 and (3.17) and Step 3, we have
\[ \limsup_{j \to \infty} \|z_{nj} - S^k_{r_k} \hat{z}\| \]
\[ \leq \limsup_{j \to \infty} \|z_{nj} - S^k_{r_k,nj} z_{nj}\| \]
\[ + \limsup_{j \to \infty} \|S^k_{r_k,nj} z_{nj} - S^k_{r_k,nj} \hat{z}\| \]
\[ + \limsup_{j \to \infty} \|S^k_{r_k,nj} \hat{z} - S^k_{r_k} \hat{z}\| \]
\[ \leq \limsup_{j \to \infty} \|z_{nj} - \hat{z}\| = 0. \] (3.18)

This implies that \( S^k_{r_k}(\hat{z}) = \hat{z} \) for every \( k \in \{1, 2, \ldots, K\} \).

Therefore, \( \hat{z} \in \text{Fix}(W) \cap (\bigcap_{k=1}^K \text{Fix}(S^k_{r_k})). \) In terms of Theorems 2.4 and 2.5, we conclude that \( \hat{z} \in (\bigcap_{k=1}^K \text{Fix}(T_i)) \cap \text{SEP}(\cdot). \) Since \( \hat{z} \in \text{Fix}(S), \) therefore, \( \hat{z} \in \mathfrak{F}. \)

Step 7. There exists a unique solution \( x^* \in \mathfrak{F} \) of the variational inequality (3.5), such that
\[ \Gamma := \limsup_n \langle (\gamma f - A)x^*, z_n - x^* \rangle \leq 0. \] (3.19)

Proof. Banach Contraction Mapping Principle guarantees that \( P_\mathfrak{F}(I - (A - \gamma f)) \) has a unique fixed point \( x^* \) which is, by Lemma 2.1, the unique solution of the variational inequality :
\[ \langle (\gamma f - A)x^*, x - x^* \rangle \leq 0 \quad (x \in \mathfrak{F}). \]

Note that, from the definition of \( \Gamma \) and the fact that \( z_n \) is a bounded sequence, we can select a subsequence \( z_{nj} \) of \( z_n \) with the following properties:
(i) \( \lim_j \langle (\gamma f - A)x^*, z_{nj} - x^* \rangle = \Gamma; \)
(ii) \( z_{nj} \) is weakly converge to a point \( \hat{z}; \)

by Step 6, we have \( \hat{z} \in \mathfrak{F} \) and then
\[ \Gamma = \lim_j \langle (\gamma f - A)x^*, z_{nj} - x^* \rangle = \langle (\gamma f - A)x^*, \hat{z} - x^* \rangle \leq 0, \]
as \( x^* \in \mathfrak{F} \) is the unique solution of (3.5).

Step 8. \( \{z_n\} \) strongly converges to \( x^*. \)

Proof. Indeed, from (3.6), (3.19) and that \( x^* \in \mathfrak{F}, \) we conclude
\[ \limsup_{n} \|z_n - x^*\|^2 \]
\[ \leq \frac{1}{\gamma - \alpha \gamma} \limsup_{n} \langle (\gamma f - A)x^*, z_n - x^* \rangle \leq 0. \]
That is \( z_n \to x^*. \)

**Theorem 3.2** Let \( H \) be a real Hilbert space, \( T \) be a nonexpansive mapping of \( C \) into itself such that \( \text{Fix}(T) \neq \emptyset, \) \( \{T_i\}_{i \in \mathbb{N}} \) be a sequence of nonexpansive mappings from \( C \) into itself such that \( T_i(\text{Fix}(T)) \subseteq \text{Fix}(T) \) for every \( i \in \mathbb{N}, \) and \( \varphi = (G_k : k = 1, 2, \ldots, K) \) be a finite family of bifunctions from \( H \times H \) into \( \mathbb{R}. \) Suppose that \( A \) is a strongly positive bounded linear operator with coefficient \( \gamma, \) and \( f \) be an \( \alpha \)-contraction on \( H. \) Moreover, let \( \{r_{k,n}\}, \{\epsilon_n\}, \{\lambda_n\} \) be real sequences such that \( r_{k,n} > 0, \) \( 0 < \epsilon_n < 1 \) and \( 0 < \lambda_n \leq b < 1, \) and \( \gamma \) is a real number such that \( 0 < \gamma \leq \frac{\gamma}{\alpha}. \) Assume that,

(i) for every \( k \in \{1, 2, \ldots, K\}, \) the function \( G_k \) satisfies (A1) - (A4) of Theorem 2.5 ,
(ii) \( \mathfrak{F} := \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \text{Fix}(T) \cap \text{SEP}(\varphi) \neq \emptyset, \)
(iii) \( \lim \epsilon_n = 0 \) and,
(iv) for every \( k \in \{1, 2, \ldots, K\}, \lim r_{k,n} \) exists and is a positive real number.
For every \( n \in \mathbb{N} \), let \( W_n \) be the mapping generated by \( \{ T_i \} \) and \( \{ \lambda_n \} \) as in (2.3), for every \( k \in \{ 1, 2, \cdot \cdot \cdot , K \} \) and \( n \in \mathbb{N} \), let \( S^k_{r_k,n} \) be the resolvent generated by \( G_k \) and \( r_{k,n} \) as in Theorem 2.5. If \( \{ z_n \} \) is the sequence generated by

\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) \frac{1}{n} \sum_{k=1}^{n} T^k W_n S^k_{r_k,n} z_n
\]

\((n \in \mathbb{N}).\)

Then \( \{ z_n \} \) strongly converges to \( x^* \in \mathcal{F} \), where \( x^* \) is the unique solution of the variational inequality

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathcal{F}).
\]

**Proof.** Let \( S = \{ 1, 2, \cdot \cdot \cdot , K \} = \{ T^i : i \in S \} \). For \( f = (z_1, z_2, \cdot \cdot \cdot ) \in B(S) \), define

\[
\mu_n(f) = \frac{1}{n} \sum_{k=1}^{n} z_k \quad (n \in \mathbb{N}).
\]

Then \( \{ \mu_n \} \) is a regular sequence of means on \( B(S) \); for more details, see [19]. Next for each \( x \in H \) and \( n \in \mathbb{N} \), we have

\[
T_{\mu_n} x = \frac{1}{n} \sum_{k=1}^{n} T^k x.
\]

Therefore, it follows from Theorem 3.1 that the sequence \( \{ z_n \} \) converges strongly to \( x^* \in \mathcal{F} \), which is the unique solution of the variational inequality:

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathcal{F}).
\]

**Theorem 3.3** Let \( H \) be a real Hilbert space, \( T \) be a nonexpansive mapping of \( C \) into itself such that \( \text{Fix}(T) \neq \emptyset \), \( \{ T_i \}_{i \in \mathbb{N}} \) be a sequence of nonexpansive mappings from \( C \) into itself such that \( T_i(\text{Fix}(T)) \subseteq \text{Fix}(T) \) for every \( i \in \mathbb{N} \), \( \varphi = \{ G_k : k = 1, 2, \cdot \cdot \cdot , K \} \) be a finite family of bifunctions from \( H \times H \) into \( \mathbb{R} \). Suppose that \( A \) is a strongly positive bounded linear operator with coefficient \( \gamma \), and \( f \) be an \( \alpha \)-contraction on

\( H. \) Moreover, let \( \{ r_k,n \}, \{ \epsilon_n \} \) and \( \{ \lambda_n \} \) be real sequences such that \( r_{k,n} > 0, 0 < \epsilon_n < 1 \) and \( 0 < \lambda_n \leq b < 1 \), and \( \gamma \) is a real number such that \( 0 < \gamma < \frac{1}{\alpha} \). Assume that,

(i) for every \( k \in \{ 1, 2, \cdot \cdot \cdot , K \} \), the function \( G_k \) satisfies (A1) – (A4) of Theorem 2.5,

(ii) \( \mathcal{F} := \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \text{Fix}(T) \cap \text{SEP}(\varphi) \neq \emptyset \),

(iii) \( \lim \epsilon_n = 0 \) and,

(iv) for every \( k \in \{ 1, 2, \cdot \cdot \cdot , K \} \), \( \lim_{n \to \infty} r_{k,n} \) exists and is a positive real number.

For every \( n \in \mathbb{N} \), let \( W_n \) be the mapping generated by \( \{ T_i \} \) and \( \{ \lambda_n \} \) as in (2.3), for every \( k \in \{ 1, 2, \cdot \cdot \cdot , K \} \) and \( n \in \mathbb{N} \), let \( S^k_{r_k,n} \) be the resolvent generated by \( G_k \) and \( r_{k,n} \) as in Theorem 2.5. If \( \{ z_n \} \) is the sequence generated by

\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) \frac{1}{n} \sum_{k=1}^{n} (a_n)^k T^k W_n S^k_{r_k,n} z_n
\]

\((n \in \mathbb{N}),\)

where \( \{ a_n \} \) is an increasing sequence in \( (0, 1) \) such that \( \lim a_n = 1 \). Then \( \{ z_n \} \) strongly converges to \( x^* \in \mathcal{F} \), where \( x^* \) is the unique solution of the variational inequality

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathcal{F}).
\]

**Proof.** Let \( S = \{ 1, 2, \cdot \cdot \cdot , K \}, \varphi = \{ T^i : i \in S \} \). For \( f = (z_1, z_2, \cdot \cdot \cdot ) \in B(S) \), define

\[
\mu_n(f) = \frac{1}{a_n} \sum_{k=1}^{\infty} (a_n)^k z_k \quad (n \in \mathbb{N}).
\]

Then \( \{ \mu_n \} \) is a regular sequence of means on \( B(S) \); for more details, see ([19], p. 79). Next for each \( x \in H \) and \( n \in \mathbb{N} \), we have

\[
T_{\mu_n} x = \frac{1 - a_n}{a_n} \sum_{k=1}^{\infty} (a_n)^k T^k x.
\]
Therefore, it follows from Theorem 3.1 that the sequence \( \{z_n\} \) converges strongly to \( x^* \in \mathfrak{F} \), which is the unique solution of the variational inequality:

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathfrak{F}).
\]

**Theorem 3.4** Let \( H \) be a real Hilbert space, and \( C \) be a nonempty closed convex subset of a Hilbert space \( H \), and \( S = R^+ = \{ t \in R : 0 \leq t < +\infty \} \), if \( \mathfrak{F} = \{ T : t \in \mathbb{R}^+ \} \), and if \( \mathfrak{G} = \{ \mathfrak{T}_i : t \in \mathbb{R}^+ \} \) be a representation of \( S \) as nonexpansive mappings of \( C \) into itself and suppose Fix(\( \mathfrak{G} \)) \( \neq \emptyset \). Let \( X \) be a left invariant subspace of \( B(\mathbb{R}^+) \) such that \( 1 \in X \) and the function \( t \mapsto \langle T(x), y \rangle \) is an element of \( X \) for each \( x \in C \), \( y \in H \), \( \{ T_i \}_{i \in \mathbb{N}} \) be a sequence of nonexpansive mappings from \( C \) into itself such that \( T_i(\text{Fix}(\mathfrak{G})) \subseteq \text{Fix}(\mathfrak{G}) \) for \( i \in \mathbb{N} \), \( \mathfrak{G} = \{ G_k : k = 1,2,\cdots,K \} \) be a finite family of bifunctions from \( H \times H \) into \( R \). Suppose that \( A \) is a strongly positive bounded linear operator with coefficient \( \tau \), and \( f \) is an \( \alpha \)-contraction on \( H \). Moreover, let \( \{ r_{k,n} \}, \{ \epsilon_n \} \) and \( \{ \lambda_n \} \) be real sequences such that \( r_{k,n} > 0, 0 < \epsilon_n < 1 \) and \( 0 < \lambda_n \leq b < 1 \), and \( \gamma \) is a real number such that \( 0 < \gamma < \frac{\alpha}{\gamma} \). Assume that,

(i) for every \( k \in \{1,2,\cdots,K\} \), the function \( G_k \) satisfies (A1) - (A4) of Theorem 2.5,

(ii) \( \mathfrak{F} := \bigcap_{n \in \mathbb{N}} \text{Fix}(T_n) \cap \text{Fix}(\mathfrak{G}) \cap \text{SEP}(\mathfrak{G}) \neq \emptyset \),

(iii) \( \lim_{n} \epsilon_n = 0 \) and,

(iv) for every \( k \in \{1,2,\cdots,K\} \), \( \lim_{n} r_{k,n} \) exists and is a positive real number.

For every \( n \in \mathbb{N} \), let \( W_n \) be the mapping generated by \( \{ T_i \} \) and \( \{ \lambda_n \} \) as in (2.3), for every \( k \in \{1,2,\cdots,K\} \) and \( n \in \mathbb{N} \), let \( S_k^n \) be the resolvent generated by \( G_k \) and \( r_{k,n} \) as in Theorem 2.5. If \( \{z_n\} \) is the sequence generated by

\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} z_n \quad (n \in \mathbb{N}),
\]

where \( \{a_n\} \) is an increasing sequence in \((0, \infty)\) such that \( \lim a_n = \infty \). Then \( \{z_n\} \) strongly converges to \( x^* \in \mathfrak{F} \), where \( x^* \) is the unique solution of the variational inequality

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathfrak{F}).
\]

**Proof.** For \( f \in B(\mathbb{R}^+) \), define

\[
\mu_n(f) = \frac{1}{a_n} \int_0^{a_n} f(t) t \quad (n \in \mathbb{N}).
\]

Then \( \mu_n \) is a regular sequence of means on \( B(\mathbb{R}^+) \); for more details, see ([19], p. 80). Next for each \( x \in H \) and \( n \in \mathbb{N} \), we have

\[
T_{\mu_n} x = \frac{1}{a_n} \int_0^{a_n} T_{\mu} x t \quad (n \in \mathbb{N}).
\]

Therefore, it follows from Theorem 3.1 that the sequence \( \{z_n\} \) converges strongly to \( x^* \in \mathfrak{F} \), which is the unique solution of the variational inequality:

\[
\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0 \quad (x \in \mathfrak{F}).
\]

**Theorem 3.5** Let \( \mathfrak{G} = \{ T_i : i \in S \} \) be a representation of \( S \) as nonexpansive mappings of \( H \) into itself such that \( \text{Fix}(\mathfrak{G}) \neq \emptyset \). Let \( X \) be a left invariant subspace of \( B(\mathbb{R}^+) \) such that \( 1 \in X \), and the function \( t \mapsto \langle T_{i}(x), y \rangle \) is an element of \( X \) for each \( x, y \in H \). Let \( \{ \mu_n \} \) be a left regular sequence of means on \( X \). Suppose that \( A \) is a strongly positive bounded linear operator with coefficient \( \tau \) and \( f \) is an \( \alpha \)-contraction on \( H \). Moreover, let \( \{ \epsilon_n \} \) and \( \{ \lambda_n \} \) be real sequences such that \( 0 < \epsilon_n < 1 \), \( \lim \epsilon_n = 0 \), \( 0 < \lambda_n \leq b < 1 \), and \( \gamma \) is a real number such that \( 0 < \gamma < \frac{\tau}{\alpha} \). If \( \{z_n\} \) is the sequence generated by

\[
z_n = \epsilon_n \gamma f(z_n) + (I - \epsilon_n A) T_{\mu_n} z_n \quad (n \in \mathbb{N}).
\]
Then \( \{z_n\} \) strongly converges to \( x^* \in \text{Fix}(\varrho) \).

Proof. Take \( G_k = 0 \) for every \( k \in \{1, 2, \cdots K\} \), \( T_i = I \) for every \( i \in \mathbb{N} \) and \( C = H \) in Theorem 3.1. Then we have \( S_{r_1,n}^1 S_{r_2,n}^2 \cdots S_{r_K,n}^K z_n = z_n \) and \( W_n = I \) for all \( n \in \mathbb{N} \). So from Theorem 3.1 the sequences \( \{z_n\} \) converges strongly to \( x^* \in \text{Fix}(\varrho) \).

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References


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