

$\mathcal{S}_S T$ decomposition method for solving fully fuzzy linear systems

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Abstract

The $\mathcal{S}_S T$ Decomposition Method for solving system of linear equations make it possible to obtain the values of roots of the system with the specified accuracy as the limit of the sequence of some vectors. In this topic we are going to consider vectors as fuzzy vectors. We have considered a numerical example and tried to find out solution vector \tilde{x} in fuzzified form using method of $\mathcal{S}_S T$ Decomposition.

Keywords : Triangular fuzzy numbers; System of Linear Equations; $\mathcal{S}_S T$ Decomposition Method.

1 Introduction

System of linear equations arise in a large number of areas, both directly in modeling physical situations and indirectly in the numerical solution of other Mathematical models. These applications occur in virtually all areas of the physical, biological and social sciences. In addition, linear systems are involved in the following: Optimization theory, solving system of non linear equations, the approximation of functions, the numerical solution of boundary value problems for ordinary differential equations, partial differential equations and integral equations, statistical inference and numerous other problems. Because of the widespread importance of linear systems, much research has been devoted to their numeri-

cal solution. In this paper we are going to discuss the $\mathcal{S}_S T$ Decomposition Method of the most common type $\tilde{A} \otimes \tilde{x} = \tilde{b}$ in fuzzified form.

Zadeh [13] introduced the concepts of fuzzy numbers and fuzzy arithmetic. Solving fuzzy $n \times n$ linear system whose coefficient matrix is crisp and the right hand side column is an arbitrary fuzzy number vector was introduced by Fridman et al [6].

Some authors Moslesh [][8] introduced LU decomposition method for solving fuzzy linear system. Muzzilo et al. [9] developed fully fuzzy linear system of the form $A_1 x + b_1 = A_2 x + b_2$ where A_1, A_2 are square matrices of fuzzy coefficients, b_1, b_2 are fuzzy numbers. Dehgan et al. [3] considered fully fuzzy linear system of the form $Ax + b$ where A is a fuzzy matrix, x is a fuzzy vector, and the constant b are vectors. Vijayalakshmi et al. [12] introduced the concept of solving fully fuzzy linear system for triangular fuzzy number matrices. Allahviranloo et al. [1] introduced the new method to obtain symmetric solutions of a fully fuzzy linear system based on a 1-cut expansion.

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Dookhitram et al. [4, 5] introduced the concept of a Preconditioning Algorithm for the Positive solution of fully fuzzy linear systems and later he introduced the another concept An Implicit partial pivoting Gauss Elimination Algorithm for linear system of equations with fuzzy parameters. A.Ghomashi et al. [7]

Anna Lee [2] introduce the concept of secondary symmetric matrix. In this paper we decomposed the coefficient matrix into secondary symmetric and triangular matrices for solving system of linear equations.

2 Basic Definitions

Definition 2.1 Let A be a classical set and $\mu_{\tilde{A}}(x)$ be a function from A to $[0, 1]$. A fuzzy set \tilde{A} with the membership function $\mu_{\tilde{A}}(x)$ is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A \text{ and } \mu_{\tilde{A}}(x) \in [0, 1]\}$

Definition 2.2 A fuzzy number \tilde{a} is a triangular fuzzy number denoted by (a_1, a_2, a_3) , where a_1, a_2 and a_3 are real numbers and its membership functions $\mu_{\tilde{A}}(x)$ is given below

$$\mu_{\tilde{A}}(x) = \begin{cases} (x - a_1)/(a_2 - a_1) & \text{for } a_1 \leq x \leq a_2 \\ (a_3 - x)/(a_3 - a_2) & \text{for } a_2 \leq x \leq a_3 \\ 0 & \text{Otherwise} \end{cases}$$

Definition 2.3 A fuzzy number \tilde{A} is called positive (negative), denoted by $\tilde{A} > 0$ ($\tilde{A} < 0$), if its membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0$ for all $x \leq 0$ (for all $x \geq 0$).

Using its mean value and left and right spreads, and shape functions, such a fuzzy number \tilde{A} is symbolically written $\tilde{A} = (a_1, a_2, a_3)$. Clearly, $\tilde{A} = (a_1, a_2, a_3)$ is positive, if and only if $a_1 - a_2 \geq 0$.

Remark 2.1 We consider $\tilde{0} = (0, 0, 0)$ as a zero triangular fuzzy number.

Remark 2.2 We show the set of all triangular fuzzy number by $F(R)$.

Definition 2.4 (Equality in fuzzy numbers) Two fuzzy numbers $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are said to be equal, if and only if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$.

Definition 2.5 Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two triangular numbers. Then

1. $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $\tilde{A} \ominus \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 + b_1)$
3. $k\tilde{A} = (ka_1, ka_2, ka_3)$, for $k \geq 0$
4. $k\tilde{A} = (ka_3, ka_2, ka_1)$, for $k < 0$
5. If $\tilde{A} > 0, \tilde{B} > 0$, then $\tilde{A} \otimes \tilde{B} = (a_1b_1, a_1b_2 + b_1a_2, a_1b_3 + b_1a_3)$

Definition 2.6 If $\tilde{A} = (a_1, a_2, a_3) > 0, \tilde{B} = (b_1, b_2, b_3) > 0$, then

$$\tilde{A} \otimes \tilde{B} = (a_1b_1, a_1b_2 + b_1a_2, a_1b_3 + b_1a_3) \quad (2.1)$$

Definition 2.7 A matrix $\tilde{A} = (\tilde{a}_{ij})$ is called a fuzzy matrix, if each element of \tilde{A} is a fuzzy number. A fuzzy matrix \tilde{A} will be positive and denoted by $\tilde{A} > 0$, if each element of \tilde{A} be positive. We may represent $n \times n$ fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$, such that $\tilde{a}_{ij} = (a_{ij}, m_{ij}, n_{ij})$, with the new notation $\tilde{A} = (A, M, N)$, where $A = (a_{ij}), M = (m_{ij})$ and $N = (n_{ij})$ are three $n \times n$ crisp matrices.

Definition 2.8 A square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ will be an upper triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0, 0, 0)$, for all $i > j$, and square fuzzy matrix $\tilde{A} = (\tilde{a}_{ij})$ will be a lower triangular fuzzy matrix, if $\tilde{a}_{ij} = \tilde{0} = (0, 0, 0)$, for all $i < j$.

Definition 2.9 (Secondary Transpose) The secondary transpose of $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ is $A^s = (b_{ij})$, where $b_{ij} = a_{n-j+1, n-i+1}$ (Simply $A^s = VA^TV$, where V is the fixed disjoint permutation matrix with units in its secondary diagonal and A^T stands for transpose of A).

Definition 2.10 (Secondary Symmetric Matrix) Let $A \in \mathcal{M}_n(\mathbb{C})$ is called Secondary Symmetric if $A^s = A$.

Definition 2.11 Consider the $n \times n$ fuzzy linear system of equations

$$\begin{aligned} (\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) &= \tilde{b}_1 \\ (\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) &= \tilde{b}_2 \\ \dots & \dots \\ (\tilde{a}_{n1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{n2} \otimes \tilde{x}_2) \oplus \dots \oplus (\tilde{a}_{nn} \otimes \tilde{x}_n) &= \tilde{b}_n \end{aligned}$$

The matrix form of the above equation is $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where the coefficient matrix $\tilde{A} = (\tilde{a}_{ij})$, $1 \leq i, j \leq n$ is a $n \times n$ fuzzy matrix and $\tilde{x}_j, \tilde{b}_j \in F(R)$. This system is called a fully fuzzy linear system.

In this paper, we are going to obtain a positive (and negative) solution of a fully fuzzy linear system $\tilde{A} \otimes \tilde{x} = \tilde{b}$, where $\tilde{A} = (A, M, N) > 0$ ($\tilde{A} = (A, M, N) < 0$), $\tilde{b} = (b, g, h) > 0$ ($\tilde{b} = (b, g, h) < 0$) and $\tilde{x} = (x, y, z) > 0$ ($\tilde{x} = (x, y, z) < 0$). So we have $(A, M, N) \otimes (x, y, z) = (b, h, g)$. Then by using Equation 2.1 we have $(Ax, Ay + Mx, Az + Nx) = (b, h, g)$.

3 $\mathcal{S}_S T$ Decomposition Method

In this method, for solving the crisp linear system of equations $Ax = b$ is reduced to diagonal matrix by Gauss elimination method and applying the back substitution to get the corresponding unknown values in the form of triangular fuzzy numbers. Given any fuzzy linear system of equations in the form of triangular fuzzy matrices that can be decomposed into the form such that $A = \mathcal{S}_S T$ decomposition method whereas \mathcal{S}_S is the secondary symmetric matrix and T is the triangular matrix. An algorithm has been introduced to rewritten A as the product of secondary symmetric matrix and triangular matrix.

3.1 Algorithm

Step.1 Consider the non singular triangular fuzzy number matrices A.

$$\text{Set } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Step.2 Decomposition the matrix A, $A = \mathcal{S}_S T$, where \mathcal{S}_S is secondary symmetric matrix and T is triangular matrix. $\mathcal{S}_S = \begin{pmatrix} a_{11} & s_{12} & s_{13} \\ a_{21} & s_{22} & s_{12} \\ a_{31} & a_{21} & a_{11} \end{pmatrix}$ and

$$T = \begin{pmatrix} 1 & t_{12} & t_{13} \\ 0 & 1 & t_{23} \\ 0 & 0 & 1 \end{pmatrix}, \text{ where}$$

$$\begin{aligned} t_{12} &= \frac{a_{32} - a_{21}}{a_{31}}, \\ s_{12} &= \frac{a_{12}a_{31} - a_{11}a_{32} + a_{11}a_{21}}{a_{31}}, \\ s_{22} &= \frac{a_{31}a_{22} - a_{21}a_{32} + a_{21}^2}{a_{31}}, \\ t_{23} &= \frac{a_{23}a_{31} - a_{21}a_{33} - a_{12}a_{31} + a_{11}a_{32}}{a_{31}s_{22} - a_{21}^2}, \\ t_{13} &= \frac{a_{33} - a_{21}t_{23} - a_{11}}{a_{31}}, \\ s_{13} &= a_{13} - a_{11}t_{13} - s_{12}t_{23}. \end{aligned}$$

Step.3 On solving $\tilde{A} \otimes \tilde{x} = \tilde{b}$, we have

$$\begin{aligned} (A, M, N) \otimes (x, y, z) &= (b, h, g) \\ \Rightarrow (Ax, Ay + Mx, Az + Nx) &= (b, h, g) \\ Ax = b &\Rightarrow x = A^{-1}b \\ Ay + Mx = h &\Rightarrow y = A^{-1}(h - Mx) \\ Az + Nx = g &\Rightarrow z = A^{-1}(g - Nx) \end{aligned}$$

Step.4 Replace $A = \mathcal{S}_S T$ we have

$$\begin{aligned} x &= T^{-1}(\mathcal{S}_S)^{-1}b \\ y &= T^{-1}(\mathcal{S}_S)^{-1}(h - Mx) \\ z &= T^{-1}(\mathcal{S}_S)^{-1}(g - Nx) \end{aligned} \tag{3.2}$$

using this formula we are finding the solution of x, y and z using $\mathcal{S}_S T$ decomposition.

4 Numerical Examples

Example 4.1 A person loves steaks, potatoes and milk products. Therefore he has decided to go on a steady diet of only these three foods for all his meals. He realizes that this is not the healthiest diet, so he wants to make sure that he eats the right quantities of the three foods to satisfy some key nutritional requirements. He has obtained the following nutritional and cost information. Now, to determine the number of daily servings of steak, potatoes and milk products that will meet the above requirements.

To represent the above problem as linear system, we represent x as a quantity of steak that will be

Ingredient	Grams of Ingredient per serving			Daily Requirement(Grams)
	Streak	Potatoes	Milk Products	
Carbohydrates	(1,2,5)	(3,4,4)	(0,1,2)	(19,68,115)
Protein	(2,3,5)	(0,1,11)	(4,5,6)	(30,77,261)
Fat	(2,5,7)	(4,6,6)	(5,7,10)	(61,167,253)

incurred daily. Similarly, y and z represent the quantity of potatoes and milk product incurred daily respectively.

$$\begin{pmatrix} (1, 2, 5) & (3, 4, 4) & (0, 1, 2) \\ (2, 3, 5) & (0, 1, 11) & (4, 5, 6) \\ (2, 5, 7) & (4, 6, 6) & (5, 7, 10) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (19, 68, 115) \\ (30, 77, 261) \\ (61, 167, 253) \end{pmatrix}$$

First we obtain the decomposition for matrix A as follows

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \mathcal{S}_S T = \begin{pmatrix} 1 & 2 & \frac{-5}{2} \\ 2 & -2 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore Equation 3.2 concludes that

$$\begin{aligned} \tilde{x} &= (1, 6, 7) \\ \tilde{y} &= (2, 12, 10) \\ \tilde{z} &= (7, 14, 12) \end{aligned}$$

Example 4.2 *The suna manufacturing company has decided to produce three products namely Product 1, Product 2, and Product 3. The available capacity of the machines that might limit output is summarized below. The number of machine hours required for each unit of the respective product is given below. Now, to determine how much of each product should produce to utilize the entire available time.*

To represent the above problem as fully fuzzy linear system, we represent \tilde{x} as a quantity of

the Product 1 that will be produced during the month. Similarly, \tilde{y} and \tilde{z} represent the quantity of Product 2 and 3 respectively.

The corresponding fully fuzzy linear system for the above problem as

$$\begin{pmatrix} (19, 1, 1) & (12, 1.5, 1.5) & (6, 0.5, 0.2) \\ (2, 0.1, 0.1) & (4, 0.1, 0.4) & (1.5, 0.2, 0.2) \\ (2, 0.1, 0.2) & (2, 0.1, 0.3) & (4.5, 0.1, 0.1) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} (1897, 427.7, 536.2) \\ (434.5, 76.2, 109.3) \\ (535.5, 88.3, 131.9) \end{pmatrix}$$

Now, we solve the above system by using the Algorithm 3.1 and Therefore Equation 3.2 concludes that

$$\begin{aligned} \tilde{x} &= (37, 7, 13.5) \\ \tilde{y} &= (62, 5.5, 4.5) \\ \tilde{z} &= (75, 10.2, 14) \end{aligned}$$

5 Conclusion

In this paper, a new methodology is applied to find the solution of fully fuzzy linear system in the form of triangular fuzzy matrices. We obtain both positive and negative solutions of fully fuzzy linear system. An algorithm is introduced to solve triangular fuzzy matrices by $\mathcal{S}_S T$ decomposition method. We have discussed by taking an example of order 3 to solve fully fuzzy linear system of triangular fuzzy matrices using $\mathcal{S}_S T$ decomposition method.

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Machine type	Available time (Machine hours per month)
Milling Machine	(1897, 427.7, 536.2)
Lathe	(434.5, 76.2, 109.3)
Grinder	(535.5, 88.3, 131.9)

Product coefficient (in machine hours per unit)

Machine Type	Product 1	Product 2	Product 3
Milling Machine	(19, 1, 1)	(12, 1.5, 1.5)	(6, 0.5, 0.2)
Lathe	(2, 0.1, 0.1)	(4, 0.1, 0.4)	(1.5, 0.2, 0.2)
Grinder	(2, 0.1, 0.2)	(2, 0.1, 0.3)	(4.5, 0.1, 0.1)

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