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An application of fuzzy logic in measuring a systems effectiveness

M. Gr. Voskoglou [*] [†]

Abstract

In the present paper we use principles of fuzzy logic to develop a general model representing several processes in a systems operation characterized by a degree of vagueness and/or uncertainty. We also introduce three alternative measures of a fuzzy systems effectiveness connected to our general model. These measures include the systems total possibilistic uncertainty, the Shannons entropy properly modified for use in a fuzzy environment and the centroid method in which the coordinates of the center of mass of the graph of the membership function involved provide an alternative measure of the systems performance. The advantages and disadvantages of the above measures are discussed and a combined use of them is suggested for achieving a worthy of credit mathematical analysis of the corresponding situation. Finally, an application of is presented for the Problem Solving process illustrating the use of our results in practice.

Keywords : Systems theory; Fuzzy sets and logic; Possibility; Uncertainty theory; Problem solving.

1 Introduction: Systems' modelling and fuzzy logic

A system is a set of interacting or interdependent components forming an integrated whole. A system comprises multiple views such as planning, analysis, design, implementation, deployment, structure, behavior, input and output data, etc. As an interdisciplinary and multi- perspective domain systems theory brings together principles and concepts from ontology, philosophy of science, information and computer science, mathematics, as well as physics, biology, engineering, social and cognitive sciences, management and economics, strategic thinking, fuzziness and uncertainty, etc. Thus, it serves as a bridge for an interdisciplinary dialogue between autonomous areas of study. The emphasis with systems theory shifts from parts to the organization of parts, recognizing that interactions of the parts are not static and constant, but dynamic processes.

Most systems share common characteristics including structure, behavior, interconnectivity (the various parts of a system have functional and structural relations to each other), sets of functions, etc. We scope a system by defining its boundary; this means choosing which entities are inside the system and which are outside of it, part of the environment.

The systems modelling is a basic principle in engineering, in natural and in social sciences. When we face a problem concerning a systems operation (e.g. maximizing the productivity of an industry, minimizing the functional costs of a company, etc) a model is required to describe and represent the systems multiple views. The model is a simplified representation of the basic characteristics of the real system including only its entities and features under concern. In this sense,

*Corresponding author. voskoglou@teipat.gr
†School of Technological Applications, Graduate Technological Educational Institute of Patras, Greece.

no model of a complex system could include all features and/or all entities belonging to the system. In fact, in this way the models structure could become very complicated and therefore its use in practice could be very difficult and sometimes impossible. Therefore the construction of the model usually involves a deep abstracting process on identifying the systems dominant variables and the relationships governing them. The resulting structure of this action is known as the assumed real system (see Figure 1). The model, being an abstraction of the assumed real system, identifies and simplifies the relationships among these variables in a form amenable to analysis. A

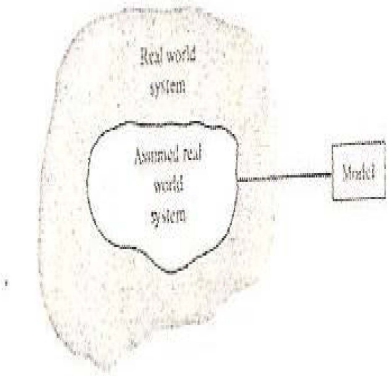

Figure 1: A graphical representation of the modelling process.

system can be viewed as a bounded transformation, i.e. as a process or a collection of processes that transforms inputs into outputs with the very broad meaning of the concept. For example, an output of a passengers bus is the movement of people from departure to destination.

Many of these processes are frequently characterized by a degree of vagueness and/or uncertainty. For example, during the processes of learning, of reasoning, of problem-solving, of modelling, etc, the human cognition utilizes in general concepts that are inherently graded and therefore fuzzy. On the other hand, from the teachers point of view there usually exists an uncertainty about the degree of students success in each of the stages of the corresponding didactic situation.

There used to be a tradition in science and engineering of turning to probability theory when one is faced with a problem in which uncertainty plays a significant role. This transition was justified when there were no alternative tools for dealing with the uncertainty. Today this is no longer the case. Fuzzy logic, which is based on fuzzy sets theory introduced by Zadeh [28] in 1965, provides a rich and meaningful addition to standard logic. The applications which may be generated from or adapted to fuzzy logic are wide-ranging and provide the opportunity for modelling under conditions which are inherently imprecisely defined, despite the concerns of classical logicians. Many systems may be modelled, simulated and even replicated with the help of fuzzy logic, not the least of which is human reasoning itself (e.g. [21], [24], [26], [27], etc). A real test of the effectiveness of an approach to uncertainty is the capability to solve problems which involve different facets of uncertainty. Fuzzy logic has a much higher problem solving capability than the standard probability theory. Most importantly, it opens the door to construction of mathematical solutions of computational problems which are stated in a natural language. In contrast, standard probability theory does not have this capability, a fact which is one of its principal limitations.

All these gave us the impulsion to introduce principles of fuzzy logic to describe in a more effective way a systems operation in situations characterized by a degree of vagueness and/or uncertainty. For general facts on fuzzy sets and logic and on uncertainty theory we refer freely to the book of Klir and Folger [4].

2 The general fuzzy model

$$U = \{a, b, c, d, e\}.$$

We are going to attach to each stage S_i a fuzzy subset, A_i of U. For this, if $n_{ia}, n_{ib}, n_{ic}, n_{id}$ and n_{ie} denote the number of entities that faced very low, low, intermediate, high and very high success at stage S_i respectively, $i = 1, 2, 3$, we define the membership function m_{Ai} for each x in U, as follows:

$$m_A(x) = \begin{cases} 1, & \frac{4n}{5} < n_{ix} \leq n, \\ 0.75, & \frac{3n}{5} < n_{ix} \leq \frac{4n}{5}, \\ 0.5, & \frac{2n}{5} < n_{ix} \leq \frac{3n}{5}, \\ 0.25, & \frac{n}{5} < n_{ix} \leq \frac{2n}{5}, \\ 0, & 0 < n_{ix} \leq \frac{n}{5}. \end{cases}$$

Then the fuzzy subset A_i of U corresponding to S_i has the form: In order to represent all possible profiles (overall states) of the systems entities during the corresponding process we consider a fuzzy relation, say R, in U^3 of the form: We assume that the stages of the process that we study are depended to each other. This means that the degree of systems success in a certain stage depends upon the degree of its success in the previous stages, as it usually happens in practice. Under this hypothesis and in order to determine properly the membership function mR we give the following definition:

Definition 2.1 *A profile $s = (x, y, z)$, with $x, y, z \in U$, is said to be well ordered if x corresponds to a degree of success equal or greater than y and y corresponds to a degree of success equal or greater than z.*

For example, (c, c, a) is a well ordered profile, while (b, a, c) is not.
We define now the membership degree of a profile s to be if s is well ordered, and 0 otherwise.
In fact, if for example the profile (b, a, c) possessed a nonzero membership degree, how it could be possible for an object that has failed during the middle stage, to perform satisfactorily at the next stage?
Next, for reasons of brevity, we shall writes m_s instead of $m_R(s)$. Then the probability p_s of the profile s is defined in a way analogous to crisp data, i.e. by

$$P_s = \frac{m_s}{\sum_{s \in U^3} m_s}.$$

We define also the possibility r_s of s by

$$r_s = \frac{m_s}{max\{m_s\}},$$

where $max\{m_s\}$ denotes the maximal value of m_s , for all s in U^3. In other words the possibility of s expresses the "relative membership degree" of s with respect to $max\{m_s\}$.
Assume further that one wants to study the combined results of behaviour of k different groups of a system's entities, $k \geq 2$, during the same process.
For this we introduce the fuzzy variables $A_1(t), A_2(t)$ and $A_3(t)$ with $t = 1, 2, \cdots, k$. The values of these variables represent fuzzy subsets of U corresponding to the stages of the process for each of the k groups; e.g. $A_1(2)$ represents the fuzzy subset of U corresponding to the first stage of the process for the second group ($t = 2$). It becomes evident that, in order to measure the degree of evidence of combined results of the k groups, it is necessary to define the probability $p(s)$ and the possibility $r(s)$ of each profile s with respect to the membership degrees of s for all groups. For this reason we introduce the pseudo-frequencies

$$f(s) = \sum_{t=1}^{k} m_s(t),$$

and we define the probability of a profile s by

$$p(s) = \frac{f(s)}{\sum_{s \in U^3} f(s)}.$$

We also define the possibility of s by

$$r_s = \frac{f(s)}{max\{f(s)\}},$$

where $max\{f(s)\}$ denotes the maximal pseudo-frequency.
Obviously the same method could be applied when one wants to study the combined results of behaviour of a group during k different situations.

3 Fuzzy measures of a system's effectiveness

There are natural and human-designed systems. Natural systems may not have an apparent objective, but their outputs can be interpreted as purposes. On the contrary, human-designed systems are made with purposes that are achieved by the delivery of outputs. Their parts must be related, i.e. they must be designed to work as a coherent entity.
The most important part of a human-designed system's study is probably the assessment, through the model representing it, of its performance. In fact, this could help the system's designer to make all the necessary modifications/improvements to the system's structure in order to increase its effectiveness.
In this paper we'll present three fuzzy measures of a systems effectiveness connected to the general fuzzy model developed above. The advantages and disadvantages of these measures will be

also discussed and an application for the problem solving process will be presented illustrating our results.

The amount of information obtained by an action can be measured by the reduction of uncertainty resulting from this action. Accordingly a systems uncertainty is connected to its capacity in obtaining relevant information. Therefore a measure of uncertainty could be adopted as a measure of a system's effectiveness in solving related problems. Within the domain of possibility theory uncertainty consists of strife (or discord), which expresses conflicts among the various sets of alternatives, and non-specificity (or imprecision), which indicates that some alternatives are left unspecified, i.e. it expresses conflicts among the sizes (cardinalities) of the various sets of alternatives ([5]; p.28). Strife is measured by the function $ST(r)$ on the ordered possibility distribution of a group of a system's entities defined by while non-specificity is measured by the function

$$N(r) = \frac{1}{log2}\left[\sum_{i=2}^{n}(r_i - r_{i+1})logi\right],$$

The sum $T(r) = ST(r)+N(r)$ is a measure of the total possibilistic uncertainty for ordered possibility distributions. The lower is the value of $T(r)$, which means greater reduction of the initially existing uncertainty, the better the system's performance.

Another fuzzy measure for assessing a systems performance is the well known from classical probability and information theory Shannon's entropy [12]. For use in a fuzzy environment, this measure is expressed in terms of the Dempster-Shafer mathematical theory of evidence in the form:

$$H = -\frac{1}{lnn}\sum_{s=1}^{n} m_s \; lnm_s,$$

([5], p. 20).

In the above formula n denotes the total number of the system's entities involved in the corresponding process. The sum is divided by lnn (the natural logarithm of n) in order to be normalized. Thus H takes values in the real interval $[0, 1]$. The value of H measures the system's total probabilistic uncertainty and the associated to it information. Similarly with the total possibilistic uncertainty, the lower is the final value of H, the better the system's performance.

An advantage of adopting H as a measure instead of $T(r)$ is that H is calculated directly from the membership degrees of all profiles s without being necessary to calculate their probabilities p_s. In contrast, the calculation of $T(r)$ presupposes the calculation of the possibilities r_s of all profiles first. However, according to Shackle [11] human reasoning can be formalized more adequately by possibility rather, than by probability theory. But, as we have seen in the previous section, the possibility is a kind of "elative probability". In other words, the "hilosophy" of possibility is not exactly the same with that of probability theory. Therefore, on comparing the effectiveness of two or more systems by these two measures, one may find non compatible results in boundary cases, where the systems' performances are almost the same.

Another popular approach is the "centroid" method, in which the centre of mass of the graph of the membership function involved provides an alternative measure of the systems performance. For this, given a fuzzy subset

$$A = \{(x, m(x)) : x \in U\},$$

of the universal set U with membership function $m : U \to [0, 1]$, we correspond to each $x \in U$ an interval of values from a prefixed numerical distribution, which actually means that we replace U with a set of real intervals. Then, we construct the graph F of the membership function $y = m(x)$.

There is a commonly used in fuzzy logic approach to measure performance with the pair of numbers (x_c, y_c) as the coordinates of the centre of mass, say F_c, of the graph F, which we can calculate using the following well-known [18] formulas:

$$x_c = \frac{\int\int_F xdxdy}{\int\int_F dxdy}, \quad y_c = \frac{\int\int_F ydxdy}{\int\int_F dxdy}, \qquad (3.1)$$

It is easy to check that, if the bar graph consists of n rectangles (in Figure 2 we have $n = 5$), the formulas (3.1) can be reduced to the following formulas:

$$x_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n}(2i-1)y_i}{\sum_{i=1}^{n} y_i}\right), \; y_c = \frac{1}{2}\left(\frac{\sum_{i=1}^{n} y_i^2}{\sum_{i=1}^{n} y_i}\right), \qquad (3.2)$$

From the above argument, where $F_i, i = 1, 2, \cdots, n$, denote the n rectangles of the bar

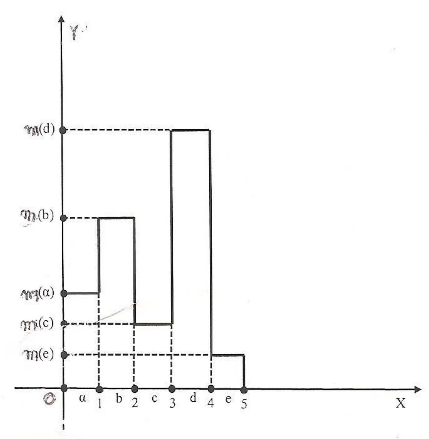

Figure 2: Bar graphical data representation

graph, it becomes evident that the transition from (3.1) to (3.2) is obtained under the assumption that all the intervals have length equal to 1 and that the first of them is the interval $[0, 1]$.
In our case ($n = 5$) formulas (3.2) are transformed into the following form:

$$x_c = \frac{1}{2}\left(\frac{y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5}{y_1 + y_2 + y_3 + y_4 + y_5}\right),$$

$$y_c = \frac{1}{2}\left(\frac{y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2}{y_1 + y_2 + y_3 + y_4 + y_5}\right).$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 1.$$

Therefore we can write:

$$x_c = \frac{1}{2}\left(y_1 + 3y_2 + 5y_3 + 7y_4 + 9y_5\right), \qquad (3.3)$$

$$y_c = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2\right).$$

with $y_i = \frac{m(x_i)}{\sum_{x \in U} m(x)}$, where $x_1 = a, x_2 = b, x_3 = c, x_4 = d$ and $x_5 = e$. But

$$0 \leq (y_1 - y_2)^2 = y_1^2 + y_2^2 - 2y_1y_2,$$

therefore

$$y_1^2 + y_2^2 \geq 2y_1y_2,$$

with the equality holding if, and only if, $y_1 = y_2$.
In the same way one finds that

$$y_1^2 + y_3^2 \geq 2y_1y_3,$$

and so on. Hence it is easy to check that

$$(y_1+y_2+y_3+y_4+y_5)^2 \leq 5(y_1^2+y_2^2+y_3^2+y_4^2+y_5^2),$$

with the equality holding if, and only if $y_1 = y_2 = y_3 = y_4 = y_5$. But $y_1 + y_2 + y_3 + y_4 + y_5 = 1$, therefore

$$1 \leq 5(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2) \qquad (3.4)$$

with the equality holding if, and only if $y_1 = y_2 = y_3 = y_4 = y_5 = \frac{1}{5}$.
Then the first of formulas (3.3) gives that $x_c = \frac{5}{2}$.
Further, combining the inequality (3.4) with the second of formulas (3.3) one finds that Therefore the unique minimum for y_c corresponds to the centre of mass $F_m(\frac{5}{2}, \frac{1}{10})$.
The ideal case is when $y_1 = y_2 = y_3 = y_4 = 0$ and $y_5 = 1$. Then from formulas (3.3) we get that $x_c = \frac{9}{2}$ and $y_c = \frac{1}{2}$. Therefore the centre of mass in this case is the point $F_i(\frac{9}{2}, \frac{1}{2})$.
On the other hand the worst case is when $y_1 = 1$ and $y_2 = y_3 = y_4 = y_5 = 0$. Then for formulas (3.3) we find that the centre of mass is the point $F_w(\frac{1}{2}, \frac{1}{2})$.
Therefore the "area" where the centre of mass F_c lies is represented by the triangle F_w F_m F_i of Figure 3. Then from elementary geometric considerations it follows that for two groups of a system's objects with the same $x_c \geq 2.5$ the group having the centre of mass which is situated closer to F_i is the group with the higher y_c; and for two groups with the same $x_c < 2.5$ the group having the centre of mass which is situated farther to F_w is the group with the lower y_c.
Based on the above considerations it is logical to

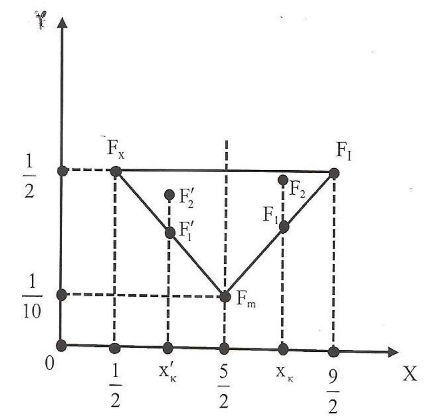

Figure 3: Graphical representation of the "area" of the centre of mass

formulate our criterion for comparing the groups performances in the following form:

- Among two or more groups the group with the biggest x_c performs better.
- If two or more groups have the same $x_c \geq 2.5$, then the group with the higher y_c performs better.
- If two or more groups have the same $x_c < 2.5$, then the group with the lower y_c performs better.

From the above description it becomes clear that the application of the 'centroid' method in practice is simple and evident and needs no complicated calculations in its final step. However, we must emphasize that this method treats differently the idea of a system's performance, than the two measures of uncertainty presented above do. In fact, the weighted average plays the main role in this method, i.e. the result of the system's performance close to its ideal performance has much more weight than the one close to the lower end. In other words, while the measures of uncertainty are dealing with the average systems performance, the 'centroid' method is mostly looking at the quality of the performance. Consequently, some differences could appear in evaluating a systems performance by these different approaches. Therefore, it is argued that a combined use of all these (3 in total) measures could help the user in finding the ideal profile of the system's performance according to his/her personal criteria of goals.

4 Modelling the process of Problem Solving (PS)

In earlier papers we have developed models similar to the general fuzzy model developed above for a more effective description of several situations involving fuzziness and uncertainty in the areas of Education (for the processes of Learning and of Mathematical modelling), of Artificial Intelligence (for Case-Based and Analogical Reasoning) and of Management (for the evaluation of the fuzzy data obtained by a markets research and for Decision Making); see for example [26] and its references. Notice also, that Subbotin et al., based on our fuzzy model for the process of learning [21], have applied the 'centroid' method on comparing students' mathematical learning abilities [16] and for measuring the scaffolding (assistance) effectiveness provided by the teacher to students [17]. Also Perdikaris has used the total possibilistic uncertainty [8] and the Shannon's entropy [9] for assessing students geometrical reasoning skills in terms of the corresponding van Hieles' levels.

In this paper we shall apply our general fuzzy model developed above for representing the Problem Solving (PS) process.

As the world economy moved from an industrial to a knowledge economy, it can be argued that the nature of many problems also changed and new problems have arisen which may require a different approach to overcome them. Educational institutions and governments have recognized long ago the importance of PS and volumes of research have been written about PS (see [3], [7], etc). Universities and other higher learning institutions are entrusted with the task of producing graduates that have such higher order thinking skills among other skills (e.g. see [1], etc).

Mathematics by its nature is a subject whereby PS forms its essence. According to Schoenfeld [14] a problem is only a problem (as mathematicians use the word) if you don't know how to go about solving it. A problem that has no 'surprises' in store, and can be solved comfortably by routine or familiar procedures (no matter how difficult!) it is an exercise. In an earlier paper [25][25] we have examined the role of problem in learning mathematics and we have attempted a review of the evolution of research on PS in mathematics education from its emergency as a self sufficient science at the end of the 1960's until today. Here is a rough chronology of that progress:

1950's 1960's: Polya's theories on the use of heuristic strategies in PS ([10], etc)

1970's: Emergency of mathematics education as a self sufficient science (research methods were almost exclusively statistical). Research on PS was mainly based on Polya's ideas.

1980's: A framework describing the PS process, and reasons for success or failure in PS (e.g. [6], [13], etc.)

1990's: Models of teaching using PS, e.g. constructivist view of learning (see [23] and its relevant references), Mathematical modelling and applications (see [22] and its references), etc.

2000's: While early work on PS focused mainly

Table 1: Profiles with non zero membership degrees (The outcomes of the above Table were obtained with accuracy up to the third decimal point).

A_1	A_2	A_3	$M_s(1)$	$r_s(1)$	$M_s(2)$	$r_s(2)$	$f(s)$	$r(s)$
b	b	b	0	0	0.016	0.258	0.016	0.129
b	b	a	0	0	0.016	0.258	0.016	0.129
b	a	a	0	0	0.016	0.258	0.016	0.129
c	c	c	0.062	1	0.062	1	0.124	1
c	c	a	0.062	1	0.062	1	0.124	1
c	c	b	0	0	0.031	0.5	0.031	0.25
c	a	a	0	0	0.031	0.5	0.031	0.25
c	b	a	0	0	0.031	0.5	0.031	0.25
c	b	b	0	0	0.031	0.5	0.031	0.25
d	d	a	0.016	0.258	0	0	0.016	0.129
d	d	b	0.016	0.258	0	0	0.016	0.129
d	d	c	0.016	0.258	0	0	0.016	0.129
d	a	a	0	0	0.016	0.258	0.016	0.129
d	b	a	0	0	0.016	0.258	0.016	0.129
d	b	b	0	0	0.016	0.258	0.016	0.129
d	c	a	0.031	0.5	0.031	0.5	0.062	0.5
d	c	b	0.031	0.5	0.031	0.5	0.062	0.5
d	c	c	0.031	0.5	0.031	0.5	0.062	0.5
e	c	a	0.031	0.5	0	0	0.031	0.25
e	c	b	0.031	0.5	0	0	0.031	0.25
e	c	c	0.031	0.5	0	0	0.031	0.25
e	d	a	0.016	0.25	0	0	0.016	0.129
e	d	b	0.016	0.25	0	0	0.016	0.129
e	d	c	0.016	0.25	0	0	0.016	0.129

on analyzing the PS process and on describing the proper heuristic strategies to be used in each of its stages, more recent investigations have focused mainly on solvers' behavior and required attributes during the PS process; e. g. [2], [15], etc.

Carlson Bloom [2] drawing from the large amount of literature related to PS developed a broad taxonomy to characterize major PS attributes that have been identifying as relevant to PS success. This taxonomy gave genesis to their 'Multidimensional Problem-Solving Framework' (MPSF), which includes the following 4 phases: Orientation, Planning, Executing and Checking. It has been observed that once the solvers oriented themselves to the problem space, the plan-execute-check cycle was usually repeated through out the remainder of the solution process; only in a few cases a solver obtained linearly the solution of a problem (i.e. he/she made this cycle only once). Thus embedded in the framework are two cycles (one cycling back and one cycling forward), each of which includes the three out of the four phases, that is planning, executing and checking. It has been also observed that, when contemplating various solution approaches during the planning phase of the PS process, the solvers were at times engaged in a conjecture-imagine-evaluate (accept/reject) sub-cycle. Therefore, apart of the two main cycles, embedded in the framework is the above sub-cycle, which is connected to the phase of planning (see Figure 4, taken from [2]).

In order to illustrate the use of our results in practice, we performed the experiments presented in the next section.

5 Applications of the model for PS

The following two experiments took place recently at the Graduate Technological Educational Institute (T.E.I.) of Patras in Greece. In the first of them our subjects were 35 students of the School of Technological Applications, i.e. future engineers, and our basic tool was a list of 10 problems (see Appendix) given to students for solution (time allowed 3 hours). Before starting the experiment we gave the proper instructions

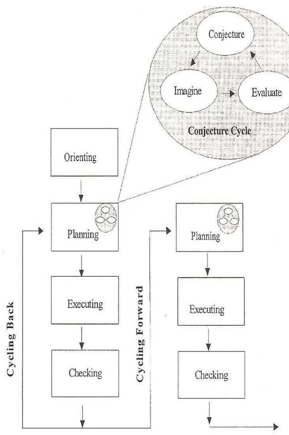

Figure 4: A graphical representation of Carlsons and Blooms MPSF

to students emphasizing among the others that we are interested for <u>all their efforts</u> (successful or not) during the PS process, and therefore they must keep records on their papers for all of them, at all stages of the PS process. This manipulation enabled as in obtaining realistic data from our experiment for each stage of the PS process and not only those based on students final results that could be obtained in the usual way of graduating their papers.

Our characterizations of students performance at each stage of the PS process involved:

- Negligible success, if they obtained (at the particular stage) positive results for less than 2 problems.
- Low success, if they obtained positive results for 2, 3, or 4 problems.
- Intermediate success, if they obtained positive results for 5, 6, or 7 problems.
- High success, if they obtained positive results for 8, or 9 problems.
- Complete success, if they obtained positive results for all problems.

$A_1 = \{(a,0),(b,0),(c,0.5),(d,0.25),(e,0,0.25)\}$,
In the same way we represented the stages of executing and checking as fuzzy sets in U by
$A_2 = \{(a,0),(b,0),(c,0,5),(d,0,25),(e,0)\}$
and
$A_3 = \{(a,0,25),(b,0,25),(c,0,25),(d,0),(e,0)\}$
respectively.
Next we calculated the membership degrees of the 5^3 (ordered samples with replacement of 3 objects taken from 5) in total possible students' profiles as it is described in section 2 (column of $m_s(1)$ in Table 1). For example, for the profile $s = (c,c,a)$ one finds that $m_s = m_{A1}(c).m_{A2}(c).m_{A3}(a) = 0,5.0,5.0,25) = 0,06225$.
It is a straightforward process then to calculate in terms of the membership degrees the Shannons entropy for the student group, which is $H \approx 0,289$.
Further, from the values of the column of $m_s(1)$ it turns out that the maximal membership degree of students' profiles is 0,06225. Therefore the possibility of each s in U^3 is given by

$$r_s = \frac{m_s}{0.06225}.$$

Calculating the possibilities of all profiles (column of $r_s(1)$ in Table 1) one finds that the ordered possibility distribution for the student group is:

$$r : r_1 = r_2 = 1, r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0.5,$$
$$r_9 = r_{10} = r_{11} = r_{12} = r_{13} = r_{14} = 0.258,$$
$$r_{15} = r_{16} = \cdots = r_{125} = 0.$$

Thus with the help of a calculator one finds that

$$ST(r) = \frac{1}{log2}\left[\sum_{i=1}^{14}(r_i - r_{i+1})log\frac{i}{\sum_{j=1}^{i} r_j}\right]$$
$$\approx \frac{1}{0.301}\left[0.5log\frac{2}{2} + 0.242log\frac{8}{5} + 0.258log\frac{14}{6.548}\right]$$
$$\approx 3,32.0,242.0,204 + 0,258.0,33 \approx 0.445$$

and

$$N(r) = \frac{1}{log2}\left[\sum_{i=1}^{14}(r_i - r_{i+1})logi\right]$$
$$= \frac{1}{log2}(0.5log2 + 0.242log8 + 0.258log14)$$

Therefore we finally have that $T(r) \approx 2,653$.
A few days later we performed the same experiment with a group of 30 students of the School of Management and Economics. Working as above we found that
$A_1 = \{(a,0),(b,0,25),(c,0,5),(d,0,25),(e,0)\}$,
$A_2 = \{(a,0,25),(b,0,25),(c,0,5),(d,0),(e,0)\}$
$A_3 = \{(a,0,25),(b,0,25),(c,0,25),(d,0),(e,0)\}$
Then we calculated the membership degrees of all possible profiles of the student group (column of $m_s(2)$ in Table 1) and the Shannon's entropy,

which is $H \approx 0,312$.
Since the maximal membership degree is again 0,06225, the possibility of each s is given by the same formula as for the first group. Calculating the possibilities of all profiles (column of $r_s(2)$ in Table 1) one finds that the ordered possibility distribution of the second group is:

$$r : r_1 = r_2 = 1, r_3 = r_4 = r_5 = r_6 = r_7 = r_8 = 0.5,$$

$$r_9 = r_{10} = r_{11} = r_{12} = r_{13} = r_{14} = 0.258,$$

$$r_{15} = r_{16} = \cdots = r_{125} = 0.$$

Finally, working in the same way as above one finds that $T(r) = 0,432 + 2,179 = 2,611$. Therefore, since $2,611 < 2,653$, it turns out that the second group had in general a slightly better performance than the first one. Notice that the values of the Shannon's entropy lead to the opposite conclusion (since 0,312¿0,289), but this, as we have already explained in section 2, is not surprising in cases, where the difference between the performances of the two groups is very small.
Further, using formulas (3.3) of section 3, one can compare the performances of the two groups by the 'centroid' method in each of the listed above phases of the PS process as follows:
Denote by A_{ij} the fuzzy subset of U attached to the phase $S_j, j = 1, 2, 3$, of the PS process with respect to the student group $i, i = 1, 2$.
In the first phase of orientation/planning we have
$A_{11} = \{(a, 0), (b, 0), (c, 0, 5), (d, 0, 25), (e, 0, 25)\}$,
$A_{21} = \{(a, 0), (b, 0, 25), (c, 0, 5), (d, 0, 25), (e, 0)\}$
and respectively

$$x_{c11} = \frac{1}{2}(5.0, 5 + 7.0, 25 + 9.0, 25) = 3, 25$$

$$x_{c21} = \frac{1}{2}(3.0, 25 + 5.0, 5 + 7.0, 25) = 2, 25$$

. By our criterion the first group demonstrates better performance. At the second stage of solution we have:
$A_{11} = \{(a, 0), (b, 0), (c, 0, 5), (d, 0, 25), (e, 0)\}$,
$A_{21} = \{(a, 0, 25), (b, 0, 25), (c, 0, 5), (d, 0), (e, 0)\}$.
$A_{11} = \{(a, 0), (b, 0), (c, 0, 67), (d, 0, 33), (e, 0)\}$,
$A_{21} = \{(a, 0, 25), (b, 0, 25), (c, 0, 5), (d, 0), (e, 0)\}$.
and respectively

$$x_{c12} = \frac{1}{2}(5.0, 67 + 7.0, 33) = 5, 66$$

$$x_{c22} = \frac{1}{2}(0, 25 + 3.0, 25 + 5.0, 25) = 3, 25$$

. By our criterion the first group demonstrates again a significantly better performance. Finally, at the third phase of checking we have

$A_{13} = A_{23} =$

$$(a, 0, 25), (b, 0, 25), (c, 0, 25), (d, 0), (e, 0),$$

which obviously means that in this phase the performances of both groups are identical. Based on our calculations we can conclude that the first group demonstrated a significantly better performance at the phases of orientation/planning and of executing, but performed identically with the second one at the phase of checking.

Remark 5.1 *In earlier papers we have also developed a stochastic model for the representation of the PS process by applying a Markov chain on the stages of Schoenfeld's 'Expert Performance Model for PS' ([19], [20]). There are many similarities between Carlson's and Blum's MPSF [2] and Schoenfeld's model [13]. However, their main qualitative difference is that, while in the former case emphasis is given to the solver's behaviour and required attributes rather, the latter is oriented towards the PS process itself (use of the proper heuristic strategies at each stage of the process).*

Our stochastic model for the PS process is self restricted to give quantitative information only through the description of the ideal behavior of a group of solvers (i.e. how they must act for the solution of a problem and not how they really act in practice).

6 Conclusion

The following conclusions can be drawn from the discussion performed in this paper:

- In studying a systems operation a model is required to describe and represent all its multiple views. An essential part of a human-designed systems study is the assessment, through the model, of its performance. In fact, this could help the system's designer to make all the necessary modifications/improvements to the systems structure in order to increase its effectiveness.

- In this paper we developed a general fuzzy model for representing processes in a systems operation involving vagueness and/or uncertainty. We also presented 3 methods of measuring a systems effectiveness connected to the above model. The first of them concerns the measurement of the total possibilistic uncertainty defined on the systems profiles ordered possibility distribution and being equal to the sum of strife and non specificity. The second concerns the measurement of the systems probabilistic uncertainty expressed by a modified version of the Shannons entropy for use in a fuzzy environment. Finally, the third one is the, so called, centroid method, in which the coordinates of the center of mass of the graph of the membership function involved provide an alternative measure of the system's performance. Each one of the above methods adheres its own advantages and disadvantages and a combined use of them could help the user in finding the ideal profile of the systems performance according to his/her personal criteria of goals.

- In earlier papers we have applied similar fuzzy models for a more effective description of several processes in the areas of Education, of Artificial Intelligence and of Management. In the present paper we applied our general fuzzy model for the description of the PS process The construction of the fuzzy model for the PS process was based on Carlsons and Blums Multidimensional PS Framework (MPSF). Two classroom experiments were also presented illustrating the use of our results in practice.

- In contrast to our stochastic (Markov chain) model for the PS process developed in earlier papers, which is restricted to give quantitative information only, our fuzzy model has the advantage of giving also a qualitative/realistic view of the PS process through the calculation of the probabilities and/or possibilities of all possible solvers profiles. Nevertheless, the characterization of the problem solvers performance in terms of a set of linguistic labels, which are fuzzy themselves, is a disadvantage of the fuzzy model, because this characterization depends on the users personal criteria. A live example about this is the different evaluations for the two groups of solvers obtained by using our fuzzy measures for the PS skills in our classroom experiments presented in section 5. Therefore the stochastic could be used as a tool for the validation of the fuzzy model in the effort of achieving a worthy of credit mathematical analysis of the PS process.

Appendix

List of the problems given for solution to students in our classroom experiment

Problem 1: We want to construct a channel to run water by folding across its longer side the two edges of an orthogonal metallic leaf having sides of length 20cm and 32 cm, in such a way that they will be perpendicular to the other parts of the leaf. Assuming that the flow of the water is constant, how we can run the maximum possible quantity of the water?

Problem 2: Given the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and a positive integer n, find the matrix A^n.

Problem 4: Let us correspond to each letter the number showing its order into the alphabet ($A = 1, B = 2, C = 3etc$). Let us correspond also to each word consisting of 4 letters a $2X2$ matrix $\begin{pmatrix} 19 & 15 \\ 13 & 5 \end{pmatrix}$ corresponds to the word SOME. Using the matrix $E = \begin{pmatrix} 8 & 5 \\ 11 & 7 \end{pmatrix}$ as an encoding matrix how you could send the message LATE in the form of a camouflaged matrix to a receiver knowing the above process and how the receiver could decode your message?

Problem 5: The demand function $P(Q_d) = 25 - Q_d^2$ represents the different prices that consumers willing to pay for different quantities Q_d of a good. On the other hand the supply function $P(Q_s) = 2Q_s + 1$ represents the prices at which different quantities Q_s of the same good will be supplied. If the markets equilibrium occurs at (Q_0, P_0), the producers who would supply at lower price than P0 benefit. Find the total gain to producers.

Problem 6: A ballot box contains 8 balls numbered from 1 to 8. One makes 3 successive drawings of a lottery, putting back the corre-

sponding ball to the box before the next lottery. Find the probability of getting all the balls that he draws out of the box different.
Problem 7: A box contains 3 white, 4 blue and 6 black balls. If we put out 2 balls, what is the probability of choosing 2 balls of the same colour?
Problem 9: A company circulates for first time in the market a new product, say K. Markets research has shown that the consumers buy on average one such product per week, either K, or a competitive one. It is also expected that 70i) Find the markets share for K two weeks after its first circulation, provided that the markets conditions remain unchanged.
ii) Find the markets share for K in the long run, i.e. when the consumers preferences will be stabilized.
Problem 10: Among all cylinders having a total surface of 180 m2, which one has the maximal volume?

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Michael Gr. Voskoglou (B.Sc., M.Sc., M.Phil. , Ph.D. in Mathematics) is a Professor of Mathematical Sciences at the Graduate Technological Educational Institute of Patras, Greece. He is the author of 8 books and of more than 260 papers published in reputed journals and proceedings of conferences of 22 countries in 5 continents, with many references from other researchers. He is also the Editor in Chief and publisher of the International Journal of Applications of Fuzzy Sets (e-journal), reviewer of the American Mathematical Society and member of the Editorial Board or referee in several mathematical journals. His research interests include Algebra, Fuzzy Sets, Markov Chains and Mathematics Education.