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A Note on Modular Hyperconvex Space

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Abstract

Recently, it has been obtained a lot of results on hyperconvex space (see [1, 2, 3, 5]). In this paper we develop some of those results for modular hyperconvex spaces.

Keywords : Hyperconvex space, Modular function, Modular hyperconvex space.

1 Introduction

The theory of hyperconvex space was initiated by Aronzajn and Panitchpakdi [1]. It was proved that a hyperconvex space is a nonexpansive absolute retract ,i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. Many interesting results and applications of the theory of hyperconvex spaces in another branch of mathematics are the base of our motivation to study this subject. For example ,it has been used in probability and mathematical statistics, boundary-value problems [3], the inverse function [10], and the existence of solutions of differential equations [9, 12]. The theory of modular function space was initiated by Nakano in 1950 in connection with the theory of order spaces and redefined and generalized by Luxemburg and Orlicz in 1959.

The organization of this paper is the following: We start with introducing the definitions and notations which will be used later. For convenience of readers, we suggest that one refers to [1, 2, 4, 5, 13]. Section two startes with the proof of the existence of the selection of Lipschitz set valued mappings T^* from a modular hyperconvex space $\mathcal{H}_\rho$ that choose their values from the space of the external modular hyperconvex subset $\mathcal{H}_\rho$ i.e $\varepsilon_\rho(\mathcal{H}_\rho)$. Also we get interesting results by considering intersection of the sets in modular hyperconvex spaces such as the intersection of a modular admissible subset and

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a externally modular hyperconvex subset with respect to the hyperconvex modular space $\mathcal{H}_\rho$ is externally modular hyperconvex with respect to $\mathcal{H}_\rho$.
Also we show that when the communion of externally modular hyperconvex subsets of a modular hyperconvex space is nonempty.

2 Preliminaries

Let X be a vector space on $\mathbb{R}$, a function $\rho : X \to [0, +\infty]$ is called modular if for every x, y in X , (i) $\rho(x) = 0$ if and only if $x = 0$, (ii) $\rho(\alpha x) = \rho(x)$, for every $\alpha \in \mathbb{R}$ where $|\alpha| = 1$, (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$, and ρ is called convex modular if , $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$. By a modular space we mean $X_\rho = \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$, where ρ is a modular function on X. Following Khamsi [3], for a modular space X_ρ, the sequence $\{x_n\}$ is called ρ-convergent to x if $\rho(x_n, x) \to x$, and it is called ρ-Cauchy if $\rho(x_n, x_m) \to 0$ as $n, m \to 0$. We will say that the modular function ρ satisfies the Fatou property if $\rho(x) \leq \liminf_n \rho(x_n)$ as $x_n \to x$,where $\{x_n\}$ is a sequence in X_ρ. A modular function ρ is called complete if every ρ- Caushy sequence $\{x_n\}$ is ρ- convergent. A subset A of X_ρ is called ρ- closed if the ρ-limit of a ρ-convergent sequence of A always belongs to A. By a ρ-ball $B_\rho(x, r)$, we mean $\{y \in X_\rho : \rho(x - y) \leq r\}$.

Finally, a subset A of X_ρ is called ρ-bounded if

$$\delta_\rho(A) = \sup\{\rho(x - y) : x, y \in A\} < \infty.$$

We note that ρ does not behave in general as a metric because ρ does not satisfy the triangle inequality. For example ρ- convergent does not imply ρ- Caushy. However, ρ-balls are ρ-closed in a modular space X_ρ if and only if they have Fatou property [5].

Definition 2.1. *A modular space X_ρ is called modular hyperconvex space if, for any collection of points $\{x_\alpha\}_{\alpha\in\Gamma}$ of X and for any collection $\{r_\alpha\}$ of non-negative real numbers such that $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$ ($\alpha, \beta \in \Gamma$), it follows that $\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset$.*

If X_ρ is a modular space we show the family of all the nonempty and bounded subset X_ρ by $B_\rho(X_\rho)$.

Definition 2.2. *Let X_ρ be a modular space such that has Fatou property. A subset A of X_ρ is called modular admissible subset if A is an intersection of ρ-closed balls in X_ρ.*

Definition 2.3. *Suppose that X_ρ is a modular space and C it's subset. We say that C is modular proximinal ,if for each $x \in X_\rho$*

$$C \cap B_\rho(x, dist_\rho(x, C)) \neq \emptyset$$

such that

$$dist_\rho(x, C) = inf\{\rho(x - y) \ : \ y \in C\}$$

Definition 2.4. *A subset E of modular space (X_ρ, ρ) is called externally modular hyperconvex (with respect to X_ρ) if for each family of elements $\{x_\alpha\}_{\alpha\in\Gamma}$ in X_ρ and each family of positive real number $\{r_\alpha\}_{\alpha\in\Gamma}$ such that for every $\alpha, \beta \in \Gamma$*

$$dist_\rho(x_\alpha, E) \leq r_\alpha \quad , \quad \rho(\frac{1}{2}(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$$

the following holds

$$\bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha) \cap E \neq \emptyset.$$

$\mathcal{A}_\rho(X_\rho)$ is the notation of all the nonempty modular admissible subsets X_ρ and $\varepsilon_\rho(X_\rho)$ is the notation of all the externally modular hyperconvex subsets X_ρ and $\mathcal{H}_\rho(X_\rho)$ is the notation of all the modular hyperconvex subsets X_ρ.

Definition 2.5. *Let A be a subset of a modular hyperconvex space X_ρ, set*

$$\begin{aligned}
r_x(A) &= \sup\{\rho(x-y) \ : \ y \in A\}, \quad x \in X_\rho;\\
r(A) &= \inf\{r_x(A) \ : \ x \in X_\rho\};\\
R(A) &= \inf\{r_x(A) \ : \ x \in A\};\\
diam(A) &= \sup\{\rho(x-y) \ : \ x,y \in A\};\\
C(A) &= \{x \in X_\rho \ : \ r_x(A) = r(A)\};\\
C_A(A) &= \{x \in A : \ r_x(A) = r(A)\};\\
cov_\rho(A) &= \bigcap\{B : B \ is \ a \ \rho - ball \ and \ B \supseteq A\};
\end{aligned}$$

$r(A)$ is called the reduce of A (relative to X_ρ), $diam(A)$ is called the diameter of A, $R(A)$ is called Chebyshev radius of A, $C(A)$ is called the Chebyshev center of A, and $cov_\rho(A)$ is called the cover of A.

The reader can see the proof of the following Lemma in [8].

Lemma 2.1. *Let A be a ρ-bounded subset of modular hyperconvex space X_ρ, then:*
1) $cov_\rho(A) = \bigcap\{B_\rho(x, r_x(A)) : x \in X_\rho\}$.
2) $r_x(cov_\rho(A)) = r_x(A)$, for any $x \in X_\rho$.
3) $r(cov_\rho(A))) = r(A)$.
4) $r(A) = 1/2(diam(A))$.
5) $diam(cov_\rho(A)) = diam(A)$.
6) If $A = cov_\rho(A)$, then $r(A) = R(A)$. In particular we have $R(A) = 1/2(diam(A))$.

If A is a subset of modular space X_ρ, we denote the $\varepsilon-$ closed neighborhood A with $N_\varepsilon(A)$ in which

$$N_\varepsilon(A) = \{x \in X_\rho \ : \ dist_\rho(x, A) \leq \varepsilon\}.$$

Definition 2.6. *Suppose that A is an arbitrary set. A map T from A to $P(A)$ where $P(A)$ is the power set of A, is called set-valued mapping.*

Definition 2.7. *Let $T^* : X_\rho \to B_\rho(X_\rho)$ is a set-valued mapping. A selection is a map such as $T : X_\rho \to X_\rho$ such that $T(x) \in T^*(x)$ for each $x \in X_\rho$.*

Definition 2.8. *Suppose that (X_1, ρ_1) and (X_2, ρ_2) are modular spaces. we say the mapping $T : X_1 \to X_2$ is $\lambda-$ lipschitzian when there exists $\lambda \geq 0$ such that for each $x, y \in X_1$ the following satisfies $\rho_2(T(x) - T(y)) \leq \lambda\rho_1(x - y)$.*

The smallest λ that satisfies in the above relation is called lipschitzian constant and we note it with $Lip(T)$. If $\lambda = 1$ then we say the above map is nonexpansive.

Lemma 2.2. *(lemma 2.2 [8]) If A is a external modular hyperconvex subset or a admissible modular subset of a modular hyperconvex space H_ρ, then A is the modular proximinal in H_ρ.*

Lemma 2.3. *(lemma 3.2 [8]) Let X_ρ be a modular hyperconvex space and $D = \bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha)$. In this case for each $\varepsilon > 0$ we have*

$$N_\varepsilon(D) = \bigcap_{\alpha\in\Gamma} B_\rho(x_\alpha, r_\alpha + \varepsilon).$$

Definition 2.9. *If A and B are bounded and closed subsets in modular space X_ρ we define Hausdorff distance ρ_H as follows:*

$$\rho_H(A, B) = inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \ \ , \ \ B \subset N_\varepsilon(A)\}.$$

Theorem 2.1. *(Theorem 2.2 [8]) If X_ρ is a modular hyperconvex space then we have*

$$\mathcal{A}_\rho(X_\rho) \subset \varepsilon_\rho(X_\rho) \subset \mathcal{H}_\rho(X_\rho).$$

3 Main results

In this section, we develop some results getting in [4, 7], for modular hyperconvex space.

Theorem 3.1. *Let H_ρ be a modular hyperconvex space , S is an arbitrary set and $T^* : S \to \varepsilon_\rho(H_\rho)$. Then there exists the map $T : S \to H_\rho$ such that for each $x \in S$, $T(x) \in T^*(x)$ and for each $x, y \in S$ we have*

$$\rho(T(x) - T(y)) \le \rho_H(T^*(x), T^*(y)).$$

Proof: Let F denote the collection of all pairs (D,T), where $D \subseteq S$ and for all $d \in D$ and for all x, y in D we have
$T : D \to H$
$T(d) \in T^*(d)$
$\rho(T(x) - T(y)) \le \rho_H(T^*(x), T^*(y))$
We note that for each $x_0 \in S$, $T(x_0) \in T^*(x_0)$, then $(\{x_0\}, T) \in F$. Thus we have $F \neq \emptyset$.
Now, we define the order relation $\le$ on F as follows

$$(D_1, T_1) \le (D_2, T_2) \Leftrightarrow D_1 \subset D_2 \ \ , \ \ T_2|_{D_1} = T_1.$$

Suppose $\{(D_\alpha, T_\alpha)\}$ is the increasing chain in $(F, \le)$. So it follows that $(\bigcup_{\alpha\in\Gamma} D_\alpha, T) \in F$ where $T|_{D_\alpha} = T_\alpha$. It is clear that this member is an upper bound for above chain. with defining order by Zorn's lemma, the maximal element such as (D, T) in $(F, \le)$ exists. Now, we show that $D = S$. Assume $D \neq S$, therefore there exists the element $x_0 \in S \backslash D$. Let $\tilde{D} = D \cup \{x_0\}$.
Now, consider the following set :

$$J = (\bigcap_{x\in D} B_\rho(T(x), \rho_H(T^*(x), T^*(x_0)))) \bigcap T^*(x_0).$$

Since $T^*(x_0) \in \varepsilon_\rho(H_\rho)$ we have $J \neq \emptyset$ if and only if for each $x \in D$:

$$dist_\rho(T(x), T^*(x_0)) \le \rho_H(T^*(x), T^*(x_0)).$$

by lemma 2.2 we have $T^*(x_0) \in \varepsilon_\rho(H_\rho)$ as a modular proximinal subset from H_ρ. The above is true if and only if for each $x \in D$

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0))) \cap T^*(x_0) \neq \emptyset.$$

By the definition of Hausdorff distance

$$T^*(x) \subset N_{\rho_H(T^*(x),T^*(x_0))+\varepsilon}(T^*(x_0)).$$

However by assumption $T(x) \in T^*(x)$ so it must be the case that for each $\varepsilon > 0$

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0)) + \varepsilon) \cap T^*(x_0) \neq \emptyset.$$

Since $T^*(x_0)$ is a modular proximinial in H_ρ, this implies that

$$B_\rho(T(x), \rho_H(T^*(x), T^*(x_0))) \cap T^*(x_0) \neq \emptyset.$$

Thus $J \neq \emptyset$. Now choose $y_0 \in J$ and define

$$\tilde{T}(x) = \begin{cases} y_0 & if x = x_0 \\ T(x) & if x \in D \end{cases}$$

On the other hand $\rho(\tilde{T}(x_0) - \tilde{T}(x)) = \rho(y_0 - T(x)) \leq \rho_H(T^*(x), T^*(x_0))$ $(\forall x \in D)$. Thus $(D \cup \{x_0\}, \tilde{T}) \in F$ and it has contradiction with maximality of (D, T). Therefore $D = S$.

Corollary 3.1. *Suppose that H_ρ is a modular hyperconvex space and (M, ρ_1) is a modular space and the set-valued mapping $T^* : M \to \varepsilon_\rho(H_\rho)$ is nonexpansive i.e, for each $x, y \in M$, $\rho_H(T^*(x), T^*(y)) \leq \rho_1(x - y)$. Then the nonexpansive map $T : M \to H_\rho$ exists such that for each $x \in M$, $T(x) \in T^*(x)$.*

Proof: By theorem 3.1 and nonexpansive the T^*, there exists the selection $T : M \to H_\rho$ that for each $x \in M$, $T(x) \in T^*(x)$ and

$$\rho(T(x) - T(y)) \leq \rho_H(T^*(x), T^*(y)) \leq \rho_1(x - y), \quad (\forall x, y \in M).$$

Therefore $\rho(T(x) - T(y)) \leq \rho_1(x - y)$, thus T is nonexpansive.

Theorem 3.2. *Let M_ρ be a bounded modular space and $(H_\beta)_{\beta \in \Gamma}$ be a decreasing family of nonempty modular hyperconvex subsets of M_ρ, where Γ is totally ordered. Then $\bigcap_{\beta \in \Gamma} H_\beta$ is nonempty and modular hyperconvex.*

Proof: Define F as the follows:

$$F = \{A = \Pi_{\beta \in \Gamma} A_\beta \ , \ A_\beta \in \mathcal{A}_\rho(H_\beta) \ \ and \ \ (A_\beta) \ \ is\ decreasing\ and\ nonempty\ \}.$$

Since M_ρ is bounded so H_β is bounded . Thus $H_\beta \in \mathcal{A}_\rho(H_\beta)$. So H_β is not empty and decreasing then $\Pi_{\beta \in \Gamma} H_\beta \in F$ and $F \neq \emptyset$.
Since H_β is modular hyperconvex then $\mathcal{A}_\rho(H_\beta)$ is compact for every $\beta \in \Gamma$. Thus F satisfies the assumptions of zorn's lemma when ordered by set inclusion. Hence for every $D \in F$ there exists a minimal element $A \in F$ such that, $A \subset D$.
We claim that if $A = \Pi_{\beta \in \Gamma} A_\beta$ is minimal then there exists $\beta_0 \in \Gamma$ such that $\delta(A_\beta) = 0$ for every $\beta \geq \beta_0$, where $\delta(A) = diam(A)$. Let $\beta \in \Gamma$ be fixed. For every $D \subset M_\rho$ define

$$cov_{\rho\beta}(D) = \bigcap_{x \in H_\beta} \beta_\rho(x, r_x(D)).$$

Consider $A' = \Pi_{\alpha\in\Gamma} A'_\alpha$ where $A'_\alpha = cov_{\rho\beta}(A_\beta) \cap A_\alpha$ if $\alpha \leq \beta$ and $A'_\alpha = A_\alpha$ if $\alpha \geq \beta$.

Since $A \in F$ then the family $(A'_{\alpha\geq\beta})$ is decreasing . Let $\alpha \leq \gamma \leq \beta$. Since $A_\gamma \subset A_\alpha$ and $A_\beta = cov_{\rho\beta}(A_\beta) \cap A_\beta$ so $A'_\gamma \subset A'_\alpha$. Hence the family (A'_α) is decreasing. On the other hand if $\alpha \leq \beta$ then $cov_{\rho\beta}(A_\beta) \cap A_\alpha \in \mathcal{A}_\rho(H_\alpha)$. Since $H_\beta \subset H_\alpha$ so $A'_\alpha \in \mathcal{A}_\rho(H_\alpha)$. Thus $A' \in F$. Since A is minimal this implies that $A = A'$ which implies

$$A_\alpha = cov_{\rho\beta}(A_\beta) \cap A_\alpha \qquad \forall\alpha \leq \beta.$$

Let $x \in H_\beta$ and $\alpha \leq \beta$. Since $A_\beta \subset A_\alpha$, then $r_x(A_\beta) \leq r_x(A_\alpha)$. Now $cov_{\rho\beta}(A_\beta) = \bigcap_{x\in H_\beta} \beta_\rho(x, r_x(A_\beta))$, then we have $cov_{\rho\beta}(A_\beta) \subset \beta_\rho(x, r_x(A_\beta))$, which implies

$$r_x(cov_{\rho\beta}(A_\beta)) \leq r_x(A_\beta).$$

Additionally $A_\alpha \subset cov_{\rho\beta}(A_\beta)$ so

$$r_x(A_\beta) \leq r_x(A_\alpha) \leq r_x(cov_{\rho\beta}(A_\beta)) \leq r_x(A_\beta).$$

Therefore we have $r_x(A_\beta) = r_x(A_\alpha)$ for every $x \in H_\beta$. Using the definition of r, we get

$$r_(A_\alpha) \leq r(A_\beta).$$

Let $a \in A_\alpha$ and $s = r_a(A_\alpha)$, then $a \in cov_{\rho\beta}(A_\beta)$. Since $A_\alpha \subset cov_{\rho\beta}(A_\beta)$ so

$$a \in \bigcap_{x\in A_\beta} \beta_\rho(x, s) \cap cov_{\rho\beta}(A_\beta).$$

By hyperconvexity of H_β,

$$S_\beta = H_\beta \cap \bigcap_{x\in A_\beta} \beta_\rho(x, s) \cap cov_{\rho\beta}(A_\beta) \neq \emptyset.$$

Let $z \in S_\beta$, then $z \in \bigcap_{x\in A_\beta} \beta_\rho(x, s)$. Since

$$A_\beta = H_\beta \cap cov_{\rho\beta}(A_\beta).$$

It follows that $r_z(A_\beta) \leq s$, which implies

$$r(A_\beta) \leq s = r_a(A_\alpha)$$

for every $a \in A_\alpha$. Hence

$$r(A_\beta) = r(A_\alpha) \qquad \forall\alpha, \beta \in \Gamma.$$

Assume that $\delta(A_\beta) > 0$ for every $\beta \in \Gamma$. Set $A''_\beta = C(A_\beta)$ for every $\beta \in \Gamma$. The family (A''_β) is decreasing. Let $\alpha \leq \beta$ and $x \in A''_\beta$, then $r_x(A_\beta) = r(A_\beta)$. Since we proved that $r_z(A_\beta) = r_z(A_\alpha)$ for every $z \in H_\beta$ then $r_x(A_\alpha) = r_x(A_\beta) = r(A_\beta) = r(A_\alpha)$, which implies that $x \in A''_\alpha$. Therefore

$$A'' = \Pi_{\beta\in\Gamma} A''_\beta \in F.$$

Since $A'' \subset A$ and A is minimal, we get that $A'' = A$. Therefore $A_\beta = C(A_\beta)$ for every $\beta \in \gamma$. This is in contradiction to hyperconvexity of H_β for each $\beta \in \Gamma$. Thus there exists $\beta_0 \in \Gamma$ such that $\delta(A_\beta) = 0$ for every $\beta \geq \beta_0$ .So $A_\beta = \{a\}$for every $\beta \geq \beta_0$, which implies that $a \in \bigcap_{\beta\in\Gamma} H_\beta \neq \emptyset$.

In order to complete the proof, we need to show that $S = \bigcap_{\beta\in\Gamma} H_\beta$ is modular hyperconvex. Let $(\beta_{\rho_i})_{i\in I}$ be a family of balls centered in S such that $\bigcap_{i\in I} \beta_{\rho_i} \neq \emptyset$. Define $D_\beta = \bigcap_{i\in I} \beta_{\rho_i} \cap H_\beta$ for all $\beta \in \Gamma$. Since H_β is modular hyperconvex and the family $(\beta_{\rho_i})_{i\in I}$ centered in H_β then D_β is not empty and $D_\beta \in \mathcal{A}_\rho(H_\beta)$. Therefore D_β is modular hyperconvex. the above proof shows that $\bigcap_{\beta\in\Gamma} D_\beta \neq \emptyset$.

Lemma 3.1. *Let H_ρ be a modular hyperconvex space and $E \subset H_\rho$ be externally modular hyperconvex related to H_ρ. Suppose A is a modular admissible subset of H_ρ. Then $E \cap A$ is externally modular hyperconvex related to H_ρ.*

Proof: Suppose $\{x_\alpha\}$ and $\{r_\alpha\}$ satisfy $\rho(\frac{1}{2}(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$ and $dist_\rho(x_\alpha, E \cap A) \leq r_\alpha$. Since A is admissible, $A = \bigcap_{i \in I} \beta_\rho(x_i, r_i)$ and since $\beta_\rho(x_\alpha, r_\alpha) \cap A \neq \emptyset$. It follows that $\rho(\frac{1}{2}(x_\alpha - x_i)) \leq r_\alpha + r_i$ for each $i \in I$. Since $A \subset \beta_\rho(x_i, r_i)$ it follows that

$$dist_\rho(x_i, E \cap A) \leq r_i \quad , \quad \rho(\frac{1}{2}(x_i - x_j)) \leq r_i + r_j \quad \forall i, j \in I.$$

Therefore by external modular hyperconvexity of E

$$(\bigcap_i \beta_\rho(x_i, r_i)) \cap (\bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha)) \cap E = \bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha) \cap (A \cap E) \neq \emptyset.$$

Thus $E \cap A$ is externally modular hyperconvex related to H_ρ.

Theorem 3.3. *Let $\{H_i\}$ be a decreasing chain of nonempty modular externally hyperconvex subsets of a bounded modular hyperconvex space H_ρ. Then $\cap_i H_i$ is nonempty and externally modular hyperconvex in H_ρ.*

Proof: By Theorem (3.2) and Theorem (2.1), we have $D = \bigcap_i H_i \neq \emptyset$. To prove D is externally modular hyperconvex, let $\{x_\alpha\} \subset H$ and $\{r_\alpha\} \subset R$ satisfy

$$\rho(\frac{1}{2}(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta \quad , \quad dist_\rho(x_\alpha, D) \leq r_\alpha.$$

Since H_ρ is modular hyperconvex we know that $A = \bigcap_\alpha \beta_\rho(x_\alpha, r_\alpha) \neq \emptyset$. also $dist_\rho(x_\alpha, D) \leq r_\alpha$ and $dist_\rho(x_\alpha, H_i) \leq r_\alpha$ for each i, so by externally modular hyperconvexity of H_i implies that for each i we have $A \cap H_i \neq \emptyset$. By Theorem 2.1 and lemma 3.1, $\{A \cap H_i\}$ is a decreasing chain of nonempty modular hyperconvex subsets of H_ρ. Now by Theorem 3.2, we have

$$\bigcap_i (A \cap H_i) = A \cap D \neq \emptyset.$$

Thus $\bigcap_i H_i$ is nonempty and externally modular hyperconvex in H_ρ.

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