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Fuzzy Laplace Transform on Two Order Derivative and Solving Fuzzy Two Order Differential Equations

S.J. Ramazannia Tolouti *, M. Barkhordari Ahmadi

Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, Bandar Abbas, Iran

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Abstract

In this paper, the laplace transform formula on the fuzzy two order derivative is investigated by using the strongly generalized differentiability concept. Then, it is used in a analytic method for fuzzy two order differential equation. The related theorems and properties are proved in detail and the method is illustrated by solving some examples.

Keywords : Fuzzy-number, Fuzzy-valued function, Generalized differentiability, fuzzy differential equation,fuzzy laplace transform, fuzzy initial value problem.

1 Introduction

A natural way to model dynamic systems under uncortainty is to use FDEs. Two order fuzzy differential equations are one of the simplest FDEs which may appear in many applications. The topic of fuzzy differential equations (FDEs) has been rapidly growing in recent years. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [22]; it was followed up by Dubois and Prade [27], who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [44] and Goetschel and Voxman [30]. Kandel and Byatt [37, 38] applied the concept of FDEs to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [35, 36], Seikkala [45], He and Yi [40], Kloeden [39] and Menda [42], and by other researchers (see [10, 15, 17, 16, 20, 26, 34]). The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 7, 32]. A thorough theoretical research of fuzzy Cauchy problems was given by Kaleva [35], Seikkala [45], Ouyang and Wu [40], and Kloeden [39] and Wu [48]. Kaleva [35] discussed the properties of differentiable fuzzy set-valued functions by means of the concept

*Corresponding author. Email address: S.J.Ramazannia@iauba.ac.ir.

of H-differentiability due to Puri and Ralescu [44], gave the existence and uniqueness theorem for a solution of the fuzzy differential equation $y' = f(t; y); \quad y(t_0) = y_0$ when f satisfies the Lipschitz condition. Further, song and Wu [46] investigate fuzzy differential equations, and generalize the main results of Kaleva [35]. Seikkala [45], defined the fuzzy derivative which is the generalization of Hukuhara derivative, and showed that fuzzy initial value problem $y' = f(t; y); \quad y(t_0) = y_0$ has a unique solution, for the fuzzy process of a real variable whose values are in the fuzzy number space (E, D), where f satisfies the generalized Lipschitz condition. Strongly generalized differentiability was introduced in [12] and studied in [10]. The strongly generalized derivative is defined for a larger class of fuzzy-valued function than the H-derivative, and fuzzy differential equations can have solutions which have a decreasing length of their support. So we use this differentiability concept in the present paper. The fuzzy Laplace transform method solves FTDEs and corresponding fuzzy two order and boundary value problems. In this way fuzzy Laplace transforms reduce the problem of solving a FTDE to an algebraic problem. This switching from operations of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and for the engineer, the fuzzy Laplace transform method is practically the most important operational method. The fuzzy Laplace transform also has the advantage that it solves problems directly, fuzzy two order value problems without first determining a general solution, and non homogeneous differential equations without first solving the corresponding homogeneous equation.
The paper is organized as follows:
Section 2 contains the basic material to be used in the paper. In section 3 fuzzy Laplace transform for two order derivative is defined and Procedure for solving FDEs by fuzzy Laplace transform is proposed. Several examples are given in section 4, and conclusions are drawn in section 5.

2 Preliminaries

We now recall some definitions needed through the paper. The basic definition of fuzzy numbers is given in [25, 31].
By R, we denote the set of all real numbers. A fuzzy number is a mapping $u : R \to [0, 1]$ with the following properties:
(a) u is upper semi-continuous,
(b) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0, 1]$,
(c) u is normal, i.e.,$\exists x_0 \in R$ for which $u(x_0) = 1$,
(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure cl(supp u) is compact.
Let E be the set of all fuzzy number on R. The r-level set of a fuzzy number $u \in E$, $o \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \geq r\} & if \quad 0 \leq r \leq 1 \\ cl(supp\ u) & if \quad r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ o & if \quad t \neq y \end{cases}$$

R can be embedded in E.

Remark 2.1. *(See [52]) Let X be Cartesian product of universes $X = X_1 \times ... \times X_n$, and $A_1, \ldots, A_n$ be n fuzzy numbers in $X_1, \ldots, X_n$, respectively. f is a mapping from X to a universe Y, $y = f(x_1, ..., x_n)$. Then the extension principle allows us to define a fuzzy set B in Y by*

$$B = \{(y, u(y)) \mid y = f(x_1, ..., x_n), (x_1, ..., x_n) \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{(x_1,...,x_n) \in f^{-1}(y)} \min\{u_{A_1}(x_1), ..., u_{A_n}(x_n))\}, & if \quad f^{-1}(y) \neq 0, \\ 0 & if \quad otherwise. \end{cases}$$

where f^{-1} is the inverse of f.
For $n = 1$, the extension principle, of course, reduces to

$$B = \{(y, u_B(y)) \mid y = f(x), x \in X\}$$

where

$$u_B(y) = \begin{cases} \sup_{x \in f^{-1}(y)} u_A(x), & if \quad f^{-1}(y) \neq 0, \\ 0 & if \quad otherwise. \end{cases}$$

According to Zadeh's extension principle, operation of addition on E is defined by

$$(u \oplus v)(x) = sup_{y \in R} min\{u(y), v(x - y)\}, \quad x \in R$$

and scalar multiplication of a fuzzy number is given by

$$(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ \widetilde{0}, & k = 0, \end{cases}$$

where $\tilde{0} \in E$.
It is well known that the following properties are true for all levels

$$[u \oplus v]_r = [u]_r + [v]_r, \qquad [k \odot u]_r = k[u]_r$$

From this characteristic of fuzzy numbers, we see that a fuzzy number is determined by the endpoints of the intervals $[u]_r$. This leads to the following characteristic representation of a fuzzy number in terms of the two "endpoint" functions $\underline{u}(r)$ and $\overline{u}(r)$. An equivalent parametric definition is also given in ([29, 41]) as:

Definition 2.1. *A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:*

1. *$\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$, and right continuous at 0,*

2. $\overline{u}(r)$ *is a bounded non-increasing left continuous function in* $(0,1]$, *and right continuous at 0,*

3. $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1.$

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \leq r \leq 1$. We recall that for $a < b < c$ which $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b-a)r$ and $\overline{u}(r) = c - (c-b)r$ are the endpoints of the r-level sets, for all $r \in [0,1]$.
For arbitrary $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and $k > 0$ we define addition $u \oplus v$, subtraction $u \ominus v$ and scaler multiplication by k as (See [29, 41])

(a) Addition:

$$u \oplus v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$$

(b) Subtraction:

$$u \ominus v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$$

(c) Multiplication:

$$u \odot v = (min\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\}, max\{\underline{u}(r)\overline{v}(r), \underline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r)\})$$

(d) Scaler multiplication:

$$k \odot u = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0. \end{cases}$$

The Hausdorff distance between fuzzy numbers given by $D : E \times E \longrightarrow R_+ \bigcup 0$,

$$D(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r)) \subset R$ is utilized (See [12]). Then, it is easy to see that D is a metric in E and has the following properties (See [43])
$(i) D(u \oplus w, v \oplus w) = D(u, v), \quad \forall u, v, w \in E,$
$(ii) D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall k \in R, u, v \in E,$
$(iii) D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e), \quad \forall u, v, w, e \in E,$
$(iV)(D, E)$ is a complete metric space.

Theorem 2.1. *(See [8]) (i) If we define* $\widetilde{0} = \chi_0$, *then* $\widetilde{0} \in E$ *is a neutral element with respect to addition, i.e.* $u \oplus \widetilde{0} = \widetilde{0} \oplus u = u$, *for all* $u \in E$.
(ii) With respect to $\widetilde{0}$, *none of* $u \in E \setminus R$, *has opposite in* E.
(iii) For any $a, b \in R$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and any* $u \in E$, *we have* $(a + b) \odot u = a \odot u \oplus b \odot u$; *for the general* $a, b \in R$, *the above property does not necessarily hold.*
(iv) For any $\lambda \in R$ *and any* $u, v \in E$, *we have* $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$;
(v) For any $\lambda, \mu \in R$ *and any* $u \in E$, *we have* $\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$;

Definition 2.2. *Let* E *be a set of all fuzzy numbers, we say that* f *is fuzzy- valued-function if* $f : R \to E$

It is well-known that the H-derivative (differentiability in the sense of Hukuhara)for fuzzy mappings was initially introduced by Puri and Ralescu([44]) and it is based on the H-difference of sets, as follows.

Definition 2.3. *Let $x, y \in E$. If there exists $z \in E$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x -^h y$.*

In this paper, the sign "$-^h$" always stands for H-difference, and also note that $x -^h y \neq x \ominus y$.
In this paper we consider the following definition which was introduced by Bede and Gal in ([12, 13]).

Definition 2.4. *Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0 (Bede-Gal differential). If there exists an element $f'(x_0) \in E$, such that*

(i) for all $h > 0$ sufficiently small,

$$\exists f(x_0 + h) \ominus f(x_0), \quad \exists f(x_0) \ominus f(x_0 - h)$$

and the limits(in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or

(ii) for all $h > 0$ sufficiently small,

$$\exists f(x_0) \ominus f(x_0 + h), \quad \exists f(x_0 - h) \ominus f(x_0)$$

and the limits(in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iii) for all $h > 0$ sufficiently small,

$$\exists f(x_0 + h) \ominus f(x_0), \quad \exists f(x_0 - h) \ominus f(x_0)$$

and the limits(in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

(iv) for all $h > 0$ sufficiently small,

$$\exists f(x_0) \ominus f(x_0 + h), \quad \exists f(x_0) \ominus f(x_0 - h)$$

and the limits(in the metric D)

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively)

Theorem 2.2. *(See e.g. [21]) Let $y : [0,a] \times R \longrightarrow R$ be continuous and $f : [0,a] \times E \longrightarrow E$, be the Zadeh s extension of y, i.e., $[f(t,x)]_r = f(t,[x]_r)$. If y is non-increasing with respect to the second argument, using the derivative in Definition (2.4), case (ii), the fuzzy solution of*

$$y' = f(t;y); \quad y(t_0) = y_0$$

whenever it exists, coincides with the solution obtained via differential inclusions.

Remark 2.2. *These case (iii) and (iv) introduced in [12], in order to ensure a differentiable switch the case (i) and case (ii) in Definition (2.4). Of course, as the authors in [12] and in [21] have stated, the cases (i) and (ii) in Definition (2.4), are more important since case (iii) and (iv) in Definition (2.4) occur only on a discrete set of points. As an example supporting these comments, let us consider $c \in E \backslash R$ be any fuzzy(non-real) constant and let $f : [0,a] \times E \longrightarrow E$, $f(t) = c \odot cost$, $t \in [0,a]$. It is natural to expect that f is differentiable everywhere in its domain. Let us observe that f is differentiable according to Definition (2.4) (ii), on each sub interval $(2k\pi, (2k+1)\pi)$ and differentiable according to Definition (2.4)(i), on each interval of the form $(2k+1)\pi, (2k)\pi$, $k \in Z$. But, at the points $\{k\pi\}$, $k \in Z$, none of the cases (i) and (ii) in Definition (2.4) are fulfilled. Namely, at these points the H-differences $f(k\pi + h) -^h f(k\pi)$ and $f(k\pi) -^h f(k\pi - h)$ may not exist simultaneously. Also, the H-differences $f(k\pi) -^h f(k\pi + h)$ and $f(k\pi - h) -^h f(k\pi)$ cannot exist simultaneously, so f is not differentiable at $k\pi$ in none of the cases (i) and (ii) of differentiability in Definition (2.4). Instead, it will be differentiable as in the cases (iii) and (iv) in Definition (2.4). Another argument for the importance of the cases (iii) and (iv) in Definition (2.4), is in the Theorem (2.2). Indeed, above stated theorem dose not cover the case when $f(t,x)$ has not constant monotonicity. In these cases (i) and (ii) of differentiability in Definition (2.4), so the cases (iii) and (iv) in Definition (2.4) may become important as switch points. In the special case when f is a fuzzy-valued function, we have the following result.*

Theorem 2.3. *(See e.g. [21]). Let $f : R \to E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0,1]$. Then*

(1) If f is (i)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and

$$f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$$

(2) If f is (ii)-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and

$$f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$$

Definition 2.5. *(See [5, 6]) Let $f : (a,b) \times E \to E$ and $x_0 \in (a,b)$. We Define the nth-order differential of f as follow: We say that f is strongly generalized differentiable of the $nth-$order at x_0. If there exists an element $f^{(s)}(x_0) \in E$, $\quad \forall s = 1, \ldots, n$, such that*

(i) for all $h > 0$ sufficiently small,

$$\exists f^{(s-1)}(x_0 + h) \ominus f^{(s-1)}(x_0), \qquad \exists f^{(s-1)}(x_0) \ominus f^{(s-1)}(x_0 - h)$$

and the limits(in the metric D)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h}=f^{(s)}(x_0)$$

or

(ii) for all $h>0$ sufficiently small,

$$\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h),\qquad \exists f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)$$

and the limits(in the metric D)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0-h)\ominus f(x_0)}{-h}=f^{(s)}(x_0)$$

or

(iii) for all $h>0$ sufficiently small,

$$\exists f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0),\qquad \exists f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)$$

and the limits(in the metric D)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0+h)\ominus f^{(s-1)}(x_0)}{h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0-h)\ominus f^{(s-1)}(x_0)}{-h}=f^{(s)}(x_0)$$

or

(iv) for all $h>0$ sufficiently small,

$$\exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h),\qquad \exists f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)$$

and the limits(in the metric D)

$$\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f^{(s-1)}(x_0)\ominus f^{(s-1)}(x_0-h)}{h}=f^{(s)}(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $\frac{-1}{h}$, respectively $\forall s=1\ldots n$)

To more detail about different cases of strongly generalized differentiability see [5, 6]

Definition 2.6. *[6] Let $f(x)$ be continuous fuzzy-value function. Suppose that $f(x)\odot e^{-px}$ improper fuzzy Rimann integrable on $[0,\infty)$, then $\int_0^\infty f(x)\odot e^{-px}dx$ is called fuzzy laplace transforms and is denoted as:*

$$L[f(x)]=\int_0^\infty f(x)\odot e^{-px}dx\qquad (p>0 \text{ and integer})$$

we have

$$\int_0^\infty f(x)\odot e^{-px}dx=(\int_0^\infty \underline{f}(x,r)\odot e^{-px}dx,\int_0^\infty \overline{f}(x,r)\odot e^{-px}dx)$$

also by using the definition of classical laplace transform:

$$\ell[\underline{f}(x,r)] \quad = \int_0^\infty \underline{f}(x,r) \odot e^{-px}dx \quad and\ \ell[\overline{f}(x,r)]$$
$$= \int_0^\infty \overline{f}(x,r) \odot e^{-px}dx$$

then, we follow:

$$L[f(x)] = (\ell[\underline{f}(x,r)], \ell[\overline{f}(x,r)]).$$

Theorem 2.4. *[6] Let $f'(x)$ be an integrable fuzzy-valued function, and $f(x)$ is the primitive of $f'(x)$ on $[0,\infty)$. Then*

$$\boldsymbol{L}[f'(x)] = p\boldsymbol{L}[f(x)] -^h f(0) \qquad where \quad f\ is\ (i)-differentiable$$

or

$$\boldsymbol{L}[f'(x)] = (-f(0)) -^h (-p\boldsymbol{L}[f(x)]) \qquad where \quad f\ is\ (ii)-differentiable$$

Theorem 2.5. *[6] Let $f(x)$, $g(x)$ be continuous-fuzzy -valued functions and c_1,c_2 are constant. Suppose that $f(x)e^{-px}$, $g(x)e^{-px}$ are improper fuzzy Rimann-integrable on $[0,\infty)$, then*

$$\boldsymbol{L}[(c_1 f(x)) \oplus (c_2 g(x))] = (c_1\boldsymbol{L}[f(x)]) \oplus (c_2\boldsymbol{L}[g(x)]).$$

Theorem 2.6. *[6] Let f be continuous fuzzy value function and $\boldsymbol{L}[f(x)] = F(p)$, Then*

$$\boldsymbol{L}[e^{ax} \odot f(x)] = F(p-a)$$

where e^{ax} is real value function and $p-a>0$.

3 Laplace transform formula on two order fuzzy derivative and its applications

In this section, by using definition of laplace transform on first-order fuzzy derivative, laplace transform formula on second-order fuzzy derivative is introduced then laplace transform method for solving second-order fuzzy differential equation is proposed.

Theorem 3.1. . *Let $f: R \to E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0,1]$. Then*

(1) If f, f' are differentiable in the first form (i) or f, f' are differentiable in the second form (ii), then $\underline{f}'(t,r)$ and $\overline{f}'(t,r)$ are differentiable functions and

$$f''(t) = (\underline{f}''(t,r), \overline{f}''(t,r))$$

(2) If f is (i)-differentiable and f' is differentiable in the second form (ii) or f is (ii)-differentiable and f' is differentiable in the first form (i) , then $\underline{f}'(t,r)$ and $\overline{f}'(t,r)$ are differentiable functions and

$$f''(t) = (\overline{f}''(t,r), \underline{f}''(t,r))$$

Proof: Since the proof procedure is similar for all two cases, we consider case (1) without loss of generality.
If f, f' is differentiable in the first form (i), then from theorem (2.3), we have:

$$f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$$

now, consider $g(t)$ as follows:

$$g(t) = f'(t)$$

if $h > 0$ and $r \in [0,1]$, we have

$$g(t+h) -^h g(t) = (\underline{g}(t+h,r) - \underline{g}(t,r), \overline{g}(t+h,r) - \overline{g}(t,r))$$

$$= (\underline{f}'(t+h,r) - \underline{f}'(t,r), \overline{f}'(t+h,r) - \overline{f}'(t,r))$$

and,multiplied by $\frac{1}{h}$, we have:

$$\frac{g(t+h) -^h g(t)}{h} = (\frac{\underline{f}'(t+h,r) - \underline{f}'(t,r)}{h}, \frac{\overline{f}'(t+h,r) - \overline{f}'(t,r)}{h})$$

similarly,

$$\frac{f'(t) -^h f'(t-h)}{h} = (\frac{\underline{f}'(t,r) - \underline{f}'(t-h,r)}{h}, \frac{\overline{f}'(t,r) - \overline{f}'(t-h,r)}{h})$$

passing to the limit, we have:

$$f''(t) = (\underline{f}''(t,r), \overline{f}''(t,r))$$

and If f, f' is differentiable in the first form (ii), then from theorem (2.3), we have:

$$f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$$

now, consider $g(t)$ as follows:

$$g(t) = f'(t)$$

if $h < 0$ and $r \in [0,1]$, we have

$$g(t+h) -^h g(t) = (\underline{g}(t+h,r) - \underline{g}(t,r), \overline{g}(t+h,r) - \overline{g}(t,r))$$

$$= (\overline{f}'(t+h,r) - \overline{f}'(t,r), \underline{f}'(t+h,r) - \underline{f}'(t,r))$$

and,multiplied by $\frac{1}{h}$, we have:

$$\frac{g(t+h) -^h g(t)}{h} = (\frac{\underline{f}'(t+h,r) - \underline{f}'(t,r)}{h}, \frac{\overline{f}'(t+h,r) - \overline{f}'(t,r)}{h})$$

similarly,

$$\frac{f'(t) -^h f'(t-h)}{h} = (\frac{\underline{f}'(t,r) - \underline{f}'(t-h,r)}{h}, \frac{\overline{f}'(t,r) - \overline{f}'(t-h,r)}{h})$$

passing to the limit, we have:

$$f''(t) = (\underline{f}''(t,r), \overline{f}''(t,r))$$

Definition 3.1. *For arbitrary $u = (\underline{u}(r), \overline{u}(r))$,$-^h u$ and $-^h(\ominus u)$ are defined as follows:*

$$-^h u = (-\underline{u}(r), -\overline{u}(r))$$

$$-^h(\ominus u) = (\overline{u}(r), \underline{u}(r))$$

Theorem 3.2. *Let $f''(x)$ be an integrable fuzzy-valued function, and $f(x), f'(x)$ are the primitive of $f'(x), f''(x)$ on $[0, \infty)$. Then*

$$\boldsymbol{L}[f''(x)] = p^2 \boldsymbol{L}[f(x)] -^h pf(0) -^h f'(0)$$

$$where \quad f \ is \ (i) - differentiable \ \ and \ \ f' \ is \ (i) - differentiable$$

or

$$\boldsymbol{L}[f''(x)] = -^h(\ominus p) -^h (\ominus p) \odot \boldsymbol{L}[f(x)] -^h -^h(\ominus p) -^h (\ominus 1) \odot f(0) -^h -^h(\ominus 1) \odot f'(0)$$

$$where \quad f \ is \ (ii) - differentiable \ \ and \ \ f' \ is \ (ii) - differentiable$$

or

$$\boldsymbol{L}[f''(x)] = -^h(\ominus p^2) \odot \boldsymbol{L}[f(x)] -^h -^h(\ominus p) \odot f(0) -^h -^h(\ominus 1) \odot f'(0)$$

$$where \quad f \ is \ (i) - differentiable \ \ and \ \ f' \ is \ (ii) - differentiable$$

or

$$\boldsymbol{L}[f''(x)] = -^h(\ominus p^2) \odot \boldsymbol{L}[f(x)] -^h -^h(\ominus p) \odot f(0) -^h f'(0)$$

$$where \quad f \ is \ (ii) - differentiable \ \ and \ \ f' \ is \ (i) - differentiable$$

Proof: By induction, it can be proved easily. we shall now discuss how the laplace transform method solves fuzzy differential equations.
Consider the following fuzzy initial value problem

$$y^{''} + ay^{'} + by = \widetilde{r}(t) \quad y(t_0) = \widetilde{k}_0, \quad y'(t_0) = \widetilde{k}_1$$

with constant a and b.
By applying the laplace transform method on fuzzy initial value problem, we have:

$$L[y^{''}] + aL[y^{'}] + bL[y] = L[\widetilde{r}(t)]$$

Then, by substituting, laplace transform formulas on first and second- order fuzzy derivative in theorem (2.4) and (3.2) we obtain the following alternatives for solving:
Case I. If we consider $y(t)$ and $y'(t)$ by using (i)-differentiable, then we have

$$p^2 \odot L[y(t)] -^h p \odot y(t_0) -^h y'(t_0) \oplus ap \odot L[y(t)] -^h a \odot y(t_0) \oplus b \odot L[y(t)] = L[\widetilde{r}(t)]$$

Case II. If we consider $y(t)$ and $y'(t)$ by using (ii)-differentiable, then we have

$$p^2 \odot L[y(t)] -^h p \odot y(t_0) \ominus y'(t_0) \oplus a(-^h(\ominus p)) \odot L[y(t)]$$

$$\ominus a \odot y(t_0) \oplus b \odot L[y(t)] = L[\widetilde{r}(t)]$$

Case III. If we consider $y(t)$ by using (i)-differentiable and $y'(t)$ by using (ii)-differentiable, then we have

$$(-^h(\ominus p^2)) \odot L[y(t)](\ominus p) \odot y(t_0) \ominus y'(t_0) \oplus ap \odot L[y(t)] -^h a \odot y(t_0) \oplus b \odot L[y(t)] = L[\widetilde{r}(t)]$$

Case IV. If we consider $y(t)$ by using (ii)-differentiable and $y'(t)$ by using (i)-differentiable, then we have

$$(-^h(\ominus p^2)) \odot L[y(t)](\ominus p) \odot y(t_0) -^h y'(t_0) \oplus (-^h \ominus p)a \odot L[y(t)]$$

$$(\ominus a) \odot y(t_0) \oplus b \odot L[y(t)] = L[\widetilde{r}(t)]$$

using this representation for four cases, we have the following examples.

4 example

In this section, we present two examples to illustrate the laplace transform method and also compare the results of this method with other method.

Example 4.1. *Consider the one-dimensional heat Let us consider the second order fuzzy differential equation*

$$\begin{cases} y'' - 3y' + 2y = \widetilde{4} \\ y'(0) = \widetilde{0} \\ y(0) = \widetilde{1} \end{cases}$$

where $\widetilde{1} = (0.8+0.2r, 1.5-0.5r)$ *and* $\widetilde{4} = (3.2+0.8r, 5-r)$*. by using fuzzy laplace transform method, we have:*

$$L[y''] \ominus 3L[y'] \oplus 2L[y] = L[\widetilde{4}]$$

in $(i)-$*differentiable, then by using case(II), we have*

$$L[\underline{y(t,r)}] = (3.2 + 0.8r)\frac{1}{p(p-1)(p-2)} + (0.8 + 0.2r)\frac{p-3}{(p-1)(p-2)}$$

$$L[\overline{y(t,r)}] = (5 - r)\frac{1}{p(p-1)(p-2)} + (1.5 - 0.5r)\frac{p-3}{(p-1)(p-2)}$$

Hence solution is as follows:

$$\underline{y}(t,r) = (3.2 + 0.8r)(\frac{1}{2} - e^t + \frac{1}{2}e^{2t}) + (0.8 + 0.2r)(2e^t - e^{2t})$$

$$\overline{y}(t,r) = (5 - r)(\frac{1}{2} - e^t + \frac{1}{2}e^{2t}) + (1.5 - 0.5r)(2e^t - e^{2t})$$

Now, if we consider $r = 1$*, then*

$$\underline{y}(t,1) = \overline{y}(t,1) = 2 - 2e^t + e^{2t}.$$

By using H-differentiability and Hukuhara differentiability concepts the following results are obtained:

$$\underline{y}(t,r) = (-1.6 - 0.4r)\cosh t - \frac{4}{3}\sinh t + (-0.8 + 0.2r)\cosh 2t - \frac{2}{3}\sinh 2t$$

$$+\frac{1}{3}(e^{2t}-e^{-2t})-\frac{1}{3}(e^{t}-e^{-t})+\frac{1}{3}(5-r)t+\frac{1}{2}(3.2+0.8r)-0.8-0.2r$$

$$\overline{y}(t,r)=-2\cosh t-(1.6+0.4r)\sinh t+\cosh 2t+(0.2r-0.8)\sinh 2t+\frac{5}{2}-\frac{1}{2}r$$

The disadvantage of strongly generalized differentiability of a function with respect to H-differentiability and Hukuhara differentiability seems to be that a fuzzy differential equation has not got a unique solution. So a fuzzy differential equation may have several solutions. the advantage of the existence of these solutions is that we can choose the solution that reflects the behavior of the modelled real-world system, in a better way.

Example 4.2. *consider the initial value problem equation*

$$\begin{cases} y''+4y=\widetilde{4}x \\ y'(0)=0 \\ y(0)=\widetilde{1} \end{cases}$$

where $\widetilde{1}=(0.8+0.2r,1.5-0.5r)$ *and* $\widetilde{4}=(3.2+0.8r,5-r)$. *in* $(ii)-$*differentiable, then by using case(I), we have*

$$L[\underline{y(t,r)}]-p\underline{y}(0,r)-\underline{y}'(0,r)+4L[\underline{y}(t,r)]=L[(3.2+0.8r)t]$$

$$L[\overline{y(t,r)}]-p\overline{y}(0,r)-\overline{y}'(0,r)+4L[\overline{y}(t,r)]=L[(5-r)t]$$

Hence solution is as follows:

$$\underline{y}(t,r)=(0.8+0.2r)(x-\frac{1}{2}sin2t+cos2t)$$

$$\overline{y}(t,r)=(5-r)(\frac{1}{4}t-\frac{1}{8}sin2t)+(1.5-0.5r)cos2t$$

Now, if we consider $r=1$, *then*

$$\underline{y}(t,1)=\overline{y}(t,1)=x-\frac{1}{2}sin2x+cos2x$$

From, examples (4) and (4.2), we see that the solution of a FDE is dependent on the selection of the derivative: the (i)-differentiable or the (ii)-differentiable.

5 Conclusion

Developing fuzzy Laplace transform, we provided solutions to fuzzy two order differential equation which was interpreted by using the strongly generalized differentiability concept. This may confer solutions which have a decreasing length of their support. The efficiency of method was illustrated by a numerical example.

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