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Latus: A New Accelerator for Generating Combined Iterative Methods in Solving Nonlinear Equation

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Abstract

In this paper, we investigate the problem of solving nonlinear equation and present a new family of combined iterative methods by the composition of Latus method and other higher-order iterative methods. Two new sixth and seventh order methods are developed. Meanwhile, the convergence analysis of the new methods is discussed and some examples are given to illustrate its efficiency.

Keywords : Latus method; Order of convergence; Root finding; Iterative methods; Combined iterative methods

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1 Introduction

We consider the problem of finding a numerical method to solve a real root α of nonlinear equation

$$f(x) = 0; \quad f :\subset R \rightarrow R.$$

The best known numerical method for solving the so-called equation is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

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Efforts have resulted into many modifications of Newton's [1-12]. Danfu Han and Peng Wu [13] considered the following iteration as a fifth order method

$$x_{n+1} = z_n - \frac{f'(x_n - \frac{f(x_n)}{f'(x_n)}) + f'(x_n)}{3f'(x_n - \frac{f(x_n)}{f'(x_n)}) - f'(x_n)} \frac{(f(z_n))}{(f'(x_n))}, \tag{1.1}$$

where z_n is defined by

$$z_n = x_n - \left(\frac{2f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})} \right). \tag{1.2}$$

Also they represented a sixth order method

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n) + L(x_n)\left(f(x_n) - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}\right) - f'(x_n))}, \tag{1.3}$$

where z_n and $L(x_n)$ are defined by

$$z_n = x_n - \frac{1}{2}\frac{f(x_n)}{f'(x_n)} + \frac{f(z_n)}{f'(x_n) + L(x_n)\left(f'\left(x_n - \frac{2}{3}\frac{(f(x_n))}{(f'(x_n))}\right) - f'(x_n))\right)}, \tag{1.4}$$

and

$$L(x_n) = \frac{3}{4} - \frac{3}{2}\frac{f'(x_n)}{f'(x_n) - 3f'\left(x_n - \frac{2}{3} \cdot \frac{f(x_n)}{f'(x_n)}\right)}. \tag{1.5}$$

Compared with the other five or six-order methods, (1) and (2) are not required to evaluate second or higher derivatives. Motivated by this fact, we investigate the problem that how can we compose the so called methods with Latus method represented in this paper. We find that, in general, the convergent order will be improved and increased 1 above the original level without the evaluation of the second derivative.

Definition 1.1. *Let the sequence $\{x_n\}$ tend to α such that*

$$\lim_{n\to\infty} \frac{(x_{n+1} - \alpha)}{(x_n - \alpha)^p} = C \neq 0, \quad n \geq 1 \tag{1.6}$$

The order of convergence of the sequence $\{x_n\}$ is p, and C is known as the asymptotic error constant. If $p = 1$, $p = 2$ or $p = 3$, the sequence is said to converge linearly, quadratically or cubically, respectively.It can be concluded that the error of a method with the rate of convergence $p \geq p$ in step n is

$$e_n \leq 10^{-pn} e_{n-1}, \tag{1.7}$$

where

$$e_n = (\alpha - a_n). \tag{1.8}$$

Definition 1.2. *The computational order of convergence can be approximated using the formula [12]*

$$p = \lim_{n\to\infty} \frac{|\ln(x_{n+1} - \alpha)/\ln(x_n - \alpha)|}{|\ln(x_n - \alpha)/\ln(x_{n-1} - \alpha)|}. \tag{1.9}$$

2 Latus method

Let

$$f(\alpha) = 0, \quad \alpha \in [a, b]. \tag{2.10}$$

Adopting a function, named $g(x)$, we suppose:

1) $\forall x_1, x_2 \in [a, b] : f'(x_1).f'(x_2) \geq 0,\ g'(x_1).g'(x_2) \geq 0,$ (Assumption 2.1)

2) $f(x), g(x) : [a, b] \subset R \to R,$ (Assumption 2.2)

3) $\forall x_1, x_2 \in [a, b] :\ f'(x_1).g'(x_2) \leq 0,$ (Assumption 2.3)

4) $\exists x_0 \in [a, b] :\ g(x_0) = f(x_0),$ (Assumption 2.4)

Consider x_0 as the initial point, suppose

$$\forall x_0 \in [a, b], \quad \exists x_1 \in [a, b] : \quad g(x_1) - g(x_0) = f(x_1).$$

Now for $n \geq 1$

$$\forall x_n \in [a, b], \exists x_{n+1} \in [a, b] : \quad g(x_{n+1}) - g(x_n) = f(x_{n+1}).$$

Hence

$$\{(x_n, f(x_n)\}_{n=0}^{\infty}, \quad \lim_{n\to\infty} x_n = \alpha,$$

for arbitrary $\epsilon : |x_{n+1} - x_n| < \epsilon$.

3 Analysis of convergence

The behavior of the convergence of Latus and the combined methods are considered in the following theorems.

Theorem 3.1. *If* $\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} < 1$, *then* $\exists h \in R :\ \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \leq = h < 1$.

Proof: Let

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} = q$$

hence $1 - q > 0$. Also, let $1 - q = \delta$, then $q + \delta d = 1$, so we have $q \leq = q + \delta dr$, and $0 \leq r < 1$, hence, $h = q + \delta dr$. In conclusion, according to assumption of Theorem (3.1) $\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} \leq h < 1$. Then the proof is completed.

Theorem 3.2. *According to (Assumption 2.1) and (Assumption 2.2), for* $\{(x_n, f(x_n)\}_{n=0}^{\infty}$ *we have*

$$\lim_{n\to\infty} x_n = \alpha. \tag{3.11}$$

Proof: For the points $(\alpha, 0)$, $(x_n, 0)$ and $m = (x_{n+1}, f(x_{n+1}) = g(x_{n+1}))$ according to (Assumption-2.2) we have

$$f'.g' \leq 0 \to \begin{cases} f' \geq 0, & g' \leq 0 \\ f' \leq 0, & g' \geq 0 \end{cases},$$

for $f' \geq 0$ and $g' \leq 0$, we obtain

$$f' \geq 0 \rightarrow \frac{f(x_{n+1})}{(x_{n+1} - \alpha)} \geq 0$$

and

$$g' \leq 0 \rightarrow \frac{(-g(x_{n+1}))}{(x_n - x_{n+1})} \leq 0$$

hence

$$\frac{f(x_{n+1})}{(x_{n+1} - \alpha a)} . \frac{(-g(x_{n+1}))}{(x_n - x_{n+1})} \leq 0,$$

and

$$\frac{(-f(x_{n+1})^2)}{((x_{n+1} - \alpha).(x_n - x_{n+1}))} \leq 0.$$

Therefore

$$(x_{n+1} - \alpha).(x_n - x_{n+1}) > 0 \rightarrow \begin{cases} (x_{n+1} - \alpha) > 0, & (x_n - x_{n+1}) > 0 \\ (x_{n+1} - \alpha) < 0, & (x_n - x_{n+1}) < 0 \end{cases}$$

then,

$$\begin{cases} x_{n+1} > \alpha, & x_n > x_{n+1} \\ x_{n+1} < \alpha, & x_n < x_{n+1} \end{cases} \rightarrow \begin{cases} \alpha < x_{n+1} < x_n \\ x_n < x_{n+1} < \alpha \end{cases}$$

hence,

$$|\alpha - x_n| > |\alpha - x_{n+1}|$$

so,

$$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|} < 1$$

Using Theorem (3.1), we have

$$\exists h \in R : \frac{|\alpha - x_{n+1})|}{|\alpha - x_n|} \leq h < 1,$$

therefore

$$|\alpha - x_1| \leq h|\alpha - x_0|,$$
$$\vdots$$
$$|\alpha - x_n| \leq h|\alpha - x_{n+1}|$$

Having multiplied these respects, we have $|\alpha a - x_n| \leq h^n|\alpha a - x_0|$, and also we have $0 < h < 1$ then

$$\lim_{n\to\infty}(h^n) = 0$$

consequently $\lim_{n\to\infty}(\alpha - x_n) = 0$, hence

$$\lim_{n\to\infty}(x_n) = \alpha$$

So, the proof is complete.
The same result is drawn by the assumption of

$$f' \leq 0, \quad g' \geq 0.$$

Theorem 3.3. *The point m if exist, is unique.*

Proof: Suppose that f(x) and g(x) have $n \geq 2$ number of intersection points. Without the loss of generality, let two points A and B be distinct consecutive intersection points. According to Mean Value Theorem we have

$$\exists \beta \in [a,b]: \quad f(B) - f(A) = f'(\beta)(B - A),$$
$$\exists \gamma \in [a,b]: \quad g(B) - g(A) = g'(\gamma)(B - A),$$

therefore

$$\frac{f(B) - f(A)}{B - A} = \frac{g(B) - g(A)}{B - A} \neq 0, \tag{3.12}$$

and

$$\frac{f(B) - f(A)}{B - A} = \frac{g(B) - g(A)}{B - A} = 0. \tag{3.13}$$

According to (3.12), we have $f'(\beta) = g'(\gamma) \neq 0$ then, $f'(\beta).g'(\gamma) > 0$. So, it does not satisfy the (Assumption 2.2). According to (3.13) and Rolle's Theorem, we have

$$f(A) = f(B),\ g(A) = g(B) \rightarrow \begin{cases} \exists m_1, m_2 \in [a,b] : f'(m_1).f'(m_2) < 0 \\ \exists m_3, m_4 \in [a,b] : g'(m_3).g'(m_4) < 0 \end{cases}$$

And it means they are not monotonic. Therefore, $n \leq 1$. So, the proof is complete.

Theorem 3.4. *Considering the following condition,*

- $tan\theta_1 = \frac{f(x_{n+1})}{L_1}$,
- $tan\theta_3 = \frac{f(x_{n+1})}{L_3}$,
- $k = \frac{(tan\theta_1)}{(tan\theta_3)}$,
- $|\alpha - x_{n+1}| = L_3$,
- $|\alpha - x_n| = L_1 + L_3$.

Then the order convergence of Latus method is 1.

Proof: According to the assumption and Eq. (1.9), we have

$$p \cong \frac{\ln |1/\left(1 + \frac{1}{k}\right)|}{\ln |1/\left(1 + \frac{1}{k}\right)|} = 1 \tag{3.14}$$

Consider $\{c_n\}$ as a sequence of the points achieved by a p= p order iterative method. Then

$$|a_1 - a_0| = k'_0(\alpha - a_0)$$

then

$$|\alpha - a_1| = (1 - k'_0).(\alpha - a_0).$$

Therefore, according to (1.7) and (1.8), we have

$$k'_n = 1 - 10^{-p^n} + 10^{-p^n} k'_0, \quad n = 0, 1, .. \tag{3.15}$$

and

$$k_n = 1 - 10^{-1} + 10^{-1} k_0, \quad n = 0, 1, .. \tag{3.16}$$

and the proof is complete.

Theorem 3.5. *Suppose $\{x_n\}$ as the sequence of points achieved by a combinational method. If the rate of convergence of the SS method is p, and the order of convergence of the OS method is 1, then the order of convergence of a combinational method is $1 + p^n$.*

Proof: For the step n of the combinational method we have

$$\begin{aligned} c_{n+1} - c_n &= (c'_n - c_n) + (c_{n+1} - c'_n) \\ &= k_n(\alpha - c_n) + k'_n(\alpha - (c_n + k_n(\alpha - c_n))) \\ &= (k'_n(1 - k_n) + k_n).(\alpha - c_n). \end{aligned}$$

Considering (3.15), (3.16), we have

$$\begin{aligned} (k'_n(1 - k_n) + k_n) &= (1 - 10^{-p^n} + 10^{-p^n} k'_0)(1 - (1 - 10^{-1} + 10^{-1}k_0)) + (1 - 10^{-1} + 10^{-1}k_0) \\ &= 10^{-1-p^n}(-1 + k_0 + k'_0 - k'_0 k_0) + 1 \end{aligned}$$

Let $k'_n(1 - k_n) + k_n = k''_n$, hence

$$10^{-1-p^n}(-1 + k''_0) + 1 = 1 - 10^{-1-p^n} + 10^{-1-p^n})k''_0$$

Therefore

$$p^n < 1 + p^n \leq (1 + p)^n.$$

The proof is complete.

Finally, the following iterative methods have been presented as the combinations of Latus method and the so called methods in [13].

$$\forall x_n \in [a, b], \quad \exists\ x'_n \in [a, b] : \quad g(x'_n) - g(x_n) = f(x'_n) \tag{3.17}$$

and

$$x_{n+1} = z_n - \frac{f'\left(x_n - \frac{f(x'_n)}{f'(x'_n)}\right) + f'(x_n))}{3f'(x_n - f(x'_n)/f'(x'_n)) - f'(x_n)}.\frac{(f(z_n))}{(f'(x'_n))}, \tag{3.18}$$

where z_n is defined by

$$z_n = x'_n - \frac{2f(x'_n)}{f'(x'_n) + f'\left(x'_n - \frac{f(x'_n)}{f'(x'_n)}\right)}. \tag{3.19}$$

And Also

$$\forall x_n \in [a, b], \quad \exists\ x'_n \in [a, b] : \quad g(x'_n) - g(x_n) = f(x'_n) \tag{3.20}$$

and

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x'_n) + L(x'_n)(f'(x'_n - 2/3\frac{f(x'_n)}{f'(x'_n)} - f'(x'n))} \tag{3.21}$$

where z_n and $L(x_n)$ are defined by

$$z_n = x_n - \frac{1}{2}\frac{f(x'n)}{f'(x'n)} + \frac{f(z_n)}{f'(x'_n) + L(x'_n).\left(f'\left(x'_n - \frac{2}{3}\frac{(f(x'_n))}{(f'(x'_n))}\right) - f'(x'_n))\right)} \tag{3.22}$$

and

$$L(x_n) = \frac{3}{4} - \frac{3}{2}.\frac{f'(x'_n)}{f'(x'n) - 3f'\left(x'n - \frac{2}{3}\frac{f(x'_n)}{f'(x'_n)}\right)} \tag{3.23}$$

4 Numerical results and conclusion

We have proved that it is possible to obtain a class of iterative methods with higher convergence order (more than three) by the composition of Latus and some other different methods. The present iterative formulas defined by (1.1) and (1.3) only add one evaluation of the function at another iterated point, while their order of convergence can be improved effectively. The most important characteristic of such methods is that they are not required to evaluate second or higher derivatives of the function in iterative processes. Thus the complexity of calculation is decreased greatly especially in high dimensioned case.

Finally, we have applied our new iterative methods to the following two examples, and have chosen Newton's method and Kou's method named NNM and NJM which are defined in [13] to compare with our methods (3.18)(LNNM) and (3.21)(LNJM).The stopping criterion used is $0 < |x_{n+1} - x_n| < 10^{-16}$. One of the possible functions of $g(x)$ is given in front of each function. All numerical tests agree with the theoretical results of this work.

Table 1
The numerical results of applying the iterative methods.

$Function$	x_0	i				
		N	NNM	NJM	$LNNM$	$LJNM$
(a)	-2	220	7	5	4	2
(b)	2	6	4	3	2	2

Test functions

(a) $f(x) = xe^{x^2} - sin^2x + 3cosx + 5$, $g(x) = f(x) - 10x^2$, $\alpha = -1.207647827130919$,

(b) $f(x) = sin^2x - x^2 + 1$, $g(x) = sin^2(x)$, $\alpha = -1.404491648215341$.

$i-$ the number of iterations to approximate the root to 16 decimal places.

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