A Generalized Model in the Performance Evaluation of Decision Making Sub-units

Z. Iravani *

Department of Mathematics, Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.

Received 28 July 2010; accepted 11 October 2010.

Abstract
Data Envelopment Analysis (DEA) evaluates the efficiency of decision making units with multiple inputs and outputs. So far, a number of DEA models have been developed: The CCR model, the BCC model and the FDH model are well known as basic DEA models. In many instance, However, the decision making units can be separated into different sub-units. In this paper, we study a generalized model for this DMUs by different sub-units.

Keywords: Data envelopment analysis; Decision making units; Sub-units; Efficiency; Generalized model

1 Introduction

Data Envelopment Analysis (DEA), originally proposed by Charnes, Cooper and Rodes (1978 and 1979)[1], has become one of the most widely used methods in management science. DEA measures the relative efficiency of comparable entities called Decision making units (DMUs) essentially performing the same task using similar multiple inputs to produce similar multiple outputs. The purpose of DEA is to empirically estimate the so-called efficient frontier based on the set of available DMUs. A DMU is efficient if there is no other unit-existing or virtual that can either produce more outputs by consuming the same amount or less of inputs or produce the same amount or more of outputs by consuming less or the same amount of inputs as the DMU under consideration. The former approach is referred to as the output oriented and the latter as the input oriented DEA. DAE provides the user with information about the efficient and inefficient units, as well as the efficiency scores and reference sets for inefficient units. The result of the DEA analysis, especially the efficiency scores, had practical applications as performance indicators of DMUs.

*Email address: zohrehiravani@yahoo.com, Tel: 09122263343
In many instances however, the decision making units can be separated into different sub-units. For example, grosskopf [3], for example, look at a multi-stage process where in intermediate product or output at one stage can be both final products and inputs to the later stages of production. Those authors are not explicitly interested in obtaining measures of efficiency at each stage, but rather are concerned with overall efficiency measurement. Another example is due to cook et al [4] and involves multi-component efficiency with shared inputs.

In this paper, we propose a generalized model for DEA when DMUs has sub-units, which can treat basic DEA models for this DMUs, specifically, the CCR model, the BCC model and the FDH model in a unified way. In addition, we show theoretical properties on relationships among this model and those DEA models by sub-units, and this model makes it possible to calculate the efficiency of DMUs incorporating various preference structure of decision makers.

The following sections of the paper provide a sub-units efficiency measurement.

2 Basic DEA models for sub-units

Assume that we have n DMUs, and a DMUj consists of b sub-units, called DMSU. Each DMUj transforms resources, or inputs into products, or outputs in particular, DMUj, 2 ≤ j ≤ b − 1, produces kj different types of outputs and consumes Ij types of external inputs and Ij types of internal inputs (i.e. a part of inputs coming from outside the whole DMU and the other part coming from inside the DMU). The internal input of DMSUj is output produced by the last DMSUj. The first DMSU1 consumes the input vector X1 and produces the output vector Y1 and the last DMSUb consumes the internal input vector Xb and the external input vector Xb produces the output vector Yb. All the DMSUj considered have the same types of outputs and internal and external inputs. Especially, DMSUj, 2 ≤ j ≤ b, consumes Ij types of external inputs Xj and Ij types of internal inputs Xj = Yj−1. Also DMSUj, 2 ≤ j ≤ b, produces kj types of outputs Yj. See the fig. 1.

For notational purpose, let yj,p, j = 1, . . . , b, denote the output vectors produced by jth sub-DMU of DMUj, in which

\[ Y_j^{(p)} = (y_j^{(p)}(1), \ldots, y_j^{(p)}(k_j)), \]

Also, let \( X_j^{(p)} \) and \( \bar{X}_j^{(p)} \), j = 2, . . . , b, denote Ij and Ij-dimensional vectors of external and internal inputs to jth sub-DMU of DMUj, respectively, in which

\[ X_j^{(p)} = (x_j^{(p)}(1), \ldots, x_j^{(p)}(I_j)), \]
\[ \bar{X}_j^{(p)} = (\bar{x}_j^{(p)}(1), \ldots, \bar{x}_j^{(p)}(I_j)) = (y_j^{(p)}(I_j + 1), \ldots, y_j^{(p)}(I_j + I_j)). \]

Hence, a measure of aggregate performance \( e_p^{(a)} \) can be represented by

\[ e_p^{(a)} = \frac{\mu_1^{(1)}T^1_1y_1^{(p)} + \mu_2^{(2)}T^2_2y_2^{(p)} + \cdots + \mu_b^{(b)}T^b_by_b^{(p)}}{\nu_1^{(1)}T^1_1x_1^{(p)} + \nu_2^{(2)}T^2_2x_2^{(p)} + \cdots + \nu_b^{(b)}T^b_bx_b^{(p)}}. \]
and performance for each sub-units of DMUp can be represented by

\[
e_p^{(1)} = \frac{\mu^{(1)}_{T} y_1^{(p)}}{v^{(1)}_{T} x_1^{(p)}},
\]

\[
e_p^{(i)} = \frac{\mu^{(i)}_{T} y_i^{(p)}}{v^{(i)}_{T} x_i^{(p)} + \mu^{(i-1)}_{T} y_{i-1}^{(p)}}, \quad i = 2, \ldots, b
\]

**Theorem 2.1.** The aggregate efficiency \(e_p^{(a)}\) is a convex combination of DMSU’s efficiency.

**Proof:** The proof is in [2].

**Theorem 2.2.** DMUp is efficiency iff all of DMUSp are efficiency.

**Proof:** The proof is straightforward.

Then we have the following mathematical programming problem:

\[
\begin{align*}
\max & \quad e_p^{(a)} \\
\text{s.t.} & \quad e_j^{(a)} \leq 1, \quad j = 1, \ldots, n \\
& \quad e_j^{(i)} \leq 1, \quad j = 1, \ldots, b, \quad j = 1, \ldots, n \\
& \quad \mu^{(i)} \in \Omega_1, \quad i = 1, \ldots, b \\
& \quad (v^{(i)}, \pi^{(i)}) \in \Omega_2, \quad i = 1, \ldots, b
\end{align*}
\]  

(2.1)

The sets \(\Omega_1\) and \(\Omega_2\) are assurance regions defined by any restrictions imposed on multipliers [4]. The model (2.1) can be expressed in the following form

\[
\begin{align*}
\max & \quad \sum_{i=1}^{b} \mu^{(i)T} y_i^{(p)} \\
\text{s.t.} & \quad \sum_{j=1}^{b} v^{(i)T} x_i^{(p)} + \sum_{j=1}^{b-1} \pi^{(i)T} y_{i-1}^{(p)} = 1, \\
& \quad \sum_{j=1}^{b} \mu^{(i)T} y_i^{(j)} - \sum_{j=1}^{b} v^{(i)T} x_i^{(j)} - \sum_{i=1}^{b-1} \pi^{(i)T} y_i^{(j)} \leq 0, \quad j = 1, \ldots, n \\
& \quad \mu^{(i)T} y_i^{(j)} - v^{(i)T} x_i^{(j)} - \pi^{(i-1)T} y_{i-1}^{(j)} \leq 0, \quad i = 2, \ldots, b, \quad j = 1, \ldots, n \\
& \quad \mu^{(i)} \in \Omega_1, \quad i = 1, \ldots, b \\
& \quad (v^{(i)}, \pi^{(i)}) \in \Omega_2, \quad i = 1, \ldots, b
\end{align*}
\]  

(2.2)
The form of $\Omega_1$ and $\Omega_2$ depends on how $\bar{\Omega}_1$ and $\bar{\Omega}_2$ are structured. Now the CCR model in present DMU’s follows as:

$$\begin{align*}
\text{Min} & \quad \theta \\
\text{S.t.} & \quad \sum_{j=1}^{n} \lambda_j x_i^{(j)} + \sum_{j=1}^{n} \lambda_{ij} x_i^{(j)} \leq \theta x_i^{(j)}, \quad i = 1, \ldots, b \\
& \quad \sum_{j=1}^{n} \lambda_j y_i^{(j)} + \sum_{j=1}^{n} \lambda_{ij} y_i^{(j)} \leq \theta y_i^{(p)}, \quad i = 1, \ldots, b-1 \\
& \quad \sum_{j=1}^{n} \lambda_j y_i^{(j)} - \sum_{j=1}^{n} \lambda_{ij} y_i^{(j)} \geq y_i^{(p)}, \quad i = 1, \ldots, b \\
& \quad \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& \quad \lambda_{ij} \geq 0, \quad i = 1, \ldots, b \quad j = 1, \ldots, n.
\end{align*}$$

(2.3)

And the BCC model in present DMU’s follows as:

$$\begin{align*}
\text{Min} & \quad \theta \\
\text{S.t.} & \quad \sum_{j=1}^{n} \lambda_j x_i^{(j)} + \sum_{j=1}^{n} \lambda_{ij} x_i^{(j)} \leq \theta x_i^{(j)}, \quad i = 1, \ldots, b \\
& \quad \sum_{j=1}^{n} \lambda_j y_i^{(j)} + \sum_{j=1}^{n} \lambda_{ij} y_i^{(j)} \leq \theta y_i^{(p)}, \quad i = 1, \ldots, b-1 \\
& \quad \sum_{j=1}^{n} \lambda_j y_i^{(j)} - \sum_{j=1}^{n} \lambda_{ij} y_i^{(j)} \geq y_i^{(p)}, \quad i = 1, \ldots, b \\
& \quad \sum_{j=1}^{n} \lambda_j + \sum_{i=1}^{b} \sum_{j=1}^{n} \lambda_{ij} = 1, \\
& \quad \lambda_j \geq 0, \quad j = 1, \ldots, n \\
& \quad \lambda_{ij} \geq 0, \quad i = 1, \ldots, b \quad j = 1, \ldots, n.
\end{align*}$$

(2.4)
The multiplier form of the BCC model in present DMSU follows as:

\[
\begin{align*}
\text{Max} & \quad \sum_{i=1}^{b} \mu^{(i)} T y_{i}^{(p)} + u_{0} \\
\text{S.t.} & \quad \sum_{i=1}^{b} v^{(i)} T x_{i}^{(p)} + \sum_{i=1}^{b-1} \pi^{(i)} T y_{i}^{(p)} = 1, \\
& \quad \sum_{i=1}^{b} \mu^{(i)} T y_{i}^{(j)} - \sum_{i=1}^{b} v^{(i)} T x_{i}^{(j)} - \sum_{i=1}^{b-1} \pi^{(i)} T y_{i}^{(j)} + u_{0} \leq 0, j = 1, \ldots, n \\
& \quad \mu^{(i)} T y_{i}^{(j)} - v^{(i)} T x_{i}^{(j)} - \pi^{(i-1)} T y_{i-1}^{(j)} + u_{0} \leq 0, \quad i = 2, \ldots, b, j = 1, \ldots, n \\
& \quad \mu^{(i)} \in \Omega_1 \quad i = 1, \ldots, b \\
& \quad (v^{(i)}, \pi^{(i)}) \in \Omega_2 \quad i = 1, \ldots, b
\end{align*}
\]

(2.5)

Fig. 1. The DMUs by DMSUs
3 A generalized model in present sub-units

In this section, we formulate the generalized model in present sub-units, based on a domination structure and define a new efficiency in this model. Next we establish relationships between this generalized model and basic DEA models mentioned in section 2. Now, we formulate a generalized DEA model in present sub-units by employing the augmented Tchebyshhev secularizing function [5]. This model, which can evaluate the efficiency in several basic models which are special cases for all DMUs, follows as:

\[ \text{Max} \quad \Delta \]

\[ \text{S.t.} \quad \Delta \leq \tilde{d}_j + \alpha \left( \sum_{i=1}^{b} \mu^{(i)} (y_i^{(p)} - y_i^{(j)}) + \sum_{i=1}^{b} v^{(i)} T \left( -x_i^{(p)} + x_i^{(j)} \right) + \sum_{i=1}^{b-1} \varpi^{(i)} \left( -y_i^{(p)} + y_i^{(j)} \right) \right) \]

\[ \mu^{(i)} y_i^{(j)} - v^{(i)} T x_i^{(j)} - \varpi^{(i-1)} T y_{i-1}^{(j)} \leq 0, \quad i = 2, \cdots, b, \quad j = 1, \cdots, n \]

\[ \sum_{i=1}^{b} \mu^{(i)} + \sum_{i=1}^{b} v^{(i)} + \sum_{i=1}^{b-1} \varpi^{(i)} = 1 \]

\[ \mu^{(i)} \geq 0, \quad i = 1, \cdots, b \]

\[ v^{(i)} \geq 0, \quad i = 1, \cdots, b \]

\[ \varpi^{(i)} \geq 0, \quad i = 1, \cdots, b - 1 \]

where \( \alpha > 0 \) is appropriately given according to given problems, and \( \tilde{d}_j (j = 1, \cdots, n) \) is defined by following:

\[ \tilde{d}_j = \max_{i=1, \cdots, b, \varpi^{(i)}_{b-1}} \left\{ \mu^{(i)} (y_i^{(p)} - y_i^{(j)}), v^{(i)} (-x_i^{(p)} + x_i^{(j)}) + \varpi^{(i)} (-y_i^{(p)} + y_i^{(j)}) \right\} \quad (3.6) \]

Note that when \( j = p \) then \( \Delta \leq 0 \).

**Definition 3.1. (\( \alpha \)-efficiency)** For a given positive number \( \alpha \), DMU\(_o\) is defined to be \( \alpha \)-efficiency if and only if the optimal value to the problem (3.6) is equal to zero. Otherwise, DMU\(_o\) is said to be \( \alpha \)-inefficiency.

**Theorem 3.1.** If \( \Delta \neq 0 \) the existence DMU where dominated DMUp.

**Proof:** Let \( \Delta \neq 0 \), by contradiction suppose that there is not DMU where dominated DMUp.

On the other hand, for all \( j \) we have

\[
\begin{bmatrix}
Y^{(j)} \\
-X^{(j)} \\
-X^{(j)}
\end{bmatrix}
\leq
\begin{bmatrix}
Y^{(p)} \\
-X^{(p)} \\
-X^{(p)}
\end{bmatrix}.
\]
We denote \( Z_j = \begin{bmatrix} Y^{(j)} \\ -X^{(j)} \\ -X^{(j)} \end{bmatrix} \). Therefore

\[
Z^{(j)} \leq Z^{(p)} \quad (\forall j).
\]

And from inequalities of the model (3.6) in present sub-units for all DMUs we have

\[
\Delta \leq \tilde{d}_j + \alpha \left( \sum_{i=1}^{b} \mu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) + \sum_{i=1}^{b} v_i^{(i)} (-x_i^{(p)} + x_i^{(j)}) + \sum_{i=1}^{b} \nu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) \right)
\]

But, \( \Delta < 0 \), and if free variable, then necessary and sufficient condition for existence above inequality (for some \( j \neq p \)) is

\[
\tilde{d}_j + \alpha \left( \sum_{i=1}^{b} \mu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) + \sum_{i=1}^{b} v_i^{(i)} (-x_i^{(p)} + x_i^{(j)}) + \sum_{i=1}^{b} \nu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) \right) < 0
\]

We have

\[
\tilde{d}_j + \alpha (\mu, \nu, \nu) \begin{bmatrix} Y_i^{(p)} - Y_i^{(j)} \\ -X_i^{(p)} + X_i^{(j)} \\ -X_i^{(p)} + X_i^{(j)} \end{bmatrix} < 0 \quad (\text{for some } j \neq p).
\]

That is we have the following

\[
\tilde{d}_j + \alpha (\mu, \nu, \nu)(Z_p^{(i)} - Z_j^{(i)}) < 0 \quad (\text{for some } j \neq p).
\]

Now by (3.8) and \( \alpha > 0 \) and \((\mu, \nu, \nu) \geq 0\) we must have

\[
\tilde{d}_j < 0 \quad (\text{for some } j \neq p).
\]

And by definition \( \tilde{d}_j \) (for some \( j \neq p \)) we have:

\[
\tilde{d}_j = \max_{i=1, \ldots, b; m=1, \ldots, b-1} \left\{ \mu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) + v_i^{(i)} (-x_i^{(p)} + x_i^{(j)}) + \nu_i^{(i)} (y_i^{(p)} - y_i^{(j)}) \right\} < 0.
\]

Hence by \((\mu, \nu, \nu) \geq 0\), \(Z_p^{(i)} - Z_j^{(i)} < 0\) (for some \( j \neq p \)). Where contradiction by (3.8). This contradiction asserts that there is not existence DMU where dominated \( DMU_p \), and the proof is complete.

4 Relationships between generalized model and BCC (CCR) model in present sub-units

In this section, we establish theoretical properties on relationships among efficiencies in the basic DEA model and generalized model in present sub-units.

**Theorem 4.1.** \( DMU_p \) is BCC-efficiency in present sub-units if and only if \( DMU_p \) is \( \alpha \)-efficiency for some sufficiently large positive number \( \alpha \).
**Proof:** Suppose that DMUp is $\alpha$-efficient for some sufficiently large positive $\alpha$. That is for all optimal solution we have:

$$0 = \Delta^* \leq \tilde{d}_j + \alpha(\mu, v, \overline{\pi})(Z_p - Z_j).$$

The necessary and sufficient condition for some sufficiently large positive number $\alpha$ for this inequality follows as:

$$
\begin{cases}
Z_p - Z_j \geq 0 \\
\tilde{d}_j \geq 0
\end{cases}
$$

Then we have

$$(\mu^*, v^*, \overline{\pi}^*)(Z_p - Z_j) \geq 0.$$ 

Therefore

$$(\mu^*, v^*, \overline{\pi}^*)Z_p - (\mu^*, v^*, \overline{\pi}^*)Z_j \geq 0. \tag{4.10}$$

Suppose that $(v, \overline{\pi}) \left[ \begin{array}{c} X^p \\ \overline{X}^p \end{array} \right] = \gamma$ then $(v^*, \overline{\pi}^*) \left[ \begin{array}{c} X^p \\ \overline{X}^p \end{array} \right] = 1.$

We denote $u_0^* = -(\mu^*, v^*, \overline{\pi}^*)Z_p$. Hence by (4.10) we have

$$(\mu^*, v^*, \overline{\pi}^*)Z_j + u_0^* \leq 0 \Rightarrow \left( \frac{\mu^*}{\gamma}, \frac{v^*}{\gamma}, \frac{\overline{\pi}^*}{\gamma} \right)Z_j + \frac{u_0^*}{\gamma} \leq 0$$

Therefore $(\frac{\mu^*}{\gamma}, \frac{v^*}{\gamma}, \frac{\overline{\pi}^*}{\gamma})$ is a feasible solution for BCC model, in present sub-units where the value of objective function is one. Then DMUp is efficient in present DMU’s. Now by additional restriction $\sum_{i=1}^{b} \mu^{(i)T}(y_i^{(p)}) = \sum_{i=1}^{b} v^{(i)T}(x_i^{(j)}) + \sum_{i=1}^{b-1} \overline{\pi}^{(i)}(y_i^{(p)})$ we study the generalized model in present sub-units for all DMUs

**Max** $\Delta$

**S.t.** $
\Delta \leq \tilde{d}_j + \alpha\left( \sum_{i=1}^{b} \mu^{(i)T}(y_i^{(p)}) - y_i^{(j)} \right) + \sum_{i=1}^{b} v^{(i)T}(-x_i^{(p)} + x_i^{(j)}) + \sum_{i=1}^{b-1} \overline{\pi}^{(i)}(-y_i^{(p)} + y_i^{(j)})$

\begin{align*}
\sum_{i=1}^{b} \mu^{(i)}(y_i^{(p)}) &= \sum_{i=1}^{b} v^{(i)}T(x_i^{(p)}) + \sum_{i=1}^{b-1} \overline{\pi}^{(i)}(y_i^{(p)}) \\
\mu^{(i)T}y_i^{(j)} - v^{(i)T}x_i^{(j)} - \overline{\pi}^{(i-1)}y_i^{(j)} &\leq 0, \\
\sum_{i=1}^{b} \mu^{(i)} + \sum_{i=1}^{b} v^{(i)} + \sum_{i=1}^{b-1} \overline{\pi}^{(i)} &= 1 \\
\mu^{(i)} &\geq 0, \quad i = 1, \ldots, b \\
v^{(i)} &\geq 0, \quad i = 1, \ldots, b \\
\overline{\pi}^{(i)} &\geq 0, \quad i = 1, \ldots, b - 1,
\end{align*}

\tag{4.11}

**Theorem 4.2.** DMUp is CCR-efficient if and only if DMUp is $\alpha$-efficient for sufficient large positive $\alpha$ is present sub-units by (3.8) model.
Proof: Suppose that DMUp is $\alpha$-efficient for some sufficient large positive $\alpha$. That is for all solution $(\hat{\Delta}, \hat{\mu}, \nu^i, \hat{\tau})$ we have $\hat{\Delta} = 0$

$$0 = \hat{\Delta} \leq \tilde{d}_j + \alpha(\hat{\mu}, \hat{\nu}, \hat{\tau})(Z^{(p)} - Z^{(j)})$$

The necessary and sufficient condition for this formula is

$$\begin{cases} 
Z^{(p)} - Z^{(j)} \geq 0 \\
\tilde{d}_j \geq 0 \forall j
\end{cases} \tag{4.12}$$

But, we suppose that $\hat{\nu}x^{(p)} + \hat{\tau}\varpi^{(p)} = \beta$ then

$$\frac{\hat{\nu}}{\beta}x^{(p)} + \frac{\hat{\tau}}{\beta}\varpi^{(p)} = 1. \tag{4.13}$$

Now $\hat{\mu}y^{(p)} = \hat{\nu}x^{(p)} + \hat{\tau}\varpi^{(p)}$ therefore $\frac{\hat{\mu}}{\beta}y^{(p)} = \frac{\hat{\nu}}{\beta}x^{(p)} + \frac{\hat{\tau}}{\beta}\varpi^{(p)}$ and by (4.13) we have

$$\frac{\hat{\mu}}{\beta}y^{(p)} = 1 \tag{4.14}$$

and by (4.12) we have

$$(\hat{\mu}, \hat{\nu}, \hat{\tau})(Z^{(p)} - Z^{(j)}) \geq 0$$

$$(\hat{\mu}, \hat{\nu}, \hat{\tau})Z^{(p)} - (\hat{\mu}, \hat{\nu}, \hat{\tau})Z^{(j)} \geq 0$$

Hence

$$-(\hat{\mu}, \hat{\nu}, \hat{\tau})Z^{(j)} \geq 0.$$  

Then

$$\frac{\hat{\mu}}{\beta}y^{(j)} - \frac{\hat{\nu}}{\beta}x^{(j)} - \frac{\hat{\tau}}{\beta}\varpi^{(j)} \leq 0$$

and

$$\sum_{i=1}^{b} \frac{\hat{\nu}^{(i)T}}{\beta}y^{(j)} - \sum_{i=1}^{b} \frac{\hat{\nu}^{(i)T}}{\beta}x^{(j)} - \sum_{i=1}^{b-1} \frac{\hat{\tau}^{(i)T}}{\beta}\varpi^{(j)} \leq 0, \quad j = 1, \ldots, n.$$ 

Therefore $(\frac{\hat{\mu}}{\beta}, \frac{\hat{\nu}}{\beta}, \frac{\hat{\tau}}{\beta})$ is a feasible solution for CCR model in present sub-units and the value of objective function is $\frac{\hat{\mu}}{\beta}y^{(p)} = 1$. Then DMUp is efficient in present sub-units.

5 Conclusion

In this paper, we have suggested the GDEA model for performance evaluation based on parametric domination structure and defined $\alpha-$efficiency in the GDEA model. The method presented here can be used for the analysis of any real situation where a DMU is separated in to several different sub-units. Then we explain relationship between generalized model and BCC(CCR) model in present sub-units.
References


