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Two Modified Jacobi Methods for M-Matrices

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Abstract

In 2009 B. Zheng et al. proposed two modified Gauss-Seidel (MGS) methods for linear system with M-matrices. In this paper, we use the preconditioners introduced by B. Zheng et al . For modified Jacobi method. The comparison theorems and numerical examples show that the proposed methods are superior to the classical Jacobi method.

Keywords : Iterative methods; M-matrix; Preconditioning; Jacobi method.

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1 Introduction

We consider the following preconditioned linear system

$$PAX = Pb \tag{1.1}$$

where $A = (a_{ij}) \in R^{n\times n}$ is an M-matrix, $P \in R^{n\times n}$ is a preconditioner and $X, b \in R^n$ are vectors. Without loss of generality, we assume that A has a splitting of the form $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively.

The preconditioner $P_{S_{max}}$ was introduced by Kotakemori et al. [4] as follows:

$$P_{S_{max}} = I + S_{max} \tag{1.2}$$

where S_{max} is defined by

$$S_{max} = (S_{ij}^m) = \begin{cases} -a_{i,k_i} & i = 1\ldots, n-1, j > i; \\ o, & Otherwise, \end{cases} \tag{1.3}$$

$$k_i = min\left\{j|max\left|a_{ij}\right|, i < n\right\}.$$

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In 2009 Zheng et al. [19] proposed the following two preconditioners:

$$P_{max} = I + S_{max} + R_{max} \tag{1.4}$$

and

$$P_R = I + S_{max} + R, \tag{1.5}$$

where

$$(R_{max})_{i,j} = \begin{cases} -a_{n,k_n} & i = n, j = k_n, \\ o, & OtherWise \end{cases} \tag{1.6}$$

with $k_n = min\{j||a_{n,j}| = max\{|a_{n,l}|, l = 1, \ldots, n-1\}\}$ and

$$R_{i,j} = \begin{cases} -a_{i,j}, & i = n, 1 \le j \le n-1, \\ 0, & OtherWise. \end{cases} \tag{1.7}$$

For the preconditioner (1.4), the preconditioned matrix $A_{max} = (I + S_{max} + R_{max})A$ can be split as follows:

$$\begin{aligned} A_{max} &= M_{max} - N_{max} & (1.8) \\ &= (I - D - \acute{D}) - (L - R_{max} + E + \acute{E} + U - S_{max} + F + S_{max}U) & (1.9) \end{aligned}$$

where D, E and F are respectively the diagonal, strictly lower and strictly upper triangular parts of $S_{max}L$, while $\acute{D}$ and $\acute{E}$ are the diagonal, strictly lower triangular parts of $R_{max}(L+U)$. If M_{max} is nonsingular, the modified Jacobi iterative matrix is defined by:

$$T_{max} = M_{max}^{-1}N_{max} = (I - D - \acute{D})^{-1}(L - R_{max} + E + \acute{E} + U - S_{max} + F + S_{max}U)$$

For the preconditoner (1.5), the preconditioned matrix $A_R = (I + S_{max} + R)A$ can be split as

$$A_R = M_R - N_R = (I - D - \acute{D}) - (L - R + E + \acute{E} + U - S_{max} + F + S_{max}U), \tag{1.10}$$

where $\acute{D}$ and $\acute{E}$ are the diagonal, strictly lower triangular parts of $R(L+U)$. If M_R is nonsingular, the modified Jacobi iterative matrix is defined by

$$T_R = M_R^{-1}N_R = (I - D - \acute{D})^{-1}(L - R + E + \acute{E} + U - S_{max} + F + S_{max}U).$$

This paper is organized as follows. In section 2, we present some notations, definitions and preliminary results. In section 3, we prove the convergence of the proposed methods and some comparison theorems. In section 4 we present some numerical examples to confirm our theoretical analysis. Finally, in Section 5, conclusion is drown.

2 Preliminaries

For $A = (a_{i,j}), B = (b_{i,j}) \in R^{n\times n}$, we write $A \ge B$ if $a_{i,j} \ge b_{i,j}$ holds for all $i, j = 1, 2, \ldots, n$. Calling A nonnegative if $A \ge 0(a_{i,j} \ge 0 i, j = 1, \ldots, n)$, where 0 is an $n \times n$ zero matrix. For the vectors $a, b \in R^{n\times 1}, a \ge b$ and $a \ge 0$ can be defined in the similar manner.

Definition 2.1. *A matrix A is L-matrix if $a_{i,i} > 0, i = 1, \ldots, n$ and $a_{i,j} \le 0$ for all $i, j = 1, \ldots, n, i \ne j$. A nonsingular L-matrix A is a nonsingular M-matrix if $A^{-1} \ge 0$.*

Lemma 2.1. *[13] Let A be a nonnegative $n \times n$ nonzero matrix. Then*

1. *$\rho(A)$, the spectral radius of A, is an eigenvalue;*
2. *A has a nonnegative eigenvector corresponding to $\rho(A)$;*
3. *$\rho(A)$ is a simple eigenvalue of A;*
4. *$\rho(A)$ increases when any entry of A increases.*

Definition 2.2. *Let A be a real matrix. Then*

$$A = M - N$$

is called a splitting of A if M is a nonsingular matrix. The splitting is called

1. *weak regular if M is nonsingular, $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;*
2. *regular if M is nonsingular, $M^{-1} \geq 0$ and $N \geq 0$;*
3. *nonnegative if $M^{-1}N \geq 0$;*
4. *M-splitting if M is a nonsingular M-matrix and $N \geq 0$.*

Definition 2.3. *We call $A = M - N$ the Jacobi splitting of A, if $M = I$ is nonsingular and $N = (L + U)$. In addition, the splitting is called*

1. *Jacobi convergent if $\rho(M^{-1}N) < 1$;*
2. *Jacobi regular if $M^{-1} = I^{-1} \geq 0$ and $N = (L + U) \geq 0$.*

Lemma 2.2. *[7] Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and if A is a nonsingular M-matrix.*

Lemma 2.3. *[15] Let A be a nonsingular M-matrix, and let $A = M_1 - N_1 = M_2 - N_2$ be two convergent splittings, the first one weak regular and the second one regular. if $M_1^{-1} \geq M_2^{-1}$ then*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

3 Convergence And Comparison Theorems

Before proving the comparison theorems, we prove the convergence of modified Jacobi method with the preconditioner $P_{S_{max}} = I + S_{max}$, the preconditioned matrix $A_{S_{max}} = (I + S_{max})A$ can be written as

$$A_{S_{max}} = M_{S_{max}} - N_{S_{max}} = (I - D) - (L + E + U - S_{max} + F + S_{max}U).$$

In which D, E and F are defined as in section 1. Hence, if $a_{i,k_i}a_{k_i,i} \neq 1 (i = 1, 2, , n - 1)$ then the modified Jacobi iterative matrix $T_{S_{max}}$ can be defined by

$$T_{S_{max}} = M_{S_{max}}^{-1} N_{S_{max}} = (I - D)^{-1}(L + E + U - S_{max} + F + S_{max}U)$$

as $(I - D)^{-1}$ exists.

Lemma 3.1. *Let $A = I - L - U$ be a nonsingular M-matrix. Assume that $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$, then $A_{S_{max}} = M_{S_{max}} - N_{S_{max}}$ is regular and Jacobi convergent.*

Proof: The elements of $A_{S_{max}}$ are $a^m_{i,j} = a_{i,j} - a_{i,k_i}a_{k_i,j}$. We observe that when $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$, the diagonal elements of $A_{S_{max}}$ are positive and $M^{-1}_{S_{max}}$ exists. It is known that (see [1]) an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + S_{max} \geq 0$ implies $A_{S_{max}}y = (I + S_{max})Ay > 0$.

Consequently, the L-matrix $A_{S_{max}}$ is a nonsingular M-matrix, which means $A^{-1}_{S_{max}} \geq 0$. Since $0 \leq a_{i,k_i}a_{k_i,i} < 1$, we have $(I - D)^{-1} \geq I$, the following inequality holds:

$$M^{-1}_{S_{max}} = (I - D)^{-1} \geq 0$$

since $U \geq S_{max} \geq 0$ clearly $N_{S_{max}} \geq 0$ holds. Therefore, $A_{S_{max}} = M_{S_{max}} - N_{S_{max}}$ is a regular and Jacobi convergent splitting by definition (2.3) and lemma (2.2).

3.1 On The Preconditioner $R_{max} = I + S_{max} + R_{max}$

Theorem 3.1. *Let A be a nonsingular M-matrix and let that $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$ and $0 \leq a_{i,k_j}a_{k_j,n} < 1, k_j = 1, \cdots, n-1$, then $A_{max} = M_{max} - N_{max}$ is a regular and Jacobi convergent splitting.*

Proof: We observe that when $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$ and $0 \leq a_{n,k_j}a_{k_j,n} < 1, k_j = 1, \cdots, n-1$ the diagonal elements of A_{max} are positive and M^{-1}_{max} exists. Similar to the proof of lemma (3.1), we can show that $A_{max} = (I + S_{max} + R_{max})A$ is a nonsingular M-matrix when A is a nonsingular M-matrix.

Thus $A^{-1}_{max} \geq 0$. When $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$ and $0 \leq a_{n,k_j}a_{k_j,n} < 1, k_j = 1, \cdots, n-1$, we have $D + \acute{D} < I$ so $(I - D - \acute{D}) \geq 0$ the following relation holds:

$$M^{-1}_{max} = (I-D-\acute{D})^{-1} = (I-(D+\acute{D}))^{-1} = \{I+(D+\acute{D})+(D+\acute{D})^2+\cdots+(D+\acute{D})^{n-1}\} \geq 0$$

Since $L \geq R_{max} \geq 0$ and $U \geq S_{max} \geq 0$, clearly $N_{max} = L - R_{max} + E + \acute{E} + U - S_{max} + F + S_{max}U \geq 0$. Then $A_{max} = M_{max} - N_{max}$ is a regular and Jacobi convergent splitting by definition (2.3) and lemma (2.2).

Theorem 3.2. *Let A be a nonsingular M-matrix. Then under the assumptions of theorem (3.1), the following relation holds,*

$$\rho(T_{max}) \leq \rho(T) < 1.$$

Proof: The iteration matrix of the classical Jacobi method for A is $T = (L + U)$. Since A is a nonsingular M-matrix, the classical Jacobi splitting $A = I - (L+U)$ is clearly regular and convergent. From theorem (3.1), $A_{max} = P_{max}A = M_{max} - N_{max}$ is a Jacobi convergent splitting. To compare $\rho(T_{max})$ with $\rho(T)$, we consider the following splitting of A :

$$A_{max} = P_{max}A = (I + S_{max} + R_{max})A = M_{max} - N_{max}$$

and hence,

$$A = (I + S_{max} + R_{max})^{-1}M_{max} - (I + S_{max} + R_{max})^{-1}N_{max}.$$

If we take $M_1 = (I + S_{max} + R_{max})^{-1}M_{max}$ and $N_1 = (I + S_{max} + R_{max})^{-1}N_{max}$, then $\rho(M_1^{-1}N_1) < 1$ since $M_{max}^{-1}N_{max} = M_1^{-1}N_1$. Also, note that

$$M_1^{-1} = M_{max}^{-1}(I+S_{max}+R_{max}) = (I-D-\acute{D})^{-1}(I+S_{max}+R_{max}) \geq (I-D-\acute{D})^{-1} \geq I^{-1}.$$

If follows from lemma (2.3) that $\rho(M_1^{-1}N_1) < \rho(M^{-1}N) < 1$. Hence $\rho(M_{max}^{-1}N_{max}) < \rho(M^{-1}N) < 1$, i.e., $\rho(T_{max}) \leq \rho(T) < 1$.

3.2 On The Preconditioner $R_R = I + S_{max} + R$

Theorem 3.3. *Let A be a nonsingular M-matrix and let that $0 \leq a_{i,k_i}a_{k_i,i} < 1, 1 \leq i \leq n-1$ and $0 \leq \sum_{k=1}^{n-1} a_{n,k}a_{k,n} < 1$, then $A_R = M_R - N_R$ is regular and Jacobi convergent splitting.*

Proof: The proof is same as the proof of theorem (3.1).

Similar to the proof theorem (3.2), we can compare $\rho(T)$ with $\rho(T_R)$. The following is a comparison result and we will state it without proof.

Theorem 3.4. *Let A be a nonsingular M-matrix. Then under the assumptions of theorem* 3.3*, the following relation holds,*

$$\rho(T_R) \leq \rho(T) < 1.$$

4 Examples

In this section, we test the following matrix,

$$A = \begin{pmatrix} 1.00 & 0.00 & -0.20 & -0.60 \\ -0.10 & 1.00 & -0.10 & -0.50 \\ -0.30 & -0.10 & 1.00 & -0.10 \\ -0.40 & -0.30 & -0.10 & 1.00 \end{pmatrix}$$

By using preconditioners $I + S_{max} + R_{max}$ and $I + S_{max} + R$, we have the following matrices :

$$A_{max} = \begin{pmatrix} 0.76 & -0.18 & -0.26 & 0.00 \\ -0.30 & 0.85 & -0.15 & 0.00 \\ -0.34 & -0.13 & 0.99 & 0.00 \\ 0.00 & -0.30 & -0.18 & 0.76 \end{pmatrix}$$

and

$$A_R = \begin{pmatrix} 0.76 & -0.18 & -0.26 & 0.00 \\ -0.30 & 0.85 & -0.15 & 0.00 \\ -0.34 & -0.13 & 0.99 & 0.00 \\ -0.06 & -0.01 & -0.11 & 0.60 \end{pmatrix}$$

By computation, we have

$$\rho(M^{-1}N) = 0.736125 > \rho(M_{max}^{-1}N_{max}) = 0.530363$$

and

$$\rho(M^{-1}N) = 0.736125 > \rho(M_R^{-1}N_R) = 0.53029.$$

Next, we test the following matrix:

$$A = \begin{pmatrix} 1.0 & -0.1 & 0.0 & -0.1 & 0.0 & -0.1 & 0.0 & -0.2 & -0.4 & 0.0 \\ -0.1 & 1.0 & -0.1 & -0.3 & -0.1 & 0.0 & 0.0 & -0.1 & -0.1 & 0.0 \\ -0.2 & -0.1 & 1.0 & -0.1 & 0.0 & -0.1 & 0.0 & 0.0 & -0.3 & -0.1 \\ -0.1 & -0.1 & 0.0 & 1.0 & 0.0 & -0.1 & -0.4 & 0.0 & -0.1 & 0.0 \\ 0.0 & -0.1 & 0.0 & -0.1 & 1.0 & -0.4 & -0.2 & 0.0 & -0.1 & -0.1 \\ -0.2 & 0.0 & -0.1 & 0.0 & 0.0 & 1.0 & 0.0 & -0.4 & -0.1 & -0.1 \\ 0.0 & -0.1 & -0.2 & -0.1 & 0.0 & -0.1 & 1.0 & 0.0 & -0.3 & -0.1 \\ -0.2 & -0.1 & -0.2 & 0.0 & 0.0 & -0.1 & 0.0 & 1.0 & -0.3 & 0.0 \\ 0.0 & -0.1 & -0.1 & 0.0 & -0.2 & 0.0 & -0.1 & -0.2 & 1.0 & -0.1 \\ -0.1 & 0.0 & 0.0 & -0.1 & 0.0 & -0.1 & -0.3 & 0.0 & -0.1 & 1.0 \end{pmatrix}$$

We have
$\rho(T) = 0.856049, \rho(T_{max}) = 0.774723$ and $\rho(T_R) = 0.774471$. Clearly, $\rho(T_{max}) < \rho(T)$ and $\rho(T_R) < \rho(T)$ holds.

5 Conclusion

In 1991, A. D. Gunawardena et al. proposed the modified Gauss-Siedel (MGS) method for solving the linear system with the preconditioned $P = I + S$ [A. D. Gunawardena, S. K. Jain, L. Snider, Modified Iterative Method For Consistent Linear System, Linear Algebra Appl. 154-156 (1991)123-143]. Based on their work, in 2009 B. Zheng et al.[19] proposed two modified Gauss-Seidel (MGS) methods for linear system with M-matrices. In this paper, we used the preconditioners introduced by B. Zheng et al . for modified Jacobi method. Also, the comparison theorems and numerical examples were shown that the proposed methods are superior to the classical Jacobi method.

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