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Fractional Differential Equations with Legendre Polynomials

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Abstract

In this paper we propose a method for computing approximations of solution of fractional differential equations using Legendre polynomials and Adomian decomposition method.

Keywords : Fractional derivative, Adomian decomposition method, Legendre polynomials.

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1 Introduction

Differential equations may involve Reimann-Liouville differential operators of fractional order $r > 0$, which have the form

$$D^r_{x_0} y(x) = \frac{1}{\Gamma(m-r)} \frac{d^m}{dx^m} \int_{x_0}^{x} \frac{y(u)}{(x-u)^{r-m+1}} du, \tag{1.1}$$

where m is the integer defined by $m - 1 \le r < m$. In order to obtain a unique solution for $D^r y(x) = f(x, y(x))$, the exact m initial value is needed. When $m = 1$, we study the following fractional initial value problem

$$D^r_{x_0} y + y = f(x), \quad y(x_0) = y_0 \tag{1.2}$$

And using Adomian decomposition method, we give a new method to find an approximate solution of Eq. (1.2) when $f(x)$ is expressed by Legendre polynomials. In recent years, fractional differential equations have found applications in many problems in Physics and engineering [4]. Also some numerical methods are used to find approximate analytical solutions, for instance Adomian decomposition method, variational iteration method, homotopy perturbation method and homotopy analysis method [1, 2, 3, 5, 6]. In this paper

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a modification of Adomian decomposition method is introduced to solve fractional initial value problems.

The organization of the paper is as follows. In Section 2 we list some basic definitions of fractional derivative and integral. In Section 3, an approximate solution for fractional initial value problems is introduced. Finally, we conclude the paper in Section 4.

2 Preliminaries

In this work, we express $f(x)$ in the Legendre series

$$f(x)=\sum_{i=0}^{\infty} c_i P_i(x)$$

Where $P_i(x)$ is the second kind of Legendre polynomial and can be found by the following recursive relation

$$\begin{aligned}&nP_n(x)=(2n-1)P_{n-1}(x)-(n-1)P_{n-2}(x), \qquad n\geq 2.\\&P_0(x)=1,\\&P_1(x)=x.\end{aligned}$$

and the coefficient c_i can be found by

$$c_i=\frac{\int_{-1}^{1} f(x)P_i(x)dx}{\int_{-1}^{1} P_i^2(x)dx}.$$

When the domain of $f(x)$ is different from $[-1,1]$, we must use suitable changing variables.

Definition 2.1. *A function $y(x), x>0$ is said to be in the space C_μ, $\mu\in\mathbb{R}$, if there exist a real number $p>\mu$ such that $y(x)=x^p y_1(x)$, where $y_1(x)\in C(0,\infty)$, and is said to be in the space C_μ^n if and only if $y^{(n)}\in C_\mu, n\in\mathbb{N}$.*

Definition 2.2. *Reimann-Liouville's fractional derivative and fractional integral of order $0<r<1$ for $y(x):\mathbb{R}\to\mathbb{R}$ are defined as*

$$y^{(r)}=\frac{1}{\Gamma(1-r)}\frac{d}{dx}\int_0^x (x-s)^{-r}u(s)ds \tag{2.3}$$

and

$$I^r y(x)=\frac{1}{\Gamma(r)}\int_0^x (x-s)^{r-1}u(s)ds. \tag{2.4}$$

For instance, when $r>0$ and $\lambda>-1$ we have

$$\frac{d^r}{dt^r}(x^\lambda)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-r)}x^{\lambda-r} \tag{2.5}$$

and

$$I^r(x^\lambda)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+r)}x^{\lambda+r}. \tag{2.6}$$

Lemma 2.1. *Let $y(x)\in C_{-1}^n, n\in\mathbb{N}$, then $D^r y, 0\leq r\leq n$ is well defined and $D^r\in C_{-1}$.*

Lemma 2.2. *Let $n-1<r\leq n, n\in\mathbb{N}$ and $y(x)\in C_\mu^n, \mu\geq -1$, then*

$$I^r D^r y(x)=y(x)-\sum_{k=0}^{n-1} y^{(k)}(0^+)\frac{x^k}{k!}$$

3 Fractional initial value problem

To perform the Adomian decomposition method, the source term $f(x)$ is usually expressed in the Taylor series with k terms, for some constant k. In this paper we use the Legendre series

$$f(x) = \sum_{0}^{\infty} c_i P_i(x)$$

and the fractional differential equation can be modeled as $Ly(x) + Ny(x) + Ry(x) = f(x)$, where $L = D^r$ therefore $L^{-1} = I^r$. Since

$$L^{-1}Ly = y - cx^{r-1}$$

then

$$y = cx^{r-1} + I^r(f(x)) - I^r(Ry) - I^r(Ny). \tag{3.7}$$

The solution y is represented as an infinite sum

$$y = \sum_{n=0}^{\infty} y_n \tag{3.8}$$

and the nonlinear term Ny will be decomposed by the infinite series of Adomian polynomials

$$Ny = \sum_{n=0}^{\infty} A_n$$

where the A_ns are obtained by writing

$$z(\lambda) = \sum_{n=0}^{\infty} \lambda^n y_n$$

$$N(z(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n$$

therefore, for any $n = 0, 1, ...$

$$A_n = \frac{1}{n!}[\frac{d^n}{d\lambda^n} N(z(\lambda))]_{\lambda=0}.$$

Then by substituting (3.8) in (3.7) we obtain the following relations

$$\sum_{i=0}^{\infty} y_i = cx^{r-1} + I^r(f(x)) - \sum_{i=0}^{\infty} I^r(Ry_i) - \sum_{i=0}^{\infty} I^r(A_i)$$

and we define $y_0, y_1, y_2, \cdots$ in a recurrent manner in which Relations (2.4) and (2.6) are applied.

$$y_0 = cx^{r-1} + I^r f(x)$$

$$y_1 = -I^r Ry_0 - I^r A_0$$

$$y_2 = -I^r Ry_1 - I^r A_1$$

$$\vdots$$

When the term $I^r f(x)$, is hard to calculate, in general we can express $f(x)$ in Taylor series

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i.$$

Suppose that we can find the coefficient c_i such that

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x)$$

Then the modified method for the fractional differential equation $y^{(r)} + y = f(x)$ can be shown as following

$$\sum_{i=0}^{\infty} y_i = cx^{r-1} + I^r(\sum_{i=0}^{\infty} c_i P_i(x)) - I^r(\sum_{i=0}^{\infty} y_i)$$

then

$y_0 = cx^{r-1} + c_0 I^r P_0$

$y_1 = c_1 I^r P_1 - I^r y_0 = c_1 I^r P_1 - cI^r x^{r-1} - c_0 I^{2r} P_0$

$y_2 = c_2 I^r P_2 - I^r y_1 = c_2 I^r P_2 - c_1 I^{2r} P_1 + cI^{2r} x^{r-1} + c_0 I^{3r} P_0$

$\vdots$

therefore

$$\begin{aligned} y &= \sum_{i=0}^{\infty} y_i \\ &= cx^{r-1} + c_0 I^r P_0 + c_1 I^r P_1 - cI^r x^{r-1} - c_0 I^{2r} P_0 + c_2 I^r P_2 - c_1 I^{2r} P_1 + cI^{2r} x^{r-1} + c_0 I^{3r} P_0 + \cdots \\ &= c(1 - I^r + I^{2r} - \cdots)x^{r-1} + c_0(I^r - I^{2r} + I^{3r} - \cdots)P_0 + c_1(I^r - I^{2r} + I^{3r} - \cdots)P_1 \\ &+ c_2(I^r - I^{2r} + I^{3r} - \cdots)P_2 + \cdots + c_k(I^r - I^{2r} + I^{3r} - \cdots)P_k + \cdots \end{aligned}$$

As an approximate solution we can use the following truncation

$$\begin{aligned} y &= \sum_{0}^{n} y_i \\ &= c(1 - I^r + I^{2r} - + \cdots + (-1)^n I^{nr})x^{r-1} \\ &+ c_0(I^r - I^{2r} + I^{3r} - + \cdots + (-1)^n I^{(n+1)r})P_0 \\ &+ c_1(I^r - I^{2r} + I^{3r} - \cdots + (-1)^{n-1} I^{nr})P_1 \\ &+ c_2(I^r - I^{2r} + I^{3r} - \cdots + (-1)^n I^{(n-1)r})P_2 + \cdots + c_n(I^r)P_n \end{aligned}$$

where

$$f(x) \simeq \sum_{0}^{n} c_i P_i(x).$$

4 Conclusion

Fractional differential equations have been studied using Legendre polynomials. Adomian decomposition method has been applied to obtain approximate solution.

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