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Solving the non-linear system of third-order boundary value problems by using He's homotopy perturbation method

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Abstract

In this paper we introduce a homotopy perturbation method (HPM) to solve non-linear system of third-order boundary value problems. Using the HPM, it is possible to find the exact solution in some cases or an approximate solution in convergent series forms. This method is a powerful devise for solving a wide variety of problems, without any discretization, linearizetion or small perturbations. Some examples are presented, and obtained results ensure that this method is very effective, simple and has less the numerical computation.

Keywords : Non-linear third-order differential system; Homotopy perturbation method; Numerical method.

1 Introduction

The non-linear differential equations are generally difficult to solve and their exact solutions are difficult to obtain, therefore, some various approximate method recently have been developed, such as homotopy perturbation method. HPM was developed and improved by J. Huan He [8, 11, 12, 13, 14]. This method has been used by many authors to handel a wide variety of scientific and engineering applications to solve various functional equations. A homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". When $p = 0$, the system of equation usually reduces to a simplified form, which admits a rather simple solution. Since the variation of p changes from 0 to 1, the problem goes through to a sequence of deformity, such that the solution of each is closer to the previous stage of the deformity. At $p = 1$, the problem takes the original form and at the end we can have the desired solution. This

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method yields a very rapid convergence of the solution series for most cases and with only few iterations produces accurate solutions. This new method is applied to non-linear oscillators with discontinuities by He's [15], bifurcation of non-linear problems, periodic solution and bifurcations of delay-differential equations, limit cycle [16, 17, 18], heat and wave like equations [19] and non-linear wave equations [9]. The authors of [23] applied HPM for a beam equation with non-linear boundary conditions of third order. S. Abbasbandy applied this method for solving functional integral equations, Riccati differential equation and non-linear Klein-Gordan equation, see [1, 2, 3]. Solving partial differential equations and second-order singular problems by using HPM is done in [20, 21]. In [6], the authors applied HPM to nonlinear coupled systems of reaction-diffusion equations. M. A. Noor and S. Tauseef Mohyud-Din suggested an efficient method for fourth-order boundary value problems [22]. Solving the Dufffing-harmonic oscillator by using HPM is done in [4]. Recently, the authors of [24, 25] have applied HPM for solving non-linear system of second-order boundary value problems and nonlinear integral and integro-differential equations. In [5], the authors have discussed about convergence of the HPM for partial differential equations.

Solving a nonlinear system of third-order boundary value problems is very difficult and requires a series of complicate stages. In this paper we consider the following non-linear system of third-order ordinary differential equations:

$$\begin{cases} u''' + a_1(x)u'' + a_2(x)u'(x) + a_3(x)u + a_4(x)v''' + a_5(x)v''+ \\ \qquad a_6(x)v' + a_7(x)v + G_1(x,u,v) = f_1(x), \\ v''' + b_1(x)'' + b_2(x)v'(x) + b_3(x)v + b_4(x)u''' + b_5(x)u''+ \\ \qquad b_6(x)u' + b_7(x)u + G_2(x,u,v) = f_2(x), \end{cases} \quad 0 \le x \le 1, \tag{1.1}$$

where G_1 and G_2 are non-linear functions in terms of u and v, $a_i(x)$ and $b_i(x)$ are given continuous functions,for i=1,...,7, f_1 and f_2 are known and also boundary conditions that are given in examples. The rest of this paper is organized as follows. In Section 2 we describe the basic formulation of HPM required for our subsequent development. In Section 3 we illustrate some examples which are solved by HPM and the numerical results are computed by mathematica package. Conclusion of this paper is given in section 4.

2 Homotopy perturbation method

In this section, we use HPM for solving Eq. (1). As [24], we use the following transformations:

$$u_1 = u, \quad u_2 = \frac{du_1}{dx}, \quad u_3 = \frac{du_2}{dx}, \quad u_4 = \frac{du_3}{dx}, \tag{2.2}$$

$$v_1 = v, \quad v_2 = \frac{dv_1}{dx}, \quad v_3 = \frac{dv_2}{dx}, \quad v_4 = \frac{dv_3}{dx}. \tag{2.3}$$

By using these equations, we can get the system of third-order boundary value problem Eq. (1) as the system of integral equations:

$$u_1(x) = u_1(0) + \int_0^x u_2(t)dt, \tag{2.4}$$

$$u_2(x) = u_2(0) + \int_0^x u_3(t)dt, \tag{2.5}$$

$$u_3(x) = A + \int_0^x u_4(t)dt, \tag{2.6}$$

$$u_4 + a_1(x)u_3 + a_2(x)u_2 + a_3(x)u_1 + a_4(x)v_4 + a_5(x)v_3 + a_6(x)v_2 + a_7(x)v_1 + G_1(x, u_1, v_1) = f_1(x), \tag{2.7}$$

$$v_1(x) = v_1(0) + \int_0^x v_2(t)dt, \tag{2.8}$$

$$v_2(x) = v_2(0) + \int_0^x v_3(t)dt, \tag{2.9}$$

$$v_3(x) = B + \int_0^x v_4(t)dt, \tag{2.10}$$

$$v_4 + b_1(x)v_3 + b_2(x)v_2 + b_3(x)v_1 + b_4(x)u_4 + b_5(x)u_3 + b_6(x)u_2 + b_7(x)u_1 + G2(x, u_1, v_1) = f_2(x), \tag{2.11}$$

where A, B are constants which we must obtain them through of the example. Clearly, we can eliminate v_4 from Eq. (7) by the combination of Eq. (7) and Eq. (11). Therefore we take $u_4 = K_1(x, u_1, u_2, u_3, v_1, v_2, v_3)$. Similarly, we can obtain $v_4 = K_2(x, u_1, u_2, u_3, v_1, v_2, v_3)$ which is described. Thus Eqs. (4)-(11) can be written as follows:

$$u_1(x) = u_1(0) + \int_0^x u_2(t)dt, \tag{2.12}$$

$$u_2(x) = u_2(0) + \int_0^x u_3(t)dt, \tag{2.13}$$

$$u_3(x) = A + \int_o^x K_1(t, u_1, u_2, u_3, v_1, v_2, v_3)dt, \tag{2.14}$$

$$v_1(x) = v_1(0) + \int_0^x v_2(t)dt, \tag{2.15}$$

$$v_2(x) = v_2(0) + \int_0^x v_3(t)dt, \tag{2.16}$$

$$v_3(x) = B + \int_0^x K_2(t, u_1, u_2, u_3, v_1, v_2, v_3)dt. \tag{2.17}$$

Also we can rewrite Eqs. (12)-(17) as follows:

$$\begin{cases} L_1(u_1, u_2, u_3, v_1, v_2, v_3) = u_1(x) - u_1(0) - \int_0^x u_2(t)dt, \\ L_2(u_1, u_2, u_3, v_1, v_2, v_3) = u_2(x) - u_2(0) - \int_0^x u_3(t)dt, \\ L_3(u_1, u_2, u_3, v_1, v_2, v_3) = u_3(x) - A - \int_0^x K_1(t, x, u_1, u_2, u_3, v_1, v_2, v_3)dt, \\ L_4(u_1, u_2, u_3, v_1, v_2, v_3) = v_1(x) - v_1(0) - \int_0^x v_2(t)dt, \\ L_5(u_1, u_2, u_3, v_1, v_2, v_3) = v_2(x) - v_2(0) - \int_0^x v_3(t)dt, \\ L_6(u_1, u_2, u_3, v_1, v_2, v_3) = v_3(x) - B - \int_0^x K_2(t, x, u_1, u_2, u_3, v_1, v_2, v_3)dt. \end{cases} \tag{2.18}$$

Then

$$L(u_1, u_2, u_3, v_1, v_2, v_3) = \begin{bmatrix} L_1(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \\ L_2(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \\ L_3(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \\ L_4(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \\ L_5(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \\ L_6(u_1, u_2, u_3, v_1, v_2, u_1, v_3) \end{bmatrix} = 0. \tag{2.19}$$

Now if we define:

$$F(u_1,u_2,u_3,v_1,v_2,v_3) = \begin{bmatrix} F_1(u_1,u_2,u_3,v_1,v_2,v_3) \\ F_2(u_1,u_2,u_3,v_1,v_2,v_3) \\ F_3(u_1,u_2,u_3,v_1,v_2,v_3) \\ F_4(u_1,u_2,u_3,v_1,v_2,v_3) \\ F_5(u_1,u_2,u_3,v_1,v_2,v_3) \\ F_6(u_1,u_2,u_3,v_1,v_2,v_3) \end{bmatrix} = \begin{bmatrix} u_1 - u_1(0) \\ u_2 - u_2(0) \\ u_3 - A \\ v_1 - v_1(0) \\ v_2 - v_2(0) \\ v_3 - B \end{bmatrix} \quad (2.20)$$

and

$$\mathbf{H}(u_1,u_2,u_3,v_1,v_2,v_3,p) = \begin{bmatrix} \mathbf{H}_1(u_1,u_2,u_3,v_1,v_2,v_3,p) \\ \mathbf{H}_2(u_1,u_2,u_3,v_1,v_2,v_3,p) \\ \mathbf{H}_3(u_1,u_2,u_3,v_1,v_2,v_3,p) \\ \mathbf{H}_4(u_1,u_2,u_3,v_1,v_2,v_3,p) \\ \mathbf{H}_5(u_1,u_2,u_3,v_1,v_2,v_3,p) \\ \mathbf{H}_6(u_1,u_2,u_3,v_1,v_2,v_3,p) \end{bmatrix}, \quad (2.21)$$

then we can give homotopy $\mathbf{H}(u_1,u_2,u_3,v_1,v_2,v_3,p)$ as follows:

$$\mathbf{H}(u_1,u_2,u_3,v_1,v_2,v_3,p) = (1-p)\mathbf{F}(u_1,u_2,u_3,v_1,v_2,v_3)+p\mathbf{L}(u_1,u_2,u_3,v_1,v_2,v_3). \quad (2.22)$$

If $p = 0$ we get

$$\mathbf{H}(u_1,u_2,u_3,v_1,v_2,v_3,0) = \mathbf{F}(u_1,u_2,u_3,v_1,v_2,v_3)$$

and if $p = 1$, we have

$$\mathbf{H}(u_1,u_2,u_3,v_1,v_2,v_3,1) = \mathbf{L}(u_1,u_2,u_3,v_1,v_2,v_3).$$

The embedding parameter p monotonically increases form 0 to 1 as the trivial problem $\mathbf{F}(u_1,u_2,u_3,v_1,v_2,v_3) = 0$ continuously deforms to the original problem $\mathbf{L}(u_1,u_2,u_3,v_1,v_2,v_3) = 0$.

The HPM uses the embedding parameter p as an expanding parameter to obtain:

$$u_i = u_{i0} + pu_{i1} + p^2u_{i2} + p^3u_{i3} + \cdots, \quad i = 1,2,3, \quad (2.23)$$

$$v_i = v_{i0} + pv_{i1} + p^2v_{i2} + p^3v_{i3} + \cdots, \quad i = 1,2,3. \quad (2.24)$$

When p monotonically increases to 1, the exact solutions of Eq. (1) with their boundary conditions can be obtained:

$$U_i = \lim_{p\longrightarrow 1} u_i = u_{i0} + u_{i1} + u_{i2} + u_{i3} + \cdots, \quad i = 1,2,3, \quad (2.25)$$

$$V_i = \lim_{p\longrightarrow 1} v_i = v_{i0} + v_{i1} + v_{i2} + v_{i3} + \cdots, \quad i = 1,2,3. \quad (2.26)$$

Now we write Eq. (22) as follows:

$$\mathbf{H}_i(u_1,u_2,u_3,v_1,v_2,v_3,p) = (1-p)\mathbf{F}_i(u_1,u_2,u_3,v_1,v_2,v_3)+$$

$$p\mathbf{L}_i(u_1,u_2,u_3,v_1,v_2,v_3), \quad i = 1,2,3,4,5,6. \quad (2.27)$$

We know that series of (23) and (24) are convergent in most cases and the rate of convergence depends on $L(u_1, u_2, u_3, v_1, v_2, v_3)$ [10]. Substituting Eq. (18), Eq. (20), Eq. (23) and Eq. (24) in Eq. (27) we have:

$$(1-p)(u_1(x)-u_1(0))+p(u_1(x)-\int_0^x u_2(t)dt)=0,$$

$$(1-p)(u_2(x)-u_2(0))+p(u_2(x)-u_2(0)-\int_0^x u_3(t)dt)=0,$$

$$(1-p)(u_4(x)-A-\int_0^x k_1(t,u_1,u_2,u_3,v_1,v_2,v_3)dt)=0,$$

$$(1-p)(v_1(x)-v_1(0))+p(v_1(x)-\int_0^x v_2(t)dt)=0,$$

$$(1-p)(v_2(x)-v_2(0))+p(v_2(x)-v_2(0)-\int_0^x v_3(t)dt)=0,$$

$$(1-p)(v_4(x)-B-\int_0^x k_2(t,u_1,u_2,u_3,v_1,v_2,v_3)dt)=0.$$

Now if we compare the coefficient of like powers of p, we obtain:

$$p^0:\begin{cases} u_{10}=u_1(0),\\ u_{20}=u_2(0),\\ u_{30}=A,\\ v_{10}=v_1(0),\\ v_{20}=v_2(0),\\ v_{30}=B, \end{cases} \tag{2.28}$$

$$p^1:\begin{cases} u_{11}=\int_0^x u_{20}(t)dt,\\ u_{21}=\int_0^x u_{30}(t)dt,\\ u_{31}=\int_0^x \frac{\partial k_1}{\partial p}\mid_{p=0}dt,\\ v_{11}=\int_0^x v_{20}(t)dt,\\ v_{21}=\int_0^x v_{30}(t)dt,\\ v_{31}=\int_0^x \frac{\partial k_2}{\partial p}\mid_{p=0}dt, \end{cases} \tag{2.29}$$

$$p^n : \begin{cases} u_{1n} = \int_0^x u_{2,n-1}(t)dt, \\ u_{2n} = \int_0^x u_{3,n-1}(t)dt, \\ u_{3n} = \frac{1}{n!} \int_o^x \frac{\partial^n k_1}{\partial p^n} \mid_{p=0} dt, \\ v_{1n} = \int_0^x v_{2,n-1}(t)dt, \qquad n = 2, 3, \cdots . \\ v_{2n} = \int_0^x v_{3,n-1}(t)dt, \\ v_{3n} = \frac{1}{n!} \int \frac{\partial^n k_2}{\partial p^n} \mid_{p=0} dt. \end{cases} \tag{2.30}$$

Combination of all terms of Eqs. (28)-(30) gives the solution of the problem. Finally using the boundary conditions $u_1(1)$ and $v_1(1)$ we can obtain A and B.

3 Applications

In this section, we explain the application of the presented method in this paper by solving two examples. Throughout this section, we assume that

$$U_n(x) = \sum_{k=0}^{n} u_{1k}(x)$$

and

$$V_n(x) = \sum_{k=0}^{n} v_{1k}(x).$$

Example 3.1. *Consider the non-linear system of third-order boundary value problem:*

$$\begin{cases} u'''(x) + 2u'(x) + xv(x) = x^5 - x^3 - 18x^2 + 12x - 18, \\ v'''(x) + \frac{u''(x)v''(x)}{6} = -18x^3 + 6x^2 + 27x - 1, \qquad 0 \le x \le 1, \end{cases} \tag{3.31}$$

with boundary conditions:

$$u(1) = v(1) = 0, \quad u(0) = v(0) = 0, \quad u'(0) = v'(0) = 0.$$

The exact solutions of this problem are $u(x) = 3x^2 - 3x^3$ and $v(x) = x^4 - x^2$. Using the transformations (2) and (3), we can rewrite Eq. (31) as

$$\begin{cases} u_1 = \int_0^x u_2(t)dt, \\ u_2 = \int_0^x u_3(t)dt, \\ u_3 = A + \int_0^x [t^5 - t^3 - 18t^2 + 12t - 18 - 2u'(t) - tv(t)]dt, \\ v_1 = \int_0^x v_2(t)dt, \\ v_2 = \int_0^x v_3(t)dt, \\ v_3 = B + \int_0^x [-18t^3 + 6t^2 + 27t - 1 - \frac{u''(t)v''(t)}{6}]dt. \end{cases}$$

Comparing the coefficient of like powers of p we have:

$$p^0 : \begin{cases} u_{10} = 0, \\ u_{20} = 0, \\ u_{30} = A, \\ v_{10} = 0, \\ v_{20} = 0, \\ v_{30} = B. \end{cases}$$

$$p^1 : \begin{cases} u_{11} = \int_0^x u_{20}(t)dt = 0, \\ u_{21} = \int_0^x u_{30}(t)dt = Ax, \\ u_{31} = \int_0^x [t^5 - t^3 - 18t^2 + 12t - 18]dt = \frac{x^6}{6} - \frac{x^4}{4} - 6x^3 + 6x^2 - 18x, \\ v_{11} = \int_0^x v_{20}(t)dt = 0, \\ v_{21} = \int_0^x v_{30}(t)dt = Bx, \\ v_{31} = \int_0^x [-18t^3 + 6t^2 + 27t - 1 - \frac{u_3(t)v_3(t)}{6}]dt = \frac{-9x^4}{2} + 2x^3 + \frac{27x^2}{2} - (1 + \frac{AB}{6})x \end{cases}$$

$$p^2 : \begin{cases} u_{12} = \int_0^x u_{21}(t)dt = \frac{x^2 A}{2}, \\ u_{22} = \int_0^x u_{31}(t)dt = \frac{x^7}{42} - \frac{x^5}{20} - \frac{3x^4}{2} - 2x^3 - 9x^2, \\ u_{32} = \int_0^x [-2u_{21}(t) - tv_{11}(t)]dt = -Ax^2, \\ v_{12} = \int_0^x v_{21}(t)dt = \frac{x^2 B}{2}, \\ v_{22} = \int_0^x v_{31}(t)dt = \frac{-9x^5}{10} + \frac{x^4}{2} + \frac{9x^3}{2} - (1 + \frac{AB}{6})\frac{x^2}{2}, \\ v_{32} = \int_0^x [\frac{-1}{6}(u_{30}(t)v_{31}(t) + u_{31}(t)v30(t))]dt = \frac{1}{6}((\frac{-9A}{10} - \frac{B}{20})x^5 + (\frac{A}{2} - \frac{3B}{2})x^4 + \\ (\frac{9A}{2} - 2)x^3 - (A + \frac{BA^2}{6} + 9B)x^2 + \frac{Bx^7}{42}). \end{cases}$$

$$p^{n+1} : \begin{cases} u_{1,n+1} = \int_0^x u_{2n}(t)dt, \\ u_{2,n+1} = \int_0^x u_{3n}(t)dt, \\ u_{3,n+1} = \int_0^x [-2u_{2n}(t) - tv_{1n}(t))]dt, \\ v_{1,n+1} = \int_0^x v_{2n}(t)dt, \\ v_{2,n+1} = \int_0^x v_{3n}(t)dt, \\ v_{3,n+1} = \int_0^x [\frac{-1}{6}\sum_{i=0}^n (u_{3r}(t)v_{3,n-r}(t))]dt. \end{cases} \qquad n = 1, 2, 3, \cdots .$$

As [24], we can compute A and B using boundary conditions $u_1(1) = v_1(1) = 0$ in $U_n(x)$ and $V_n(x)$. For instance, with $n = 8$ we have $A = -2.70958$ and $B = -5.08784$. In Tables 1 and 2 the absolute errors $|U_n(x) - u(x)|$ and $|V_n(x) - v(x)|$ are calculated for different values of n and x.

Table 1
Absolute error $|U_n(x) - u(x)|$ *in example 1.*

x	n=4	n=8	n=10	n=12
0.1	0.00608633	0.00111688	0.00881573	8.7425×10^{-8}
0.2	0.0229477	0.00435769	0.033324	0.0000341082
0.3	0.0466485	0.00940003	0.0680392	0.0000735812
0.4	0.0714291	0.0157194	0.0104831	0.00012321
0.5	0.534215	0.602429	0.490702	0.624822
0.6	0.0988013	0.0288978	0.147537	0.000231537
0.7	0.0916183	0.0331111	0.138212	0.0000275507
0.8	0.068867	0.0327119	0.104873	0.000292801
0.9	0.0349456	0.0237291	0.0534512	0.000239684

Table 2
Absolute error $|V_n(x) - v(x)|$ *in example 1.*

x	n=4	n=8	n=10	n=12
0.1	0.00480988	0.000758348	0.00881573	0.0000134169
0.2	0.0171815	0.0029441	0.033324	0.0000522346
0.3	0.0319383	0.00626768	0.0680392	0.0001120224
0.4	0.053283	0.106233	0.104831	0.0961854
0.5	0.0440397	0.0141966	0.490702	0.200026221
0.6	0.0332988	0.0174643	0.147537	0.000330775
0.7	0.0124618	0.0193376	0.138212	0.000378034
0.8	0.0106721	0.018885	0.104873	0.000386227
0.9	0.0217865	0.141057	0.0534512	0.000311011

We compare numerical solutions $U_{10}(x)$ *and* $V_{10}(x)$ *with exact solutions in Figs. 1-2.*

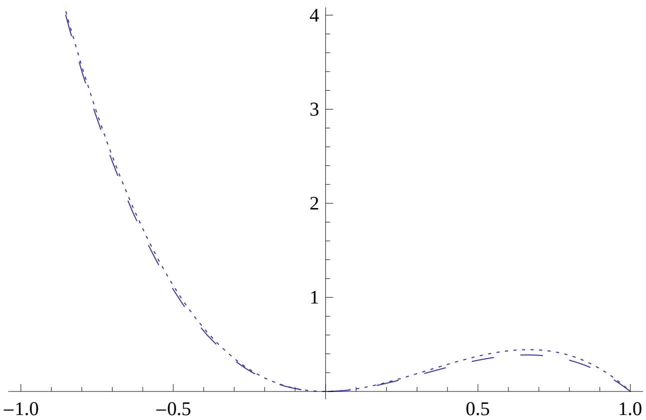

Fig. 1. Numerical solution $(\cdots)$ $U_{10}(x)$ *and exact solution* $(---)$ $u(x)$ *of example 1.*

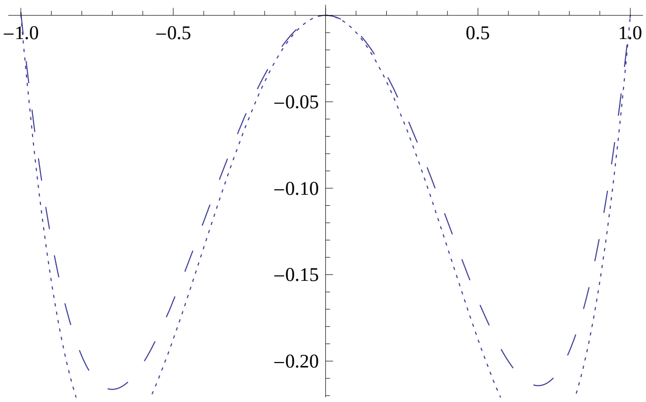

Fig.2. Numerical solution $(\cdots)$ $V_{10}(x)$ *and exact solution* $(---)$ $v(x)$ *of example 1.*

Example 3.2. *Consider the non-linear system:*

$$\begin{cases} u'''(x) - 4v'' + u''v' = f_1(x), \\ v'''(x) + 4v'(x) - u''(x) + u'v'' = f_2(x), \end{cases} \quad 0 \le x \le 1, \tag{3.32}$$

where $u(0) = v(0) = 0$, $u(1) = v(1) = 1$, $u'(0) = v'(0) = 0$, $f_1(x) = 36x^4$ *and* $f_2(x) = 12x^4 + 6$. *The exact solutions of this problem are* $u(x) = x^4$ *and* $v(x) = x^3$. *By the same manipulation as in previous example, we obtain:*

$$p^0 : \begin{cases} u_{10} = 0, \\ u_{20} = 0, \\ u_{30} = A, \\ v_{10} = 0, \\ v_{20} = 0, \\ v_{30} = B. \end{cases}$$

$$p^1 : \begin{cases} u_{11} = \int_0^x u_{20}(t)dt = 0, \\ u_{21} = \int_0^x u_{30}(t)dt = Ax, \\ u_{31} = \int_0^x [f_1(t) + 4v_{30}(t) - u_{30}(t)v_{20}(t)]dt = \frac{36x^5}{5} + 4Bx, \\ v_{11} = \int_0^x v_{20}(t)dt = 0, \\ v_{21} = \int_0^x v_{30}(t)dt = Bx, \\ v_{31} = \int_0^x [f_2(t) - 4v_{20}(t) + u_{30}(t) - u_{20}(t)v_{30}(t)]dt = \frac{12x^5}{5} + (6 + A)x. \end{cases}$$

$$p^2 : \begin{cases} u_{12} = \int_0^x u_{21}(t)dt = \frac{Ax^2}{2}, \\ u_{22} = \int_0^x u_{31}(t)dt = \frac{6x^6}{5} + 2Bx^2, \\ u_{32} = \int_0^x [4v_{31}(t) - (u_{30}(t)v_{21}(t) + u_{31}(t)v_{20}(t))]dt \\ \quad = \frac{8x^6}{5} + (12 + 2A - \frac{AB}{2})x^2, \\ v_{12} = \int_0^x v_{21}(t)dt = \frac{x^2 B}{2}, \\ v_{22} = \int_0^x v_{31}(t)dt = \frac{2x^6}{5} + 3x^2 + \frac{Ax^2}{2}, \\ v_{32} = \int_0^x [-4v_{21}(t) + u_{31}(t) - (u_{20}(t)v_{31}(t) + u_{21}(t)v_{30}(t))]dt \\ \quad = \frac{6x^6}{5} - \frac{ABx^2}{2}. \end{cases}$$

$$p^{n+1}: \begin{cases} u_{1,n+1} = \int_0^x u_{2n}(t)dt, \\ u_{2,n+1} = \int_0^x u_{3n}(t)dt, \\ u_{3,n+1} = \int_0^x [4v_{3n}(t) - v_{2n}(t)]dt, \\ v_{1,n+1} = \int_0^x v_{2n}(t)dt, \\ v_{2,n+1} = \int_0^x v_{3n}(t)dt, \\ v_{3,n+1} = \int_0^x [u_{3n}(t) - 4v_{2n}(t)]dt. \end{cases} \qquad n = 1, 2, \cdots .$$

In Tables 3 and 4 the absolute errors $|U_n(x) - u(x)|$ and $|V_n(x) - v(x)|$ are calculated for different values of n and x.

Table 3

Absolute error $|U_n(x) - u(x)|$ in example 2.

x	n=4	n=8	n=10	n=12
0.1	0.0153857	0.029996	0.0000847296	8.7425×10^{-8}
0.2	0.0571451	0.103101	0.000328555	0.00108399
0.3	0.116909	0.197837	0.00219904	0.00471442
0.4	0.184052	0.2966	0.0063004	0.0114828
0.5	0.245982	0.384298	0.0133507	0.0218865
0.6	0.288685	0.44673	0.0230742	0.035986
0.7	0.297618	0.468067	0.0341012	0.0687692
0.8	0.259037	0.426808	0.0426808	0.0766527
0.9	0.161837	0.289366	0.0377946	0.0620685

Table 4

Absolute error $|V_n(x) - v(x)|$ in example 2.

x	n=4	n=8	n=10	n=12
0.1	0.00545714	0.0366141	0.00126765	0.000673806
0.2	0.0196564	0.139956	0.00503541	0.00282326
0.3	0.0393304	0.298852	0.0112292	0.00661192
0.4	0.0611635	0.500794	0.0196771	0.0121045
0.5	0.0816964	0.729633	0.0299394	0.0191209
0.6	0.0971432	0.958728	0.0409858	0.0269597
0.7	0.103094	0.133708	0.0509034	0.0339306
0.8	0.00832658	0.06651	0.156693	0.0657775
0.9	0.0629259	0.88407	0.0431735	0.0288236

Here we present $U_{10}(x)$ and $V_{10}(x)$ as approximation of exact solutions $u(x)$ and $v(x)$, respectively. For comparing numerical and exact solutions, see Figs. 3-4.

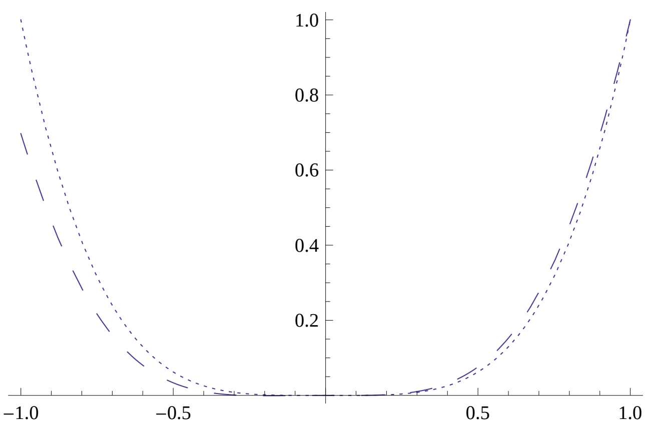

Fig. 3. Numerical solution $(\cdots)$ $U_{10}(x)$ *and exact solution* $(---)$ $u(x)$ *of example 2.*

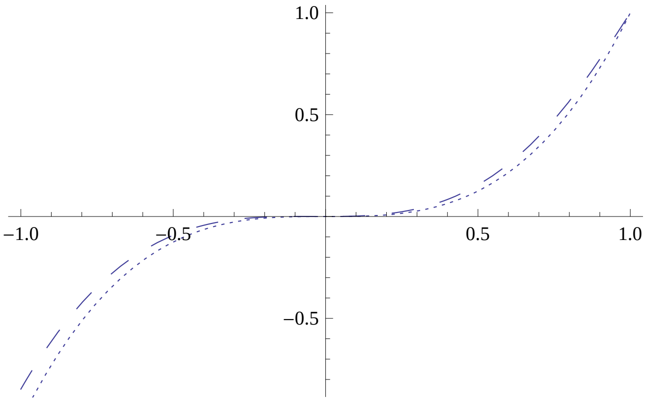

Fig. 4. Numerical solution $(\cdots)$ $V_{10}(x)$ *and exact solution* $(---)$ $v(x)$ *of example 2.*

4 Conclusion

We illustrated that the homotopy perturbation method can be used successfully for solving the system of third-order boundary value problems. Also this method can be used for solving the problem without any requirment for discretizing the variables. Therefore, this method is very effective for finding an accurate approximation of the exact solution.

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