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On Inconsistency of a Pairwise Comparison Matrix

A. Davoodi *

Department of Mathematics, Islamic Azad University, Neyshabur Branch, Neyshabur, Iran.

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Abstract

Analytical Hierarchy Process is a method of solving a Multiple Attribute Decision Making. This method is based on the pairwise comparisons between the alternatives. These comparisons form a matrix named pairwise comparison matrix or judgment matrix. The weights of mentioned alternatives will be determined from this mentioned above matrix. In the case of consistency of the judgments, the cited weights are reliable and trusty. On the other hand, if the consistency of the judgment matrix is not acceptable, the result sufficient level of trust. In this case it is requested that the decision maker revise the judgment. In this paper we propose a method to approximate a consistent judgment matrix from an inconsistent one without the revision of the judgments. This method is based on decreasing the effects of the mistakes made by the decision maker.

Keywords : Decision analysis; Analytical hierarchy process; Consistency.

1 Introduction

Analytical Hierarchy Process (AHP) is a method for solving a multiple criteria decision making problem and is widely used now (See Poh and Ang [12] , Chang et al [2], Wong and Li [19] and Vidal et al [17]). In this method the Decision Maker expresses his/her concept(opinion) as a matrix named Pairwise Comparison Matrix (Saaty [14]). A typical form of a hierarchy with k criteria and n alternatives is presented in Fig. 1.

*Email address: alirzd@yahoo.com, Tell:(+9891)53000437

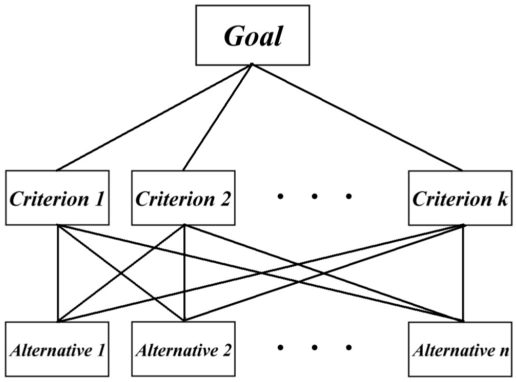

Fig. 1. A typical hierarchy with 3 levels.

In this method every pair of alternative (criterion) must be compared with each other with respect to the common alternative at their higher level. In each comparison this question must be answered: which one is more important and how much? Saaty [14] introduced the numbers $\{1, 2, ..., 9\}$ to express the rating preferences between each pair of stimulus. The pairwise comparison matrices have some interesting properties: 1)The elements on the main diagonal are equal to 1 to indicate the preference of each alternative over itself. 2) If the preference of A_i(alternative i) over A_j(alternative j) is equal to h, then the preference of A_j over A_i is equal to $\frac{1}{h}$. This property is called *reciprocally.* These properties show the elements of the judgment matrix are positive. Also suppose $a_{ij}, i, j = 1, ..., n$ are the elements of a pairwise comparison matrix for n alternatives. This matrix is called *consistent* if the equation $a_{ij} = a_{ik} * a_{kj}$ is held for all $i, j, k \in \{1, ..., n\}$. But in reality the comparison of each three alternatives may lead to creating inconsistency.

Saaty [14] also introduced a Consistency Ratio(CR) for a pairwise comparison matrix. By this definition if CR exceeds 10%, he recommended that the decision maker revise his/her elicited preferences (Saaty and Mariano [13] , Wind and Saaty [18]). Some of the limits on AHP were presented by Murphy [11] including the errors created by the scale of pairwise comparison. Inconsistency causes errors and lack of certainty to get the logical and true results. There are some approaches trying to convert an inconsistent matrix into a consistent one(or by CR less than 10%) as the modified matrix has the least difference compared to the primeval matrix (Koczkodaj and Orlowski [8]). Kwiesielewicz and Uden [9] have presented the relationship between inconsistent and contradictory of matrices of data and it shows that even if a matrix can pass a consistency test successfully, it may be contradictory. A new definition of inconsistency described in (Koczkodaj [7] , Duszak and Koczkodaj [5]) allows us to locate the roots of inconsistency and it is claimed that this definition is easier to interpret than the current one. There are two sources for inconsistency of a judgment matrix have been introduced by Dadkhah and Zahedi [3] including the decision maker's preference and the way of eliciting the decision maker's preference. Furthermore, this paper provides an algorithm for moving towards consistency and shows mathematically why the algorithm works. Several open mathematical questions about the structure of the set of positive reciprocal matrices were considered by Deturck [4]. Moreover, Aczel and Saaty [1] found the geometric mean to be the way to synthesize the judgments of several individuals to conform with the reciprocal property of paired comparisons. Also a new consistency measure, the harmonic consistency index that has been introduced by Stein and Mizzi [15], was obtained for every positive reciprocal matrix in the AHP and it was shown that this index varies with changes in any matrix element.

In this paper we propose a simple approach to get a consistent approximation of an inconsistent matrix without the revision of decision maker's preferences. Section 2 presents some preliminaries in AHP. In section 3 our method and its properties are stated. In section 4 this method is applied to an inconsistent matrix. Finally section 5 concludes.

2 Preliminaries

AHP is a method for solving an MCDM problem in discrete space. This method includes 3 steps: the first step creats a hierarchy that shows the problem graphically. The goal is at the top level, the criteria are in the intermediate levels and the alternatives are in the lowest level. The elements of each level will be compared to the elements in that level with respect to the common elements of their higher level. These comparisons lead to create pairwise comparison matrices showing the ratio preferences and value of alternatives. These ratio preferences are stated as numbers of the set $\{1, ..., 9\}$. AHP is based on calculating local weights and final weights. The local weights of each comparison matrix are determined by the eigenvector corresponding to the greatest eigenvalue of the cited matrix. In other words if A is a consistent pairwise comparison matrix, then the vector W determined by

$$AW = \lambda_{max} W$$

is the vector of local weights. The final weight of each alternative is calculated as follows:
Final weight of $A_j = \sum_i$(local weight of criterion i)$\times$(local weight of A_j with respect of criterion i).
Alternatives could be sorted by these final weights. The more the final weight, the more the priority.
Suppose P_{n*n} is a typical pairwise comparison matrix of n alternatives and $w_1, w_2, ...w_n$ are their corresponding weights. The element at the row i and the column j is $\frac{w_i}{w_j}$, so the element at the row j and the column i will be $\frac{w_j}{w_i}$(reciprocally property). Also the elements on the main diagonal are equal to 1. Moreover this matrix is consistent if the equation $\frac{w_i}{w_j} = \frac{w_i}{w_k} * \frac{w_k}{w_j}$ is held for all i, j, k, otherwise it is inconsistent. The Consistency Ratio (CR) has been introduced to show the value of the consistency of a matrix. It has been shown that for a consistent judgment matrix, its greatest eigenvalue is equal to the dimension of the matrix. So the distance between λ_{max} and n can be used as a measure for inconsistency. The consistency ratio is calculated as CR=$\frac{\lambda_{max}-n}{(n-1)*RI}$, where RI is a random inconsistency index, whose value varies with the order of pairwise comparison matrix. Table.1 shows this index for some orders. RI is the basis for defining the consistency ratio. It has been discussed by Lane and Verdini(1989) and Golden and Wang(1990). Also Tummala and Wan (1994) presented a closed-form expression for the largest eigenvalue of a three-dimensional random pairwise comparison matrix and the exact value of the mean and variance of the cited matrix. If CR ≤ 0.1, the pairwise comparison matrix is thought to have an acceptable consistency otherwise, it needs to be revised. For a consistent matrix all the methods of deriving weights have the same results. In this case the elements of each column of normalized corresponding matrix are the weights.

Table 1
Random inconsistency index for pairwise comparison matrices with the order from 1 to 10

n	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

3 An Approach to get consistency

First we present an important discussion that is useful for the rest of this paper.

3.1 Inconsistency

Suppose the following judgment matrix between three alternatives:

$$R = \begin{bmatrix} 1 & 3 & 5 \\ 1/3 & 1 & x \\ 1/5 & 1/x & 1 \end{bmatrix},$$

and w_1, w_2, w_3 are the corresponding weights respectively. Suppose the Decision Maker first presents his/her opinion about the preference of alternative 1 over others by filling row 1. Since the matrix must be reciprocal, it is sufficient to fill the elements above the main diagonal. If this matrix is consistent the following equation must be held:
$\frac{w_2}{w_3} = \frac{w_2}{w_1} * \frac{w_1}{w_3} = \frac{1}{\frac{w_1}{w_2}} * \frac{w_1}{w_3} = (1/3) * 5 = 5/3$
So if $x = 5/3$ the judgment will be consistent. Here the first reason of inconsistency appears. The general scale of AHP to compare two stimulus has the form of $\{1/9, 1/8, ..., 1/2, 1, 2, ..., 9\}$, and 5/3 does not belong to this scale. This example shows one of the reasons of inconsistency is the lack of an appropriate scale. To overcome this problem, it is more appropriate to use the numbers of the form $\{a/b; a, b \in \{1, ..., 9\}\}$. This modified scale allows the *clever* decision makers to present a consistent judgment. Now suppose the matrix R again. Decision maker judged three times: Comparison between alternative 1 and alternative 2, 2 and 3, and finally between 1 and 3. In spite of using the new scale numbers, if the matrix is still inconsistent, it can be concluded that the decision maker judged wrong at least in one of his/her comparisons. Now consider the following three situations: (a) The comparisons between alternative 1 and alternative 2 and between 1 and 3 are logically correct, so the mistake(error) has happened in comparison between alternative 2 and alternative 3. (b)The comparison between 1 and 2, 2 and 3 are correct, so the mistake happened in comparison between 1 and 3. (c)The mistake occurred while alternatives 1 and 2 is compared. Situation (a) states that the first row of the pairwise comparison matrix involves the reliable data(its elements are consistent with the subjective judgment of decision maker). Now consider situation (b), since by comparison of alternatives 1 and 2 the preference of alternative 2 over alternative 1 is recognized(by the reciprocally), as a result the elements of the second row of the pairwise comparison matrix are known. We should keep in mind that elements on the main diagonal are equal to 1. Considering the preceding discussion, the elements of the third row of the matrix are known by situation (c). Here we show that in general if the preferences of just one alternative over the others are apparent, then a consistent pairwise comparison matrix will be created. In other words, each row of a matrix conduces to a consistent matrix.
Consider an $n-$dimension pairwise comparison matrix and suppose the elements of the row

$o, o \in \{1, ..., n\}$ are known. These elements are $\frac{w_o}{w_1}, \frac{w_o}{w_2}, ..., \frac{w_o}{w_n}$, in which $w_i, i = 1, ..., n$ is the weight of alternative i. With the consistency condition, we can determine the elements of the row k as follows:
$\frac{w_k}{w_i} = \frac{w_k}{w_o} * \frac{w_o}{w_i} = \frac{1}{\frac{w_o}{w_k}} * \frac{w_o}{w_i}$, in which $\frac{w_o}{w_i}, \forall i$ and the $\frac{w_o}{w_k}$ are known by row k. We can repeat this procedure for $i = 1, ..., n \quad i \neq o$. Here the reciprocally property of matrix helps us a lot.
Example: Consider a pairwise comparison matrix with 4 rows and 4 columns. The second row is as

$$\begin{bmatrix} 3 & 1 & 2 & 1/4 \end{bmatrix}$$

we now try to create three other rows as:
$\frac{w_1}{w_1} = 1$, $\frac{w_1}{w_2} = \frac{1}{\frac{w_2}{w_1}} = 1/3$, $\frac{w_1}{w_3} = \frac{w_1}{w_2} * \frac{w_2}{w_3} = (1/3) * 2 = 2/3$, $\frac{w_1}{w_4} = \frac{w_1}{w_2} * \frac{w_2}{w_4} = (1/3) * (1/4) = 1/12$
Third row: $\frac{w_3}{w_1} = \frac{w_3}{w_2} * \frac{w_2}{w_1} = \frac{1}{\frac{w_2}{w_3}} * \frac{w_2}{w_1} = (1/2) * 3 = 3/2$, $\frac{w_3}{w_2} = \frac{1}{\frac{w_2}{w_3}} = 1/2$, $\frac{w_3}{w_3} = 1$, $\frac{w_3}{w_4} = \frac{w_3}{w_2} * \frac{w_2}{w_4} = (1/2) * (1/4) = 1/8$

In the same manner the elements of row 4 will be 12, 4, 8 and 1. So the created consistent matrix is as:

$$T = \begin{bmatrix} 1 & 1/3 & 2/3 & 1/12 \\ 3 & 1 & 2 & 1/4 \\ 3/2 & 1/2 & 1 & 1/8 \\ 12 & 4 & 8 & 1 \end{bmatrix}$$

This example shows an interesting note. If a typical pairwise comparison matrix is consistent, the elements of each row are in fact *confirmation* for other rows. For instance if we want to make a consistent matrix by the forth row as

$$\begin{bmatrix} 12 & 4 & 8 & 1 \end{bmatrix},$$

the produced matrix is the same as matrix T. If that matrix is consistent, the decision maker certifies his/her opinion iteratively by filling each row of the judgment matrix, .
Now consider an inconsistent pairwise comparison matrix again. The decision maker made at least one mistake while filling this matrix, but where? The goal is to decrease the effects of this mistake during the decision process. It is not logically true to request the decision maker to revise his/her first opinion. Since if he/she was able to recognize the mistake, probably he/she could be able to remove that in the first place. Here we propose our method to decrease the effects of carelessness of decision maker. Consider an n-dimensional inconsistent pairwise comparison matrix(P_0). Suppose the decision maker judged right at comparing the first alternative with the others. In other words the elements of the first row are logically true. We may set this row as *criterion* row. By the method described in the previous discussion, we can construct a consistent pairwise comparison matrix named P_1. Now matrix P_2 is created by setting the second row as the criterion row. The matrices $P_3, ..., P_n$ will be constructed in the same manner. Finally these matrices must be combined with each other. We propose the geometric mean as follows: consider matrix $P = [p_{ij}], i, j = 1, ..., n$. Now define $p_{ij} = (p^1_{ij} * p^2_{ij} * ... * p^n_{ij})^{\frac{1}{n}}$, in which $P_k = [p^k_{ij}]$, for $i, j, k = 1, ..., n$.

Theorem *3.1.1*. *The matrix P created by the geometric mean satisfy the following properties:*
a) elements on the main diagonal equal to 1

b) reciprocally
c) consistency

proof: Suppose w_i^k is the weight of the alternative i determined by the matrix P_k. Since matrices $P_1, ..., P_n$ are consistent pairwise comparison matrices, they are satisfied in conditions *a)* and *b)*, so
$p_{ii} = (1 * 1 * ... * 1)^{\frac{1}{n}} = 1$, for $i = 1, ..., n$ thus the first condition is held. Also
$p_{ij} = (p_{ij}^1 * p_{ij}^2 * ... * p_{ij}^n)^{\frac{1}{n}} = (\frac{w_i^1}{w_j^1} * \frac{w_i^2}{w_j^2} * ... * \frac{w_i^n}{w_j^n})^{\frac{1}{n}} = (\frac{1}{\frac{w_j^1}{w_i^1}} * \frac{1}{\frac{w_j^2}{w_i^2}} * ... * \frac{1}{\frac{w_j^n}{w_i^n}})^{\frac{1}{n}} = \frac{1}{(\frac{w_j^1}{w_i^1} * \frac{w_j^2}{w_i^2} * ... * \frac{w_j^n}{w_i^n})^{\frac{1}{n}}} = \frac{1}{p_{ji}}$,

this is the reciprocally condition. For the third condition we must show equation $p_{ij} = p_{ik} * p_{kj}$ is held for $i, j, k = 1, ..., n$.
$p_{ij} = (\frac{w_i^1}{w_j^1} * \frac{w_i^2}{w_j^2} * ... * \frac{w_i^n}{w_j^n})^{\frac{1}{n}} = ((\frac{w_i^1}{w_k^1}\frac{w_k^1}{w_j^1}) * (\frac{w_i^2}{w_k^2}\frac{w_k^2}{w_j^2}) * ... * (\frac{w_i^n}{w_k^n}\frac{w_k^n}{w_j^n}))^{\frac{1}{n}} = (\frac{w_i^1}{w_k^1} * \frac{w_i^2}{w_k^2} * ... * \frac{w_i^n}{w_k^n})^{\frac{1}{n}} * (\frac{w_k^1}{w_j^1} * \frac{w_k^2}{w_j^2} * ... * \frac{w_k^n}{w_j^n})^{\frac{1}{n}} = p_{ik} * p_{kj}$ for $i, j, k = 1, ..., n$. In which the second equality is held by the consistency property of each P_k for $k = 1, ..., n$.

Here we summarize these discussions as the following algorithm:
Algorithm:
Input: An n dimensional inconsistent pairwise comparison matrix P_0.
output: Matrix P that is a consistent approximation of the matrix P_0.
For $i = 1$ to n
Set row i as the criterion row and construct matrix P_i.
End for
By the geometric mean construct matrix P by matrices $P_1, P_2, ..., P_n$.

This method in fact decreases the effects of errors that were created by the decision maker's mistakes. Moreover the elements of the final matrix do not necessarily need to be numbers $1, ..., 9$. At the example in the next section we will study this situation.

4 Numerical Example

Matrix P_0 is inconsistent and its ratio inconsistency is equal to 0.84 (Also the value of the greatest eigenvalue is equal to 8.77). Although the matrix is inconsistent, we calculate its corresponding weight vector for comparison with the one calculated by the proposed algorithm. The (inconsistent) vector of weights is equal to (0.117, 0.099, 0.344, 0.204, 0.232). We now deal with our algorithm to get a consistent matrix. First matrices $P_1, ..., P_5$ are created and then the final matrix P will be determined by the geometric mean of these matrices.

$$P_0 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 3 \\ 2 & 1 & 1/4 & 2 & 1/5 \\ 3 & 4 & 1 & 1/3 & 8 \\ 4 & 1/2 & 3 & 1 & 1/6 \\ 1/3 & 5 & 1/8 & 6 & 1 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 3 \\ 2 & 1 & 2/3 & 1/2 & 6 \\ 3 & 3/2 & 1 & 3/4 & 9 \\ 4 & 2 & 4/3 & 1 & 12 \\ 1/3 & 1/6 & 1/9 & 1/12 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 1/2 & 1/8 & 1 & 1/10 \\ 2 & 1 & 1/4 & 2 & 1/5 \\ 8 & 4 & 1 & 8 & 4/5 \\ 1 & 1/2 & 1/8 & 1 & 1/10 \\ 10 & 5 & 5/4 & 10 & 1 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 1 & 4/3 & 1/3 & 1/9 & 8/3 \\ 3/4 & 1 & 1/4 & 1/12 & 2 \\ 3 & 4 & 1 & 1/3 & 8 \\ 9 & 12 & 3 & 1 & 24 \\ 3/8 & 1/2 & 1/8 & 1/24 & 1 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 1 & 1/8 & 3/4 & 1/4 & 1/24 \\ 8 & 1 & 6 & 2 & 1/3 \\ 4/3 & 1/6 & 1 & 1/3 & 1/18 \\ 4 & 1/2 & 3 & 1 & 1/6 \\ 24 & 3 & 18 & 6 & 1 \end{bmatrix}, P_5 = \begin{bmatrix} 1 & 15 & 3/8 & 18 & 3 \\ 1/15 & 1 & 1/40 & 6/5 & 1/5 \\ 8/3 & 40 & 1 & 48 & 8 \\ 1/18 & 5/6 & 1/48 & 1 & 1/6 \\ 1/3 & 5 & 1/8 & 6 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0.910 & 0.330 & 0.660 & 0.631 \\ 1.099 & 1 & 0.362 & 0.725 & 0.693 \\ 3.030 & 2.762 & 1 & 2 & 1.913 \\ 1.515 & 1.379 & 1/2 & 1 & 0.956 \\ 1.585 & 1.443 & 0.523 & 1.046 & 1 \end{bmatrix}$$

It is easy to show that the ratio inconsistency for intermediate matrices P_1, P_2, P_3, P_4, P_5 is equal to zero. The value of the greatest eigenvalue for all intermediate and final matrices is equal to 5. The numbers in the matrix P have been rounded. Moreover this matrix is satisfied in the conditions of theorem 3.1.1. Finally the revised vector of weights is equal to (0.121, 0.133, 0.368, 0.184, 0.192). This vector must be replaced by the first inconsistent vector in the decision process.

5 Conclusion

The existence of the right judgments of the decision maker has an important role in creating the pairwise comparison matrix in Analytical Hierarchy Process. If the judgment matrix does not have an acceptable level of consistency, the results are not reliable. There are some approaches producing consistent approximation of an inconsistent matrix. In this paper we proposed a simple objective to determine a consistent judgment matrix from an inconsistent one. This method is based on decreasing the effects of the mistakes made by the decision maker. In fact decision maker with a correct and logical judgment certifies his/her opinion by filling each row of the judgment matrix. So the elements of the matrix can be determined directly from just one row. But in the case of inconsistency of the judgment matrix, elements of some rows can not be determined from others, moreover the elements of some rows have conflicts with some others. In this case it is not clearly known where the decision maker has some mistakes. The method in this paper can decrease the effects of these mistakes in a sequence of simple computations.

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