Some Results on Modular Hyperconvex Spaces

H. R. Rahimi a, M. Firoozmasab b

(a) Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, Tehran, Iran.
(b) Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

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Abstract
In recent years, many authors have focused on hyperconvex space and obtained a lot of valuable results (see [1, 2, 3, 5]). In this paper we develop some of those results for modular hyperconvex spaces. As a consequence we show that $A_{\rho}(X_{\rho}) \subseteq \varepsilon_{\rho}(X_{\rho}) \subseteq H_{\rho}(X_{\rho})$ where, $A_{\rho}(X_{\rho}), \varepsilon_{\rho}(X_{\rho})$, and $H_{\rho}(X_{\rho})$ are modular admissible subsets, modular externally hyperconvex subsets and modular hyperconvex subsets in $X_{\rho}$, respectively.

Keywords: Hyperconvex space; Modular function; Modular hyperconvex space.

1 Introduction

Hyperconvex space, modular hyperconvex space and Fixed point theory play an important role in several subject of mathematics. For instance, it has been used in probability and mathematical statistics, boundary-value problems [3], the inverse function [9], the existence of equilibria in economics [11, 12], and the existence of solutions of differential equations [6, 10].

For the discussion of the following sections, we state here some definitions, notations and known results. For convenience of readers, we suggest that one refer to [1, 2, 4, 5, 8] for details.

Let $X$ be a vector space on $\mathbb{R}$, a function $\rho : X \to [0, +\infty]$ is called modular if for every $x, y$ in $X$, (i) $\rho(x) = 0$ if and only if $x = 0$, (ii) $\rho(\alpha x) = \rho(x)$, for every $\alpha \in \mathbb{R}$ where $|\alpha| = 1$, (iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$, and $\rho$ is called convex modular if $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ if $\alpha + \beta = 1$ and $\alpha \geq 0, \beta \geq 0$. By a modular space we mean $X_{\rho} = \{x \in X : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$, where $\rho$ is a modular function on $X$.

Following [4], for a modular space $X_{\rho}$, the sequence $\{x_n\}$ is called $\rho$-convergent to $x$ if

*Corresponding author. Email address: h_rahimi2004@yahoo.com
Let $\rho(x_n, x) \to x$, and it is called $\rho$-Cauchy if $\rho(x_n, x_m) \to 0$ as $n, m \to 0$. We will say that the modular function $\rho$ satisfy the Fatou property if $\rho(x) \leq \liminf_{n} \rho(x_n)$ as $x_n \to x$, where \{x_n\} is a sequence in $X_\rho$.

A modular function $\rho$ is called complete if every $\rho$-Cauchy sequence \{x_n\} is $\rho$-convergent. A subset $A$ of $X_\rho$ is called $\rho$-closed if the $\rho$-limit of any $\rho$-convergent sequence of $A$ always belong to $A$. By a $\rho$-ball $B_\rho(x, r)$, we mean $\{y \in X_\rho : \rho(x - y) \leq r\}$.

Finally, a subset $A$ of $X_\rho$ is called $\rho$-bounded if

$$\delta_\rho(A) = \{\rho(x - y) : x, y \in A\} < \infty$$

In general we note that $\rho$ does not a metric because $\rho$ does not satisfy the triangle inequality. For example $\rho$-convergent does not imply $\rho$-Cauchy. However, $\rho$-balls are $\rho$-closed in a modular space $X_\rho$ if and only if they have Fatou property, [5].

## 2 Main Results

In this section, we begin with basic definitions and notation. Then we discuss with more general properties on modular hyperconvexity, say, completeness of modular hyperconvex space and then we prove some technical results in modular hyperconvex spaces.

**Definition 2.1.** A modular space $X_\rho$ is called modular hyperconvex space if, for any collection of points $\{x_\alpha\}_{\alpha \in \Gamma}$ of $X$ and for any collection $\{r_\alpha\}$ of non-negative real such that $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$ ($\alpha, \beta \in \Gamma$), it follows that $\bigcap_{\alpha \in \Gamma} B_\rho(x_\alpha, r_\alpha) \neq \emptyset$.

**Theorem 2.1.** Any modular hyperconvex space is complete.

**Proof.** Let $X_\rho$ be modular hyperconvex space and $\{x_n\}_{n \geq 1}$ be a $\rho$-Cauchy sequence in $X_\rho$. For any $n \geq 1$, set $r_n = \sup_{m \geq n} \rho(x_n - x_m)$. Consider the collection of balls $\{B_\rho(x_n, r_n)\}_{n \geq 1}$. Then

$$x_{n_k} \in B_\rho(x_{n_1}, r_{n_1}) \cap B_\rho(x_{n_2}, r_{n_2}) \cap \ldots \cap B_\rho(x_{n_k}, r_{n_k})$$

where $n_1 < n_2 < \ldots < n_k$. So

$$\rho(1/2(x_{n_k} - x_{n_j})) = \rho(1/2x_{n_k} - 1/2x_{n_j} + 1/2x_{n_j} - 1/2x_{n_k})$$

$$= \rho(1/2(x_{n_k} - x_{n_j}) + 1/2(x_{n_j} - x_{n_k}))$$

$$\leq \rho(x_{n_k} - x_{n_j}) + \rho(x_{n_j}, x_{n_k})$$

$$\leq r_{n_k} + r_{n_j}$$

Now, $X_\rho$ is a modular hyperconvex space, so $\bigcap_{n \geq 1} B_\rho(x_n, r_n) \neq \emptyset$. Since $\{x_n\}_{n \geq 1}$ is a $\rho$-Cauchy sequence, $\lim_{n \to \infty} r_n = 0$, and so the intersection $\bigcap_{n \geq 1} B_\rho(x_n, r_n)$ is reduced to one point $x$ which is the $\rho$-limit of the sequence $\{x_n\}_{n \geq 1}$. 

Now we introduce some notation which will be used throughout the next Lemma.
**Definition 2.2.** Let $A$ be a subset of a modular hyperconvex space $X_{\rho}$, set

$$
\begin{align*}
  r_s(A) & = \{\sup\{d_{\rho}(x,y) : y \in A\} : x \in X_{\rho}\}; \\
  r(A) & = \inf \{r_s(A) : x \in X_{\rho}\}; \\
  R(A) & = \inf \{r_s(A) : x \in A\}; \\
  \text{diam}(A) & = \{\sup\{d_{\rho}(x,y) : x,y \in A\}\}; \\
  C(A) & = \{x \in X_{\rho} : r_s(A) = r(A)\}; \\
  C_A(A) & = \{x \in A : r_s(A) = r(A)\}; \\
  \text{cov}_{\rho}(A) & = \bigcap\{B : B \text{ is a } \rho \text{- ball and } B \supseteq A\};
\end{align*}
$$

$r(A)$ is called the reduce of $A$ (relative to $X_{\rho}$), diam$(A)$ is called the diameter of $A$, $R(A)$ is called Chebyshev radius of $A$, $C(A)$ is called the Chebyshev center of $A$, and $\text{cov}_{\rho}(A)$ is called the cover of $A$.

**Lemma 2.1.** Let $A$ be a $\rho$-bounded subset of modular hyperconvex space $X_{\rho}$, then:

1) $\text{cov}_{\rho}(A) = \bigcap\{B_{\rho}(x,r_s(A)) : x \in X_{\rho}\}.$

2) $r_s(\text{cov}_{\rho}(A)) = r_s(A)$, for any $x \in X_{\rho}$.

3) $r(\text{cov}_{\rho}(A)) = r(A)$.

4) $r(A) = 1/2(\text{diam}(A))$.

5) $\text{diam}(\text{cov}_{\rho}(A)) = \text{diam}(A)$.

6) If $A = \text{cov}_{\rho}(A)$, then $r(A) = R(A)$. In particular we have $R(A) = 1/2(\text{diam}(A))$.

**Proof.** 1) We note that $A \subseteq B_{\rho}(x,r_s(A))$ for each $x \in X_{\rho}$, so $\text{cov}_{\rho}(A) \subseteq \bigcap\{B_{\rho}(x,r_s(A)) : x \in X_{\rho}\}$. On the other hand, if $A \subseteq B_{\rho}(x,r)$ then $r_s(A) \leq r$, so $B_{\rho}(x,r_s(A)) \subseteq B_{\rho}(x,r)$.

Thus

$$\bigcap\{B_{\rho}(x,r_s(A)) : x \in X_{\rho}\} \subseteq B_{\rho}(x,r)$$

This implies that $\text{cov}_{\rho}(A) = \bigcap\{B_{\rho}(x,r_s(A)) : x \in X_{\rho}\}$.

2) By (1), $r_s(\text{cov}_{\rho}(A)) = \sup\{\rho(x-y) : y \in \bigcap_{x \in X_{\rho}} B_{\rho}(x,r_s(A))\}$. Now if $y \in \text{cov}_{\rho}(A)$ implies $y \in B_{\rho}(x,r_s(A))$ for any $x \in X_{\rho}$. Thus $r_s(\text{cov}_{\rho}(A)) \leq r_s(A)$.

On the other hand $A \subseteq \text{cov}_{\rho}(A)$ so, $r_s(A) \leq r_s(\text{cov}_{\rho}(A))$. Thus $r_s(\text{cov}_{\rho}(A)) = r_s(A)$.

3) By (2) and definition of $r$, we have $r(A) = \inf\{r_s(A) : x \in X_{\rho}\} = \inf\{r_s(\text{cov}_{\rho}(A)) : x \in X_{\rho}\}$.

4) Consider the collection $\{B_{\rho}(a,\delta/2) : a \in A\}$ where $\delta = \text{diam}(A)$. If $a,b \in A$ then $\rho(a-b) \leq \delta = (\delta/2) + (\delta/2)$ so by modular hyperconvexity,

$$\bigcap_{a \in A} B_{\rho}(a,\delta/2) \neq \emptyset$$

If $x$ is a point in this intersection then $\rho(x-a) \leq \delta/2$ so, $r_s(A) \leq \delta/2$.

On the other hand for each $a,b \in A$, $z \in X_{\rho}$ we have

$$\rho(a-b) \leq \rho(a-z) + \rho(z-b)$$

so, $\delta \leq 2r_s(A)$ imply $\delta \leq 2r(A)$. Thus $\delta \leq 2r(A) \leq 2r_s(A) \leq \delta$. Therefore $r(A) = \delta/2$.

5) By (3), (4) we have

$$\text{diam}(A) = 2r(A) = 2r(\text{cov}_{\rho}(A)) = \text{diam}(\text{cov}_{\rho}(A))$$
6) Since $1/2 \text{diam}(A) \leq r(A) \leq R(A)$ and $A = \text{cov}_\rho(A)$, so we can write $A = \bigcap_{i \in I} B_{p_i}$ where $B_{p_i}$ is $\rho$-balls in $X_\rho$ (for each $i \in I$). Now, by (4), $\bigcap_{x \in A} B_{p_i}(a, \delta/2) \neq \emptyset$ where $\delta = \text{diam}(A)$. Thus any two $\rho$-ball drawn from the collection $\{B_{p_i} : i \in I\} \cup \{B_{\rho}(a, \delta/2) : a \in A\}$ have nonempty intersection, so by hyperconvexity of $X_\rho$, $C = A \cap \{B_{\rho}(a, \delta/2) : a \in A\} = \{B_{p_i} : i \in I\} \cap \{B_{\rho}(a, \delta/2) : a \in A\} \neq \emptyset$.

Now, if $x \in C$ then, $r_x(A) \leq \delta/2$ and therefore $\delta/2 \leq r(A) \leq R(A) \leq r_x(A) \leq \delta/2$. Hence

$$r(A) = R(A) = 1/2(\text{diam}(A)) \quad \square$$

**Definition 2.3.** Let $X_\rho$ be a modular space such that has Fatou property. A subset $A$ of $X_\rho$ is called modular admissible set if $A$ is an intersection of $\rho$-closed balls in $X_\rho$.

The collection of all modular admissible sets in $X_\rho$ is denoted by $A_\rho(X_\rho)$.

**Definition 2.4.** Let $X_\rho$ be a modular space. A subset $C$ of $X_\rho$ is called modular proximal if $C \cap B_{\rho}(x, \text{dist}_\rho(x, c)) \neq \emptyset$ where $x \in X_\rho$ and

$$\text{dist}_\rho(x, c) = \inf \{\rho(x - y) : y \in C\}.$$ 

**Definition 2.5.** A subset $E$ of modular space $X_\rho$ is called modular externally hyperconvex (relative to $X_\rho$) if given any family $\{x_\alpha\}$ of point in $X_\rho$ and any family $\{r_\alpha\}$ of real positive numbers satisfying $\rho(1/2(x_\alpha - x_\beta)) \leq r_\alpha + r_\beta$ (for all $\alpha, \beta \in \Gamma$) and $\text{dist}_\rho(x_\alpha, E) \leq r_\alpha$ then it follows

$$\bigcap_{\alpha \in \Gamma} B_{\rho}(x_\alpha, r_\alpha) \cap E \neq \emptyset$$

The class of all modular externally hyperconvex subsets of $X_\rho$ is denoted by $\varepsilon_\rho(X_\rho)$ and the class of all modular hyperconvex of $X_\rho$ is denoted by $H_\rho(X_\rho)$.

**Lemma 2.2.** If $E$ is either a modular admissible or modular externally hyperconvex of a modular hyperconvex $X_\rho$. Then $E$ is modular proximal in $X_\rho$.

**Proof.** We write the proof for the case $E$ is a modular admissible subset. Other case is similar. Let $A = \bigcap_{i \in I} B_{p_i}$, then for any $\epsilon > 0$, there exists $a_\epsilon \in E$ such that $\rho(x - a_\epsilon) \leq \text{dist}_\rho(x, A) + \epsilon$.

Clearly this implies

$$\bigcap_{i \in I} B_{p_i} \cap B_{\rho}(x, \text{dist}_\rho(x, A) + \epsilon) \neq \emptyset$$

We note $a_\epsilon$ belong to the above intersection for any $\epsilon > 0$. Thus

$$A \cap B_{\rho}(x, \text{dist}_\rho(x, A)) = \bigcap_{i \in I} B_{p_i} \cap \left(\bigcap_{\epsilon > 0} B_{\rho}(x, \text{dist}_\rho(x, A) + \epsilon)\right) \neq \emptyset$$

This implies that $E$ is a modular proximal in $X_\rho$. \quad \square

**Theorem 2.2.** If $X_\rho$ is modular hyperconvex, then

$$A_\rho(X_\rho) \subseteq \varepsilon_\rho(X_\rho) \subseteq H_\rho(X_\rho)$$
This implies that there exist

Thus

Lemma 2.2.

Further, we need the following Lemma, that is similar to Lemma due to R. Sine, [7].

Lemma 2.3. If $X_\rho$ is a modular hyperconvex space and $D = \bigcap_{\alpha} B_\rho(x_\alpha, r_\alpha)$, then for any $\epsilon > 0$

Theorem 2.3. If $X_\rho$ is a modular hyperconvex space and if $A$ is a modular externally hyperconvex subset of $X_\rho$. Then $N_\epsilon(A)$ is a modular externally hyperconvex in $X_\rho$ for each $\epsilon > 0$.

Proof. Let $\{x_\alpha\}$ be sequences in $X_\rho$ and $\{r_\alpha\}$ be a sequence in $R$ such that $\rho(x_\alpha - x_\beta) \leq r_\alpha + r_\beta$, $dist(x_\alpha, N_\epsilon(A)) \leq r_\alpha$. Therefore $\rho(x_\alpha, A) \leq r_\alpha + \epsilon$. Since $A$ is modular externally hyperconvex, this implies

By Sine’s Lemma

Thus

This implies that there exist $y \in A$ such that

On the other hand, $\bigcap_{\alpha} B_\rho(x_\alpha, r_\alpha)$ is modular admissible and so is modular proximal. Thus there exist $b \in \bigcap_{\alpha} B_\rho(x_\alpha, r_\alpha)$ such that

\[
\rho(y, b) = \rho(y - b) \leq \epsilon
\]
Hence
\[ \text{dist}_\rho(b - A) = \inf \{ \rho(b - a) : a \in A \} \leq \rho(b - y) \leq \varepsilon \]
So \( b \in \mathcal{N}_\varepsilon(A) \cap (\bigcap \alpha B_\rho(x_\alpha, r_\alpha)) \). Thus
\[ \mathcal{N}_\varepsilon(A) \cap (\bigcap \alpha B_\rho(x_\alpha, r_\alpha)) \neq \emptyset \]
This means that \( \mathcal{N}_\varepsilon(A) \) is a modular externally hyperconvex in \( X_\rho \).

\[ \square \]

References


