



# Biconjugate Decomposition Using ABS Algorithms

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## Abstract

ABS method provides the general solution of a system by computing a solution and a matrix, called the Abaffian matrix, with rows generating the null space of the coefficient matrix. We present an algorithm for computing a biconjugate pair  $(V,P)$ , such that  $V^TAP = \Omega$  is a diagonal and nonsingular matrix, using ABS algorithm. Then we propose an algorithm for computing an equivalent diagonal form of a matrix  $A$  by using the extended ABS algorithm.

*Keywords* : ABS algorithms; Extended ABS algorithm; Biconjugate pair; Matrix decomposition; Biconjugate decomposition; Equivalent diagonal form.

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## 1 Introduction

ABS methods constitute a large class of methods, first introduced by Abaffy et al. [1], for solving linear algebraic systems, and later extended to solve least square problems, nonlinear algebraic equations, optimization problems [2, 8, 9] and recently to Diophantine systems [5, 7]. ABS methods are a direct iterative class of methods for solving linear equations. Each method in the class provides the general solution of the system by computing a particular solution and a matrix, the Abaffian matrix, with rows generating the null space of the coefficient matrix. The method starts with an initial vector  $x_1 \in R^n$  (arbitrary) and a nonsingular matrix  $H_1 \in R^{n \times n}$  (Spedicato's parameter). Given  $x_i$  as a solution of the first  $i - 1$  equations, and the Abaffian matrix  $H_i$  with rows generating the null space of the first  $i - 1$  equations, the ABS algorithm computes  $x_{i+1}$  and  $H_{i+1}$  as the solution and null space generator of the first  $i$  equations, respectively. The choices of

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- (1) Choose  $H_1 \in R^{n \times n}$ , arbitrary and nonsingular. Let  $i=1$ , and  $r_i = 0$ .
- (2) Compute  $s_i = H_i a_i$ .
- (3) **If**  $s_i = 0$  **and** , **then** let  $H_{i+1} = H_i$ ,  $r_{i+1} = r_i$  and **go to** step (6).
- (4) Compute search vector  $p_i$  by  $p_i = H_i^T z_i$ , where  $z_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T z_i \neq 0$ .
- (5) Update matrix  $H_i$  by,
$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$
where  $w_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T w_i \neq 0$ .
- (6) **If**  $i = m$  **then Stop** ( $r_{m+1}$  is the rank of  $A$ ), **else** let  $i = i + 1$  and **go to** step (2).

Matrices  $H_i$ , which are generalizations of projection matrices, have been called Abaffians.

Chen et al. [4] introduced a generalization of the ABS algorithms, called extended ABS (EABS) class of algorithms for the real case, which differs from the ABS algorithms only in updating the Abaffian matrices  $H_i$ . In the EABS algorithms, the Abaffian matrices  $H_i$  are updated as follows:

- $H_{i+1} = G_i H_i$ , where  $G_i \in R^{j_{i+1} \times j_i}$  is such that we have  $G_i x = 0$  if and only if  $x = \lambda H_i a_i$ , for some  $\lambda \in R$ .

For the (not necessarily independent) rows of  $H_{i+1}$  to be the generator of the null space of the first  $i$  rows of  $A$ , we must have  $\text{rank}(G_i) \geq n - r_{i+1}$ , where  $r_{i+1}$  is the rank of the first  $i$  rows of  $A$ . It can be easily verified that if in the EABS algorithm, we let  $j_1 = \dots = j_i = \dots = n$  and  $G_i = I - H_i a_i w_i^T / w_i^T H_i a_i$ ,  $1 \leq i \leq m$ , where  $w_i$  satisfies  $w_i^T H_i a_i \neq 0$ , then the EABS algorithm turns into a basic ABS algorithm. Furthermore, as in the ABS algorithms, in the EABS algorithms, for every  $i$ ,  $1 \leq i \leq m$ , we have  $H_i a_i \neq 0$  if and only if  $a_i$  is linearly independent of  $a_1, a_2, \dots, a_{i-1}$ . Indeed, the general solution of the first  $i - 1$  equations of the system is  $x_i + H_i^T y_i$ ,  $y_i \in R^{j_i}$  (see [4]).

**Remark 2.1.** *We can see that the extended ABS algorithms can always be tuned to produce a basis for the null space of the coefficient matrix. Let  $G_i \in R^{n-r_i \times n-r_i+1}$ , then,  $H_{i+1} = G_i H_i$  is a full row rank matrix and generates a basis for the null space of the first  $i$  rows of  $A$ .*

We recall some properties of the ABS class, assuming that  $A$  has full rank.

**p1:** The vector  $H_i a_i$  is zero if and only if  $a_i$  is linearly dependent on  $a_1, \dots, a_{i-1}$ .

**p2:** The vector  $H_i^T w_i$  is zero if and only if  $w_i$  is linearly dependent on  $w_1, \dots, w_{i-1}$ .

**p3:** Define  $A_i = (a_1, \dots, a_i)$  and  $W_i = (w_1, \dots, w_i)$ . Then,

$$H_{i+1} A_i = 0, \quad H_{i+1}^T W_i = 0. \quad (2.2)$$



$$MAP = L = \begin{pmatrix} l \\ 0 \end{pmatrix}, \quad (3.6)$$

where matrix  $l \in R^{r \times r}$  is a nonsingular lower triangular matrix.

Here, we state a theorem and then show how to choose parameters of the ABS algorithm, in **phase 2** to compute a biconjugate decomposition of  $A$ .

**Theorem 3.1.** *Let  $A \in R^{n \times n}$  be strongly nonsingular (i.e., all principal submatrices are nonsingular). Then, the choices  $H_1 = I$  and  $w_i = e_i$  are well defined and the following properties hold:*

- (a) *The first  $r$  rows of  $H_{i+1}$  are identically zero.*
- (b) *The last  $n - i$  columns of  $H_{i+1}$  are equal to the last  $n - i$  columns of  $H_1$ .*
- (c)  *$P = (p_1, \dots, p_n)$  is an upper triangular matrix.*

**Proof.** See [2].

Let  $B = L^T$ , since the submatrix  $l$  is strongly nonsingular. Now, we apply the ABS algorithm with coefficient matrix  $B$ . By Theorem (3.1), we can compute Abaffian matrices and search vectors as follow:

Let  $R_1 = I_{m,m}$ , update  $R_i$  by,

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T H_i}{e_i^T R_i b_i},$$

where  $b_i$  is the  $i$ th row of  $B$ , for  $i = 1, \dots, r$ . Let  $Q = (q_1, \dots, q_r)$  where  $q_i = R_i^T e_i$ . According to Theorem (3.1),  $Q$  is an upper triangular matrix thus,  $BQ$  is also an upper triangular matrix and by ABS properties (**p4**),  $BQ$  is a lower triangular matrix, therefore,  $BQ$  is a nonsingular and diagonal matrix and, we have,

$$\Omega = Q^T B^T = Q^T MAP = V^T AP,$$

is a biconjugate decomposition of  $A$ , where,  $V = M^T Q$ . Moreover, we have,

$$\text{rank}(A) = \text{rank}(V) = \text{rank}(P) = r, \quad V \in R^{m \times r}, \quad P \in R^{n \times r}.$$

Now, we ready to present an algorithm.

### Algorithm 2. A biconjugate decomposition by ABS algorithm

#### First Phase:

- (1) Choose  $H_1 \in R^{n \times n}$ , arbitrary and nonsingular, and  $M = I_{m,m}$ . Let  $i=1$ , and  $r_i = 0$ .
- (2) Compute  $s_i = H_i a_i$ . **If**  $s_i = 0$ , **then** let  $H_{i+1} = H_i$ ,  $r_{i+1} = r_i$ , shift the  $i$ th row of  $A$  and  $M$  to the end of the matrices, and **go to** step (5) (the  $i$ th equation is redundant) **else**  $r_{i+1} = r_i + 1$ .



(3) Compute  $b_i = Ap_i$  and search vector  $v_i$  by,  $v_i = R_i^T e_i$ , where  $e_i \in R^m$  is the  $i$ th unit vector.

(4) Update matrices  $H_i$  and  $R_i$  by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$

where  $w_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T w_i \neq 0$ , and

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T R_i}{e_i^T R_i b_i}.$$

(5) If  $i < m$ , then let  $i = i + 1$ , go to step (2), else let

$V = (v_1, \dots, v_m)$  and  $P = (p_1, \dots, p_m)$ . Then

$$\Omega = V^T AP,$$

is a biconjugate decomposition of  $A$ .

(6) Stop.

### An equivalent diagonal of a matrix by ABS algorithm

**Definition 3.2.** Matrices  $A, D \in R^{m,n}$  are said to be equivalent if there exist nonsingular matrices  $V \in R^{m,m}$  and  $U \in R^{n,n}$ , such that

$$V^T AU = D \tag{3.7}$$

Here, we present a two-phase algorithm based on the extended ABS algorithm for computing nonsingular matrices  $V$  and  $U$ , such that  $V^T AU \in R^{m \times n}$  is a diagonal matrix. Assume that  $A \in R^{m \times n}$  is of rank  $r$ . For the aim of computation of nonsingular matrices  $V$  and  $U$ , in the first phase we apply extended the ABS algorithm and obtain the full rank matrix  $H_{m+1}$  as a basis for the null space of  $MA$  ( $M$  is a permutation matrix such that the first  $r$  rows of  $MA$  are independent). Then,  $U = (p_1, \dots, p_r, H_{m+1}^T)$  is a nonsingular matrix and  $MAU = L \in R^{m \times n}$  is a lower triangular matrix. Let  $B = L^T$ . Now, apply the ABS algorithm with coefficient matrix  $B$ . By Theorem (3.1), we can compute Abaffian matrices and search vectors as follows:

Let  $R_1 = I_{m,m}$ , update  $R_i$  by,

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T H_i}{e_i^T R_i b_i},$$

where  $b_i$  is the  $i$ th row of  $B$ , for  $i = 1, \dots, r$ . According to Theorem (3.1), first  $r$  rows of  $R_{m+1}$  equal zero, then we delete the zero rows for generating a basis for the null space of  $B$ . Let  $Q = (q_1, \dots, q_r, R_{r+1}^T)$ , where  $q_i = R_i^T e_i$ , then  $Q$  is a nonsingular upper triangular matrix and  $BQ$  is a diagonal matrix. Therefore,

$$D = Q^T B^T = Q^T MAU = V^T AU,$$





## 4 Examples

In this section, we compute a biconjugate decomposition of matrix  $A$  using the proposed algorithms.

**Example 4.1.** Consider the following matrix:

$$A = \begin{bmatrix} 75 & 50 & 75 & 100 & 50 \\ 50 & 50 & 100 & 75 & 100 \\ 100 & 50 & 50 & 50 & 50 \\ 25 & 75 & 50 & 100 & 25 \\ 75 & 25 & 100 & 100 & 50 \end{bmatrix}.$$

Upon an application of our proposed algorithm to compute a biconjugate decomposition of  $A$ , we obtain the following results.

$$V = \begin{pmatrix} 1 & -0.7 & -2 & 1 & -1.7 \\ 0 & 1 & 1 & -3 & -0.3 \\ 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0.7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 1 & -0.7 & -1 & 1 & 0 \\ 0 & 1 & -0.5 & -3 & -0.3 \\ 0 & 0 & 0 & 1 & -1.3 \\ 0 & 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\Omega = V^T A U = \begin{pmatrix} 75 & 0 & 0 & 0 & 0 \\ 0 & 16.7 & 0 & 0 & 0 \\ 0 & 0 & -75 & 0 & 0 \\ 0 & 0 & 0 & -150 & 0 \\ 0 & 0 & 0 & 0 & -25 \end{pmatrix},$$

is an biconjugate decomposition of  $A$ .

## 5 Conclusion

In this paper, we presented some algorithms for computing an diagonal form of matrix  $A$  based on ABS algorithms. We presented a two-phase algorithm for computing a biconjugate decomposition of a matrix  $A$  with arbitrary rank, making use of ABS algorithms. Also, we proposed an algorithm for computing an equivalent diagonal form of a matrix  $A$  using the extended ABS algorithm.

## References

- [1] J. Abaffy, C.G Broyden, E. Spedicato, A class of direct methods for linear systems, Numerische Mathematik 45 (1984) 361-376.

