



# Biconjugate Decomposition Using ABS Algorithms

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## Abstract

ABS method provides the general solution of a system by computing a solution and a matrix, called the Abaffian matrix, with rows generating the null space of the coefficient matrix. We present an algorithm for computing a biconjugate pair  $(V,P)$ , such that  $V^TAP = \Omega$  is a diagonal and nonsingular matrix, using ABS algorithm. Then we propose an algorithm for computing an equivalent diagonal form of a matrix  $A$  by using the extended ABS algorithm.

*Keywords* : ABS algorithms; Extended ABS algorithm; Biconjugate pair; Matrix decomposition; Biconjugate decomposition; Equivalent diagonal form.

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## 1 Introduction

ABS methods constitute a large class of methods, first introduced by Abaffy et al. [1], for solving linear algebraic systems, and later extended to solve least square problems, nonlinear algebraic equations, optimization problems [2, 8, 9] and recently to Diophantine systems [5, 7]. ABS methods are a direct iterative class of methods for solving linear equations. Each method in the class provides the general solution of the system by computing a particular solution and a matrix, the Abaffian matrix, with rows generating the null space of the coefficient matrix. The method starts with an initial vector  $x_1 \in R^n$  (arbitrary) and a nonsingular matrix  $H_1 \in R^{n \times n}$  (Spedicato's parameter). Given  $x_i$  as a solution of the first  $i - 1$  equations, and the Abaffian matrix  $H_i$  with rows generating the null space of the first  $i - 1$  equations, the ABS algorithm computes  $x_{i+1}$  and  $H_{i+1}$  as the solution and null space generator of the first  $i$  equations, respectively. The choices of

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the parameters within the ABS class determine particular methods [8]. The implicit QR factorization of the coefficient matrix  $A$  via Gram-Schmidt algorithms [2], the implicit LU factorization of  $A$  via Gaussian elimination techniques [2], LX factorization [10], Krylov's methods [2], Broyden's methods [3] and recently Rosser's algorithm for solving Diophantine systems [6, 7] all belong to this class.

Matrix factorizations or decompositions reign supreme in providing practical numerical algorithms and theoretical linear algebra insights. Matrix factorizations are examples of perhaps the most important strategy of numerical analysis: replace a relatively difficult problem with a much easier one. Diagonal systems are easier to solve than full systems. The purpose of this paper is to compute a decomposition, ending in a diagonal matrix, making use of ABS algorithms.

The remainder of the paper is organized as follows. In Section 2, we describe the ABS method for computing the general solution of a linear system. In Section 3, we present two algorithms for computing a diagonal form of matrix  $A$ . The first algorithm, use of ABS algorithm for computing a biconjugate decomposition of  $A$ , and the second one, use of the extended ABS algorithm and proposes a method for computing an equivalent diagonal form of  $A$ . In Section 4 we report a numerical result. We conclude the paper in Section 5.

## 2 ABS algorithms for linear equations

Classes of ABS algorithms have been introduced for solving linear systems of equations (see [1, 8]) based on the basic ABS algorithms for solving real linear systems of equations. Consider the following determined or underdetermined linear system, where  $\text{rank}(A)$  is arbitrary and  $A = (a_1, \dots, a_n)^T$ ,

$$Ax = b, \quad x \in R^n, \quad A \in R^{m \times n}, \quad b \in R^m, \quad m \leq n \quad (2.1)$$

An ABS method starts with an arbitrary initial vector  $x_1 \in R^n$  and a nonsingular matrix  $H_1 \in R^{n \times n}$ , Spedicato's parameter. Given  $x_i$ , a solution of the first  $i-1$  equations, and  $H_i$ , a matrix with rows generating the null space of the first  $i-1$  rows of the coefficient matrix, an ABS algorithm computes  $x_{i+1}$  as a solution of the first  $i$  equations and  $H_{i+1}$ , with rows generating the null space of the first  $i$  rows of the coefficient matrix, by performing the following steps (see [1, 2]):

- (1) Determine  $z_i$  (Broyden's parameter) such that  $z_i^T H_i a_i \neq 0$  and set  $p_i = H_i^T z_i$ .
- (2) Update the Abaffian matrix  $H_i$  by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i},$$

with  $w_i \in R^n$  (Abaffy's parameter) satisfying  $w_i^T H_i a_i \neq 0$ .

In an ABS algorithm, we have  $s_i = H_i a_i \neq 0$  if and only if  $a_i$  is linearly independent of  $a_1, a_2, \dots, a_{i-1}$  (or equivalently,  $s_i = 0$  if and only if  $a_i$  is linearly dependent on  $a_1, a_2, \dots, a_{i-1}$ ).

System (2.1) can be solved by the following ABS class of algorithms [8] (note that below,  $r_i$  would give the rank of the first  $i-1$  rows of  $A$ ).

**Algorithm 1. ABS algorithm .**

- (1) Choose  $H_1 \in R^{n \times n}$ , arbitrary and nonsingular. Let  $i=1$ , and  $r_i = 0$ .
- (2) Compute  $s_i = H_i a_i$ .
- (3) **If**  $s_i = 0$  **and** , **then** let  $H_{i+1} = H_i$ ,  $r_{i+1} = r_i$  and **go to** step (6).
- (4) Compute search vector  $p_i$  by  $p_i = H_i^T z_i$ , where  $z_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T z_i \neq 0$ .
- (5) Update matrix  $H_i$  by,
$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$
where  $w_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T w_i \neq 0$ .
- (6) **If**  $i = m$  **then Stop** ( $r_{m+1}$  is the rank of  $A$ ), **else** let  $i = i + 1$  and **go to** step (2).

Matrices  $H_i$ , which are generalizations of projection matrices, have been called Abaffians.

Chen et al. [4] introduced a generalization of the ABS algorithms, called extended ABS (EABS) class of algorithms for the real case, which differs from the ABS algorithms only in updating the Abaffian matrices  $H_i$ . In the EABS algorithms, the Abaffian matrices  $H_i$  are updated as follows:

- $H_{i+1} = G_i H_i$ , where  $G_i \in R^{j_{i+1} \times j_i}$  is such that we have  $G_i x = 0$  if and only if  $x = \lambda H_i a_i$ , for some  $\lambda \in R$ .

For the (not necessarily independent) rows of  $H_{i+1}$  to be the generator of the null space of the first  $i$  rows of  $A$ , we must have  $\text{rank}(G_i) \geq n - r_{i+1}$ , where  $r_{i+1}$  is the rank of the first  $i$  rows of  $A$ . It can be easily verified that if in the EABS algorithm, we let  $j_1 = \dots = j_i = \dots = n$  and  $G_i = I - H_i a_i w_i^T / w_i^T H_i a_i$ ,  $1 \leq i \leq m$ , where  $w_i$  satisfies  $w_i^T H_i a_i \neq 0$ , then the EABS algorithm turns into a basic ABS algorithm. Furthermore, as in the ABS algorithms, in the EABS algorithms, for every  $i$ ,  $1 \leq i \leq m$ , we have  $H_i a_i \neq 0$  if and only if  $a_i$  is linearly independent of  $a_1, a_2, \dots, a_{i-1}$ . Indeed, the general solution of the first  $i - 1$  equations of the system is  $x_i + H_i^T y_i$ ,  $y_i \in R^{j_i}$  (see [4]).

**Remark 2.1.** *We can see that the extended ABS algorithms can always be tuned to produce a basis for the null space of the coefficient matrix. Let  $G_i \in R^{n-r_i \times n-r_i+1}$ , then,  $H_{i+1} = G_i H_i$  is a full row rank matrix and generates a basis for the null space of the first  $i$  rows of  $A$ .*

We recall some properties of the ABS class, assuming that  $A$  has full rank.

**p1:** The vector  $H_i a_i$  is zero if and only if  $a_i$  is linearly dependent on  $a_1, \dots, a_{i-1}$ .

**p2:** The vector  $H_i^T w_i$  is zero if and only if  $w_i$  is linearly dependent on  $w_1, \dots, w_{i-1}$ .

**p3:** Define  $A_i = (a_1, \dots, a_i)$  and  $W_i = (w_1, \dots, w_i)$ . Then,

$$H_{i+1} A_i = 0, \quad H_{i+1}^T W_i = 0. \quad (2.2)$$



$$MAP = L = \begin{pmatrix} l \\ 0 \end{pmatrix}, \quad (3.6)$$

where matrix  $l \in R^{r \times r}$  is a nonsingular lower triangular matrix.

Here, we state a theorem and then show how to choose parameters of the ABS algorithm, in **phase 2** to compute a biconjugate decomposition of  $A$ .

**Theorem 3.1.** *Let  $A \in R^{n \times n}$  be strongly nonsingular (i.e., all principal submatrices are nonsingular). Then, the choices  $H_1 = I$  and  $w_i = e_i$  are well defined and the following properties hold:*

- (a) *The first  $r$  rows of  $H_{i+1}$  are identically zero.*
- (b) *The last  $n - i$  columns of  $H_{i+1}$  are equal to the last  $n - i$  columns of  $H_1$ .*
- (c)  *$P = (p_1, \dots, p_n)$  is an upper triangular matrix.*

**Proof.** See [2].

Let  $B = L^T$ , since the submatrix  $l$  is strongly nonsingular. Now, we apply the ABS algorithm with coefficient matrix  $B$ . By Theorem (3.1), we can compute Abaffian matrices and search vectors as follow:

Let  $R_1 = I_{m,m}$ , update  $R_i$  by,

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T H_i}{e_i^T R_i b_i},$$

where  $b_i$  is the  $i$ th row of  $B$ , for  $i = 1, \dots, r$ . Let  $Q = (q_1, \dots, q_r)$  where  $q_i = R_i^T e_i$ . According to Theorem (3.1),  $Q$  is an upper triangular matrix thus,  $BQ$  is also an upper triangular matrix and by ABS properties (**p4**),  $BQ$  is a lower triangular matrix, therefore,  $BQ$  is a nonsingular and diagonal matrix and, we have,

$$\Omega = Q^T B^T = Q^T MAP = V^T AP,$$

is a biconjugate decomposition of  $A$ , where,  $V = M^T Q$ . Moreover, we have,

$$\text{rank}(A) = \text{rank}(V) = \text{rank}(P) = r, \quad V \in R^{m \times r}, \quad P \in R^{n \times r}.$$

Now, we ready to present an algorithm.

### **Algorithm 2. A biconjugate decomposition by ABS algorithm**

#### **First Phase:**

- (1) Choose  $H_1 \in R^{n \times n}$ , arbitrary and nonsingular, and  $M = I_{m,m}$ . Let  $i=1$ , and  $r_i = 0$ .
- (2) Compute  $s_i = H_i a_i$ . **If**  $s_i = 0$ , **then** let  $H_{i+1} = H_i$ ,  $r_{i+1} = r_i$ , shift the  $i$ th row of  $A$  and  $M$  to the end of the matrices, and **go to** step (5) (the  $i$ th equation is redundant) **else**  $r_{i+1} = r_i + 1$ .



(3) Compute  $b_i = Ap_i$  and search vector  $v_i$  by,  $v_i = R_i^T e_i$ , where  $e_i \in R^m$  is the  $i$ th unit vector.

(4) Update matrices  $H_i$  and  $R_i$  by

$$H_{i+1} = H_i - \frac{H_i a_i w_i^T H_i}{w_i^T H_i a_i}$$

where  $w_i \in R^n$  is arbitrary save for the condition  $a_i^T H_i^T w_i \neq 0$ , and

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T R_i}{e_i^T R_i b_i}.$$

(5) If  $i < m$ , then let  $i = i + 1$ , go to step (2), else let

$V = (v_1, \dots, v_m)$  and  $P = (p_1, \dots, p_m)$ . Then

$$\Omega = V^T AP,$$

is a biconjugate decomposition of  $A$ .

(6) Stop.

### An equivalent diagonal of a matrix by ABS algorithm

**Definition 3.2.** Matrices  $A, D \in R^{m,n}$  are said to be equivalent if there exist nonsingular matrices  $V \in R^{m,m}$  and  $U \in R^{n,n}$ , such that

$$V^T AU = D \tag{3.7}$$

Here, we present a two-phase algorithm based on the extended ABS algorithm for computing nonsingular matrices  $V$  and  $U$ , such that  $V^T AU \in R^{m \times n}$  is a diagonal matrix. Assume that  $A \in R^{m \times n}$  is of rank  $r$ . For the aim of computation of nonsingular matrices  $V$  and  $U$ , in the first phase we apply extended the ABS algorithm and obtain the full rank matrix  $H_{m+1}$  as a basis for the null space of  $MA$  ( $M$  is a permutation matrix such that the first  $r$  rows of  $MA$  are independent). Then,  $U = (p_1, \dots, p_r, H_{m+1}^T)$  is a nonsingular matrix and  $MAU = L \in R^{m \times n}$  is a lower triangular matrix. Let  $B = L^T$ . Now, apply the ABS algorithm with coefficient matrix  $B$ . By Theorem (3.1), we can compute Abaffian matrices and search vectors as follows:

Let  $R_1 = I_{m,m}$ , update  $R_i$  by,

$$R_{i+1} = R_i - \frac{R_i b_i e_i^T H_i}{e_i^T R_i b_i},$$

where  $b_i$  is the  $i$ th row of  $B$ , for  $i = 1, \dots, r$ . According to Theorem (3.1), first  $r$  rows of  $R_{m+1}$  equal zero, then we delete the zero rows for generating a basis for the null space of  $B$ . Let  $Q = (q_1, \dots, q_r, R_{r+1}^T)$ , where  $q_i = R_i^T e_i$ , then  $Q$  is a nonsingular upper triangular matrix and  $BQ$  is a diagonal matrix. Therefore,

$$D = Q^T B^T = Q^T MAU = V^T AU,$$





## 4 Examples

In this section, we compute a biconjugate decomposition of matrix  $A$  using the proposed algorithms.

**Example 4.1.** Consider the following matrix:

$$A = \begin{bmatrix} 75 & 50 & 75 & 100 & 50 \\ 50 & 50 & 100 & 75 & 100 \\ 100 & 50 & 50 & 50 & 50 \\ 25 & 75 & 50 & 100 & 25 \\ 75 & 25 & 100 & 100 & 50 \end{bmatrix}.$$

Upon an application of our proposed algorithm to compute a biconjugate decomposition of  $A$ , we obtain the following results.

$$V = \begin{pmatrix} 1 & -0.7 & -2 & 1 & -1.7 \\ 0 & 1 & 1 & -3 & -0.3 \\ 0 & 0 & 1 & 0.5 & 0 \\ 0 & 0 & 0 & 1 & 0.7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$U = \begin{pmatrix} 1 & -0.7 & -1 & 1 & 0 \\ 0 & 1 & -0.5 & -3 & -0.3 \\ 0 & 0 & 0 & 1 & -1.3 \\ 0 & 0 & 1 & 0 & 0.7 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\Omega = V^T A U = \begin{pmatrix} 75 & 0 & 0 & 0 & 0 \\ 0 & 16.7 & 0 & 0 & 0 \\ 0 & 0 & -75 & 0 & 0 \\ 0 & 0 & 0 & -150 & 0 \\ 0 & 0 & 0 & 0 & -25 \end{pmatrix},$$

is an biconjugate decomposition of  $A$ .

## 5 Conclusion

In this paper, we presented some algorithms for computing an diagonal form of matrix  $A$  based on ABS algorithms. We presented a two-phase algorithm for computing a biconjugate decomposition of a matrix  $A$  with arbitrary rank, making use of ABS algorithms. Also, we proposed an algorithm for computing an equivalent diagonal form of a matrix  $A$  using the extended ABS algorithm.

## References

- [1] J. Abaffy, C.G Broyden, E. Spedicato, A class of direct methods for linear systems, Numerische Mathematik 45 (1984) 361-376.

