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New family of Two-Parameters Iterative Methods for Non-Linear Equations with Fourth-Order Convergence

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Abstract

In this paper, we present a new two-parameters family of iterative methods for solving non-linear equations and prove that the order of convergence of these methods is at least four. Per iteration of these new methods require two evaluations of the function and two evaluations of its first derivative. Several numerical examples are given to illustrate the performance of the presented methods.

Keywords: Newton's method; Iterative methods; Non-linear equations; Weerakoom-Fernando's method; Fourth-order.

1 Introduction

Solving non-linear equations is one of the most important problems in numerical analysis. In this paper, a family of iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ of a non-linear equation $f(x) = 0$ is presented, where $f : I \to \mathbb{R}$ for an open interval I is a scalar function.

Newton's method for a non-linear equation is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.1}$$

this is an important and basic method, which converges quadratically.
A modification of Newton's method with third-order convergence due to Weerakoom and Fernando [6], defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n - \frac{f(x_n)}{f'(x_n)}) + f'(x_n)}. \tag{1.2}$$

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In this paper, (1.1) and (1.2) are used for the construction of the new iterative methods. The organization of paper as follows:
In Section 2 the methods based on Weerakoom-Fernando's method are given then the order of convergence is analyzed. In section 3 their better performance is also illustrated by numerical results.

2 The methods and their analysis of convergence

The following iterative method is considered

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \tag{2.3}$$

$$z_n = x_n - \frac{2f(x_n)}{f'(x_n - \frac{f(x_n)}{f'(x_n)}) + f'(x_n)}. \tag{2.4}$$

Our aim is to find a correction term for (2.3) and (2.4) that will yield a family with fourth-order convergence. To do this, first consider fitting the function $f(x)$ around the point $(x_n, f(x_n))$ with the third-degree polynomial

$$g(x) = ax^3 + bx^2 + cx + d. \tag{2.5}$$

Using the tangency condition at the $n-$th iterate x_n

$$g'(x_n) = f'(x_n), \tag{2.6}$$

so, from (2.5) and (2.6) we can obtain c as follows:

$$c = f'(x_n) - 3ax_n^2 - 2bx_n, \tag{2.7}$$

which the first derivative of the approximating is as follows: polynomial

$$g'(x) = 3ax^2 + 2bx + f'(x_n) - 3ax_n^2 - 2bx_n. \tag{2.8}$$

Now, we get $f'(z_n) \approx g'(z_n)$ and when z_n is defined by (2.4), it is clear that

$$f'(z_n) \approx \frac{f'(x_n)(f'(y_n) + f'(x_n)) + 4(\mu - \lambda x_n)f(x_n) + 4\lambda f^2(x_n)}{f'(y_n) + f'(x_n)}, \tag{2.9}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ and $\mu = -b$ and $\lambda = 3a$, then by considering (2.3) and (2.4), our new methods are

$$x_{n+1} = z_n - \frac{f(z_n)(f'(y_n) + f'(x_n))}{f'(x_n)(f'(y_n) + f'(x_n)) + 4(\mu - \lambda x_n)f(x_n) + 4\lambda f^2(x_n)}, \tag{2.10}$$

$$z_n = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}, \tag{2.11}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2.12}$$

where $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.
For the methods defined by (2.10)-(2.12), we consider the following theorem:

Theorem 2.1. *Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \to \mathbb{R}$ for an open interval I, then the methods defined by (2.10)-(2.12), have a minimum order of convergence equal to four and it satisfies the following error equation:*

$$e_{n+1} = [c_2c_3 + 2c_2^3 + (\frac{\mu - \lambda\alpha}{f'(\alpha)})(2c_2^2 + c_3)]\ e_n^4 + O(e_n^5), \tag{2.13}$$

where $c_2 = \frac{f''(\alpha)}{2f'(\alpha)}$, $c_3 = \frac{f'''(\alpha)}{6f'(\alpha)}$, $\mu \in \mathbb{R}$ *and* $\lambda \in \mathbb{R}$.

Proof: Let $e_n = x_n - \alpha$. Using Taylor expansion and taking $f(\alpha) = 0$ into account

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{2.14}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + \cdots], \tag{2.15}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!\ f'(\alpha)}$, $k = 2, 3, \cdots$. Dividing (2.14) by (2.15) gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + \cdots. \tag{2.16}$$

Now, by using $f'(x) = f'(\alpha)[1 + 2c_2(x - \alpha) + 3c_3(x - \alpha)^2 + \cdots]$ and (2.16), we get

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + \cdots], \tag{2.17}$$

then

$$f'(y_n) + f'(x_n) = f'(\alpha)[2 + 2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_2c_3 - c_2^3 + c_4)e_n^3 + \cdots], \tag{2.18}$$

and

$$\frac{1}{f'(y_n) + f'(x_n)} = \frac{1}{2f'(\alpha)}[1 - c_2e_n - \frac{3}{2}c_3e_n^2 + (c_2c_3 + 3c_2^3 - 2c_4)e_n^3 + \cdots]. \tag{2.19}$$

From (2.14) and (2.19) we obtain the following expansion

$$\frac{2f(x_n)}{f'(y_n) + f'(x_n)} = e_n - (c_2^2 + \frac{c_3}{2})e_n^3 + (-c_4 - \frac{3}{2}c_2c_3 + 3c_2^3)e_n^4 + \cdots. \tag{2.20}$$

Now, by using $f(x) = f'(\alpha)[(x - \alpha) + c_2(x - \alpha)^2 + c_3(x - \alpha)^3 + \cdots]$ and above equations, the following expansions is concluded

$$f(z_n) = f'(\alpha)[(c_2^2 + \frac{c_3}{2})e_n^3 + (c_4 + \frac{3}{2}c_2c_3 - 3c_2^3)e_n^4 + \cdots], \tag{2.21}$$

$$f(z_n)(f'(y_n) + f'(x_n)) = f'^2(\alpha)[(2c_2^2 + c_3)e_n^3 + (4c_2c_3 - 4c_2^3 + 2c_4)e_n^4 + \cdots], \tag{2.22}$$

$$f'(x_n)(f'(y_n) + f'(x_n)) = f'^2(\alpha)[2 + 6c_2e_n + (6c_2^2 + 9c_3)e_n^2 + (16c_2c_3 + 12c_4)e_n^3 + \cdots], \tag{2.23}$$

$$4(\mu - \lambda x_n)f(x_n) = f'^2(\alpha)[4Ae_n + 4Be_n^2 + 4Ce_n^3 + \cdots], \tag{2.24}$$

where $A = \frac{(\mu - \lambda\alpha)}{f'(\alpha)}$, $B = \frac{(\mu c_2 - \lambda\alpha c_2 - \lambda)}{f'(\alpha)}$ and $C = \frac{(\mu c_3 - \lambda\alpha c_3 - \lambda c_2)}{f'(\alpha)}$,

and

$$4\lambda f^2(x_n) = f'^2(\alpha)[4\lambda e_n^2 + 8\lambda c_2e_n^3 + \cdots], \tag{2.25}$$

therefore

$$4\lambda f^2(x_n) + 4(\mu - \lambda x_n)f(x_n) + f'(x_n)(f'(y_n) + f'(x_n)) = 2f'^2(\alpha)[1 + (2A + 3c_2)e_n + (2\lambda + 2B + \frac{9}{2}c_3 + 3c_2^2)e_n^2 + \cdots]. \tag{2.26}$$

Now, dividing (2.22) by (2.26) and equation (2.10), get the following result

$$e_{n+1} = [c_2c_3 + 2c_2^3 + (\frac{\mu - \lambda\alpha}{f'(\alpha)})(2c_2^2 + c_3)]\ e_n^4 + O(e_n^5),$$

and this ends the proof. ■

3 Numerical Examples

All done computations by **MATHEMATICA** software has 120 digit floating point arithmetic (Digits:=120). An approximate solution quite is accepted as exact root, depending on the precision (ϵ) of the computer. Criterias $|x_{n+1} - x_n| < \epsilon$ and $|f(x_{n+1})| < \epsilon$ are used for computer programs, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α. For numerical illustrations, the fixed stopping criterion $\epsilon = 10^{-15}$, is used.

In comparison the Newton's method (NM) with the well-known fourth-order Ostrowski's method [5], (OM), defined by

$$x_{n+1} = x_n - (1 + \frac{f(y_n)}{f(x_n) - 2f(y_n)})\ \frac{f(x_n)}{f'(x_n)}, \tag{3.27}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.28}$$

and (CM), [2], defined by

$$x_{n+1} = y_n - [\frac{f(x_n)}{f(x_n) - f(y_n)}]^2\ \frac{f(y_n)}{f'(x_n)}, \tag{3.29}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.30}$$

the other method, (KM1), [4],

$$x_{n+1} = x_n - (1 + R(x_n) + 2R^2(x_n) + 5R^3(x_n) + \cdots)\frac{f(x_n)}{f'(x_n)}, \tag{3.31}$$

$$R(x_n) = \frac{f(y_n)}{f(x_n)}, \tag{3.32}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.33}$$

the Kou et al.'s method [3], (KM2),

$$x_{n+1} = x_n - \frac{f^2(x_n) + f^2(y_n)}{f'(x_n)(f(x_n) - f(y_n))}, \tag{3.34}$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3.35}$$

and (AEM) defined by (2.10)-(2.12), in the present contribution, the same examples in Changbum Chun [1] are used.

$f_1(x) = x^3 + 4x^2 - 10,\ f_2(x) = x^2 - e^x - 3x + 2,\ f_3(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5,$ $f_4(x) = \sin(x)e^x + \ln(x^2+1),\ f_5(x) = (x-1)^3 - 2,\ f_6(x) = (x+2)e^x - 1,\ f_7(x) = \sin^2(x) - x^2 + 1.$

Table 1. Comparison of the number of iterations (NIT) in (NM), (OM), (CM), (KM1), (KM2) and (AEM) methods

	NIT					
$f(x)$	NM	OM	CM	KM1	KM2	AEM
$f_1(x), x_0 = 1$	6	4	4	4	4	4
$f_1(x), x_0 = 2$	6	4	4	4	4	4
$f_2(x), x_0 = 1$	5	3	3	3	3	4
$f_2(x), x_0 = 3$	7	4	4	4	4	5
$f_3(x), x_0 = -1$	6	4	4	5	4	4
$f_3(x), x_0 = -2$	9	5	5	5	6	6
$f_4(x), x_0 = -2$	7	4	4	4	5	4
$f_4(x), x_0 = 5$	7	5	5	5	5	5
$f_5(x), x_0 = 3$	7	4	4	4	4	4
$f_5(x), x_0 = 4$	8	5	5	5	5	5
$f_6(x), x_0 = 2$	9	5	5	5	6	6
$f_6(x), x_0 = 4$	12	6	7	7	7	8
$f_7(x), x_0 = 1$	7	4	4	5	5	4
$f_7(x), x_0 = 2.5$	7	4	4	4	4	4

4 Conclusion

In this paper, a family of new iterative methods were defined and analyzed for solving non-linear equations and also it was proved that the order of convergence of these methods is at least four.

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