Sensitivity and Stability Radius in Data Envelopment Analysis

A. Gholam Abri ^{a}*, N. Shoja ^{a}, M. Fallah Jelodar ^{a}

(a) *Department of Mathematics, Islamic Azad University, Firoozkooh Branch,Firoozkooh,Iran.*

Abstract

One important issue in DEA which has been studied by many DEA researchers is the sensitivity analysis of a specific DMU_o, the unit under evaluation.

Moreover, we know that in most models of DEA, the best DMUs have the efficiency score of unity. In some realistic situations, the performance of some inefficient DMUs is similar to that of efficient ones.

In this paper, we define a new efficiency category, namely '$quasi - efficient$'. Then, we develop a procedure for performing a sensitivity analysis of the efficient and quasi-efficient decision making units.

The procedure yields an exact '$stability\ radius$' within which data variations will not alter a DMU's classification from efficient or quasi-efficient to inefficient status (or vice versa).

Keywords : Data Envelopment Analysis, Sensitivity, Stability, Efficiency, Quasi-efficient.

1 Introduction

Data envelopment analysis (DEA), which was introduced by Charnes et al.[6] (CCR) and extended by Banker et al.[3] (BCC), is a useful method to evaluate the relative efficiency of multiple-input and multiple-output units based on observed data.

The sensitivity analysis has received much attention in recent years from researches, and so many researches have been carried out in this regard. Sensitivity analysis in DEA has been deliberated on from various points of view.

The first DEA sensitivity analysis paper by Charnes et al.[5] examined change in a single output. This was followed by a sensitivity analysis article by Charnes and Neralic [8] in which sufficient conditions for preserving efficiency are determined. Another type of DEA sensitivity analysis is based on the super-efficiency DEA approach in which the DMU under evaluation is not included in the reference set[1, 15]. Charnes et al.[7, 9]

---

*Corresponding author. amirgholamabri@gmail.com, Tell:(+989123043297)

developed a super-efficiency DEA sensitivity analysis technique for the situation where simultaneous proportional change is assumed in all inputs and outputs for a specific DMU under consideration. This data variation condition is relaxed in Zhu [16] and Seiford[15] to a situation where inputs or outputs can be changed individually and the largest stability region that encompasses that of Charnes et al.[7] is obtained.

Especially, some valuable researches have been on sensitivity analysis of extreme efficient units that lead to reaching the stability regions of these units. The first attempt was made to reach the input and output stability region for extreme efficient units by Seiford and Zhu[14]. These regions are those within which variations of inputs or outputs cause no change in the DMU class. In other words, after any kind of interior variation, the extreme efficient unit under evaluation remains efficient. Jahanshahloo et al.[13] proposed a method that requires a less complex computational process and overcomes some difficulties in the previous method.

The DEA sensitivity analysis methods we have just reviewed are all developed for the situation where data variations are only applied to the efficient DMU under evaluation and the data for the remaining DMUs are assumed fixed.

In this paper, we develop a procedure for performing a sensitivity analysis of the efficient and quasi-efficient DMUs.

The procedure yields an exact *'stability radius'* within which data variations will not alter a DMU's classification from efficient or quasi-efficient to inefficient status (or vice versa). The current paper proceeds as follows. Section 2 discusses the basic DEA models. Section 3 develops our proposed method for finding the *'stability radius'*, and section 4 provides a numerical example. Finally, conclusions are given in section 5.

2 DEA Background

Data Envelopment Analysis (DEA) is a technique that has been used widely in the supply chain management literature. This non-parametric, multi-factor approach enhances our ability to capture the multi-dimensionality of performance discussed earlier. More formally, DEA is a mathematical programming technique for measuring the relative efficiency of decision making units (DMUs), where each DMU has a set of inputs used to produce a set of outputs [2].

Consider $DMU_j, (j = 1, ..., n)$, where each DMU consumes m inputs to produce s outputs. Suppose the observed input and output vectors of DMU_j are $X_j = (x_{1j}, ..., x_{mj})$ and $Y_j = (y_{1j}, ..., y_{sj})$, respectively, and let $X_j \geq 0$, $X_j \neq 0$, $Y_j \geq 0$, and $Y_j \neq 0$.

The production possibility set T_c is defined as:

$$T_c = \left\{ (X, Y) \mid X \geq \sum_{j=1}^{n} \lambda_j X_j, \ Y \leq \sum_{j=1}^{n} \lambda_j Y_j \ , \ \lambda_j \geq 0, \ \ j = 1, \ldots, n \right\}.$$

By the stated definition, the CCR model is as follows:

$$
\begin{aligned}
Min \quad & \theta \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \qquad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \qquad r = 1, \ldots, s \\
& \lambda_j \geq 0, \qquad\qquad\quad j = 1, \ldots, n
\end{aligned}
\tag{2.1}
$$

a DMU's classification from efficient to inefficient status (or vice versa).

Similarly the model proposed for finding the ·*stability radius*· of inefficient DMUs by Charnes et al. [7] is as follows:

$$
\begin{aligned}
Max \qquad & \delta \\
s.t. \quad \sum_{j=1}^{n} & \lambda_j x_{ij} + s_i^- + \delta = x_{io} \qquad , i = 1, \cdots, m \\
\sum_{j=1}^{n} & \lambda_j y_{rj} - s_r^+ - \delta = y_{ro} \qquad , r = 1, \cdots, s \\
\sum_{j=1}^{n} & \lambda_j = 1
\end{aligned}
\tag{2.5}
$$

Here, again, all variables are constrained to be non-negative.

The above formulations pertain to an inefficient DMU, which continues to be inefficient for all data alterations which yield improvements from x_{io} to $x_{io} - \delta^*$ and from y_{ro} to $y_{ro} + \delta^*$.

This means that no reclassification to efficient status will occur within the open set defined by the value of $\delta^* > 0$,[10].

3 Proposed Model

In some realistic situations, we know the performance of some inefficient DMUs is similar to that of efficient DMUs. This similarity leads us to suggesting a new definition.

For this purpose, first consider the following data set. The data are summarized in Table 1 and are illustrated in Figure 1:

Table 1

Data

DMUs	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Input	1	2	4	7	1.5	2.1	4	6	7	1.8	6.5	4	5	7
Output	1.5	3	5	7	2	2.7	4.5	6	6.5	2.2	6	2	3.5	4.5

the quasi-efficient DMUs whose efficiency scores equal one, and repeat step (3).

Step 4: Determine the quasi-efficient DMUs whose efficiency scores equalled one in the previous steps (whether it be step 2 or step 3).

Let $\Omega_4 = \{DMU_1, DMU_2, ..., DMU_h\}$ be the set of these DMUs. By applying the BCC multiplier model to the members of Ω_4, the new frontier is constructed.

Step5: Let

$$\begin{cases} \Omega = (\Omega_1 \bigcup \Omega_2)/\Omega_4 = \{DMU_1, DMU_2, ..., DMU_l\} \\ \Omega' = (\Omega_3 \bigcup \Omega_4) = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}\} \end{cases}$$

Next, add each member of Ω to Ω' one by one, which is done as follows:

$\Gamma_1 = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_1\}$

$\Gamma_2 = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_2\}$

$\cdots$

$\Gamma_l = \{DMU_{j1}, DMU_{j2}, ..., DMU_{je}, DMU_l\}$

Then, we use model (2.4) for Γ_i, for each $i \in \{1, 2, ..., l\}$, and we obtain the stability radii for $DMU_1, DMU_2, ..., DMU_l$ $(\delta_1^*, \delta_2^*, ..., \delta_l^*)$.

4   Example

Recall the above mentioned example. First, we apply model (2.2).
The results are summarized in Table 2:

Table 2

Results of step 1.

DMUs	A	B	C	D	E	F	G	H	I	J	K	L	M	N
θ^*	1	1	1	1	0.889	0.857	0.875	0.917	0.893	0.815	0.846	0.334	0.500	0.500

By assuming $\alpha = 0.7$, it can be seen that units A,B,C,D are efficient and units E,F,G,H,I,J,K are quasi-efficient. Moreover, L,M,N are completely inefficient. So we set:

$\Omega_1 = \{A, B, C, D\}$.

$\Omega_2 = \{E, F, G, H, I, J, K\}$.

$\Omega_3 = \{L, M, N\}$.

Next, we remove A,B,C,D (efficient DMUs) and use model (2.2) for the remaining DMUs. The results are summarized in Table 3:

Table 3

Results of step 2.

DMUs	E	F	G	H	I	J	K	L	M	N
θ^*	1	1	1	1	1	0.929	0.923	0.375	0.589	0.571

Then, we omit E,F,G,H,I and apply model (2.2) for the remaining DMUs. The results are summarized in Table 4:

Table 4

Results of step 3.

DMUs	J	K	L	M	N
θ^*	1	1	0.450	0.681	0.663