Efficiency Measurement of Multiple Components units in Data Envelopment Analysis Using Common Set of Weights

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Abstract
The current paper presents a mathematical programming model for use to the measure of efficiency where multiple performance measures are needed to examine the performance and productivity changes. In many applications of data envelopment analysis (DEA), the existing models are designed to obtain a single efficiency measure. However, in many real situations, the units under consideration may perform several different functions or can be separated into different components. In these cases, some inputs are often shared among those components and all components are involved in producing some outputs and all components have exclusive inputs and outputs. Therefore, measuring the efficiency of each component and measuring the aggregated efficiency of each unit are important. In this paper, car factories’ efficiency is analyzed using data from 19 car factories in Iran. First, a DEA-efficiency analysis of multi-component DMUs, which proposed by Cook et al is presented and then by using a "common set of weights (csw)", a new model is proposed to measure the efficiency of each component and aggregated efficiency of units. One of the most important advantages of this model is that by solving only one linear programming problem all efficiency measures may be obtained. Secondly, by grouping the branches according to their organizational designation, their efficiency is measured.

Keywords: Data envelopment analysis, Efficiency, Multi-component efficiency, Aggregated efficiency, Common set of weights

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1 Introduction

Data envelopment analysis is a method to determine the relative efficiency of a set of organizational units such as schools or bank branches when there are multiple incommensurate inputs and outputs. The applications of DEA present a range of issues relating to the homogeneity of the units under consideration. Efficiency measurement using tools such as DEA, as proposed by Charnes et al. [3], has tended to concentrate on achieving a single measure of efficiency for each member of a set of decision making units (DMUs). Note that a mathematical programming based method data envelopment analysis has proven an effective tool for evaluating the relative efficiency of peer decision making units when multiple performance measures are presented. In most real applications, a single measure of production efficiency, provided by DEA methodology, has been an adequate and useful means of comparing units and identifying best performance.

However, in most real situations, the DMU involved, can be separated into different components. In these situations, inputs, in particular resources, are often shared among those components. Also, all components are involved in producing some outputs. In such cases, we have to determine the performance of DMUs in each component. Therefore, measuring models that can deal with multiple performance measures and provide an integrated performance measure are needed. The idea of measuring efficiency relative to certain components of a DMU is not new. Fare and Grosskopf [5], for example, looked at a multi-stage process wherein intermediate outputs at one stage can be both final outputs and inputs to a later stage of production. This application of multi-component efficiency measurement does not involve shared resources. Cook et al. [4] proposed a method for measuring multi-component efficiency, which involved shared inputs.

This paper, first, addresses a model for deriving an "aggregate measure of efficiency" with component measurement based on a common set of weights, and derives a model for measuring the efficiency score of each component and aggregated efficiency of the units under consideration. Secondly, an application to Iranian car factories is presented. The Iranian car factories, like those of many countries, have undergone considerable change in operational policies in recent years. Because of these changes, the flexible evaluation of performance that tells managers whether or not they are "doing things right" has become an important topic to managing reality. This paper is structured as follows:

A summary of basic DEA models and multiple component efficiency is given in Section 2. Mathematical models for implementing the measures are presented in Section 3. The fourth section details the analytical results obtained from the application of the model to a real data set involving the data of 19 Iranian car factories. Conclusions appear in the final section.

2 DEA Background

Consider $n$ DMUs with $m$ inputs and $s$ outputs. The input and output vectors of $DMU_j$, $(j = 1, \ldots, n)$, are $X_j = (x_{1j}, \ldots, x_{mj})^t$, $Y_j = (y_{1j}, \ldots, y_{sj})^t$, where $X_j \geq 0$, $X_j \neq 0$, $Y_j \geq 0$, $Y_j \neq 0$.

By using the non-empty, constant-returns-to-scale, convexity and possibility postulates,
the production possibility set (PPS) is made as follows:

\[ T_c = \left\{ (X, Y) : X \geq \sum_{j=1}^{n} \lambda_j X_j, \ Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \ \lambda_j \geq 0, \ j = 1 \ldots, n \right\} \]

Let \( DMU_o \) be the DMU under consideration. The envelopment form of the CCR model in the input-oriented case is as follows:

\[
\begin{align*}
\text{Min} \quad & \theta - \varepsilon \left[ \sum_{i=1}^{n} s_i^- + \sum_{r=1}^{s} s_r^+ \right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \ldots, s \\
& \lambda_j \geq 0, \quad j = 1, \ldots, n
\end{align*}
\]

(2.1)

and its dual, the multiplier form of the CCR model in the input oriented case is:

\[
\begin{align*}
\text{Max} \quad & \sum_{r=1}^{s} u_r y_{ro} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \quad j = 1, \ldots, n \\
& u_r \geq \varepsilon, \quad r = 1, \ldots, s \\
& v_i \geq \varepsilon, \quad i = 1, \ldots, m
\end{align*}
\]

(2.2)

Clearly:

(1) \( DMU_o \) is efficient in model 1 if and only if \( \theta^* = 1, S^- = 0 \) and \( S^+ = 0 \). (* means the optimal solution)

(2) \( DMU_o \) is efficient in model 2 if and only if there exists \( (U^*, V^*) > 0 \) such that \( U^* y_o = 1 \).

Component efficiency The DEA technique is based on mathematical programming and used for evaluating the relative efficiency of homogenous unit. The relative efficiency of each DMU is a function of its inputs and outputs. A DMU is technically efficient if it uses inputs in the best way. In other word, it does not have any waste in inputs and shortfall in outputs. When a multiple component system is evaluated as efficient, we can say that all of its components are efficient. But if it is inefficient by evaluation, using standard DEA models with regard to which its components are inefficient is not appropriate. Note that neither technical efficiency nor any other type of efficiency, including cost efficiency,
profit efficiency and total efficiency which are evaluated by standard DEA models, can be discussed in this field.

For the first time, Cook et al. [4] published a paper on component efficiency in 2001. Also, Beasley [1] conducted a research independently about instructional efficiency of London University whose results led to the publication of a paper in 2002. Then, Cook himself completed his paper. Jahanshahloo et al. [7] considered the shared inputs and outputs as non-discretionary factors. The following subsection contains the organized and modified models that have been presented up to now.

2.1 Measuring the efficiency of DMUs with multiple components

In this subsection, we are going to extend the previous method for the cases in which the evaluated units have multiple components. In this regard, consider the following figure:

![Diagram of DMU with k components]

Consider $DMU_j$, and note that $x^l_j = (x^l_{ij}, x^l_{mj}), \ l = 1, ..., k$ is the exclusive input of the $l$th component and $\bar{y}^l_j = (\bar{y}^l_{ij}, ..., \bar{y}^l_{pj})$ is the shared input for all components. Also, $y^l_j = (y^l_{ij}, ..., y^l_{sj})$ is the exclusive output of the $l$th component and $\bar{y}^l_j = (\bar{y}^l_{ij}, ..., \bar{y}^l_{pj})$ is the shared output for all components. Let $\alpha^l = (\alpha^l_1, ..., \alpha^l_k) \ l = 1, ..., p$, be the contribution of shared inputs for all components. Now, consider the vector $(\alpha^l_{1}, ..., \alpha^l_{p})$, where $\alpha^l_{i}$ is the contribution of the $i$th shared input for the $l$th component. Note that

$$\sum_{l=1}^{k} \alpha^l_{i} = 1, \ \alpha^l_{i} \geq 0, \ i = 1, ..., p; \ l = 1, ..., k$$

For more explanation, consider the following matrix:

$$\alpha = \begin{bmatrix} \alpha^1_{1} & \alpha^1_{2} & ... & \alpha^1_{p} \\ \alpha^2_{1} & \alpha^2_{2} & ... & \alpha^2_{p} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^k_{1} & \alpha^k_{2} & ... & \alpha^k_{p} \end{bmatrix} = [\alpha^1, ..., \alpha^p]$$
Note that the sum of each column is one. Further, let $\beta^r = (\beta^1_r, ..., \beta^k_r)$, $r = 1, ..., t$ be the contribution of shared outputs for all components. Now, consider the vector $(\beta^1_1 y_j, ..., \beta^k_1 y_j)$, $r = 1, ..., t$, where $\beta^j_1 y_{j1}$ is the contribution of the $r$th shared output for the $l$th component. For more convenience, consider the following matrix:

$$
\beta = \begin{pmatrix}
\beta^1_1 & \beta^1_2 & \cdots & \beta^1_t \\
\beta^2_1 & \beta^2_2 & \cdots & \beta^2_t \\
\vdots & \vdots & \ddots & \vdots \\
\beta^k_1 & \beta^k_2 & \cdots & \beta^k_t 
\end{pmatrix} = [\beta^1, ..., \beta^t]
$$

Note that the sum of each column of this matrix is also equal to one. According to the previous statements, the input vector of the $l$th component is as follows:

$$(x^l, \bar{x}_j^l) = (x^l_{1j}, ..., x^l_{mj}, \alpha^1_l x^1_{ij}, ..., \alpha^p_l x^p_{ij}), \quad l = 1, ..., k$$

And the output vector of the $l$th component is:

$$(y^l, \bar{y}_l^r) = (y^l_{1j}, ..., y^l_{sj}, \beta^1_l y_{lj}, ..., \beta^k_l y_{lj}), \quad l = 1, ..., k$$

If we denote the efficiency of the $l$th component by $e^l_j$, then:

$$e^l_j = \frac{\sum_{r=1}^{s} u^l_r y^l_{rj} + \sum_{r=1}^{t} \bar{u}^l_r \beta^l_j y^l_{rj}}{\sum_{i=1}^{m} u^l_i x^l_{ij} + \sum_{i=1}^{l} \bar{u}^l_i \alpha^l_i \bar{x}^l_{ij}}, \quad l = 1, ..., k \quad (2.3)$$

and the aggregated efficiency of $DMU_j$ is as follows:

$$e^a_j = \frac{\sum_{j=1}^{n} \sum_{l=1}^{k} u^l_j y^l_{rj} + \sum_{j=1}^{n} \sum_{r=1}^{t} \bar{u}^l_r \beta^l_j y^l_{rj}}{\sum_{i=1}^{m} \sum_{j=1}^{n} u^l_i x^l_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{u}^l_i \alpha^l_i \bar{x}^l_{ij}} \quad (2.4)$$

Clearly, the aggregated efficiency of each $DMU$ is less than or equal to 1, i.e., $e^a_j \leq 1$. Also, the aggregated efficiency of each $DMU$ is the convex combination of the efficiency of all its components, i.e.,

$$e^a_j = \sum_{l=1}^{k} \lambda_l e^l_j, \quad \sum_{l=1}^{k} \lambda_l = 1, \quad \lambda_l \geq 0, \quad l = 1, ..., k$$

(See [4,7]). Therefore, the following model is obtained to evaluate $DMU_o$:

$$\text{Max} \quad e^a_o$$

s.t. $\quad e^a_j \leq 1, \quad l = 1, ..., k$

$$\quad e^a_j \leq 1, \quad j = 1, ..., n$$

$$\quad \sum_{l=1}^{k} \lambda^l_i = 1, \quad i = 1, ..., p$$

$$\quad \sum_{l=1}^{k} \beta^r_l = 1, \quad r = 1, ..., t$$

$$\quad \text{all variables are nonnegative} \quad (2.5)$$
Clearly, the last problem is not linear. By the following transformation
\[
\begin{align*}
\overline{u}_r^l \beta_r^l &= \tilde{u}_r^l, \quad l = 1, ..., k; \quad r = 1, ..., t \\
\overline{v}_i^l \alpha_i^l &= \tilde{v}_i^l, \quad i = 1, ..., m; \quad l = 1, ..., k
\end{align*}
\] (2.6)
and Charnes-Cooper transformation [2], the following linear programming problem is obtained:

\[
\begin{align*}
\text{Max} & \quad \sum_{l=1}^{k_s} \sum_{r=1}^{s} u_r^l y_{rio} + \sum_{k=1}^{s} \sum_{p=1}^{t} \tilde{u}_r^l y_{rio} \\
\text{s.t.} & \quad \sum_{l=1}^{s} \sum_{i=1}^{m} p_i^l x_{i,o} + \sum_{p=1}^{t} \tilde{v}_i^l x_{i,o} = 1 \\
& \quad \sum_{r=1}^{k} \sum_{l=1}^{m} \sum_{i=1}^{t} u_r^l y_{ri} + \sum_{r=1}^{k} \sum_{p=1}^{t} \tilde{u}_r^l y_{ri} - \sum_{i=1}^{m} \sum_{l=1}^{p} \tilde{v}_i^l x_{i,o} \leq 0, \quad l = 1, ..., k \\
& \quad \sum_{l=1}^{k} \sum_{i=1}^{t} \sum_{j=1}^{p} u_r^l y_{rj} + \sum_{l=1}^{k} \sum_{i=1}^{t} \tilde{u}_r^l y_{rj} - \sum_{i=1}^{m} \sum_{l=1}^{p} \tilde{v}_i^l x_{ij} \leq 0, \quad j = 1, ..., n \\
\end{align*}
\]
\[
\begin{align*}
\overline{u}_r^l & \geq 0, \quad l = 1, ..., k; \quad r = 1, ..., s. \\
\overline{v}_i^l & \geq 0, \quad l = 1, ..., k; \quad i = 1, ..., m. \\
\tilde{u}_r^l & \geq 0, \quad r = 1, ..., t. \\
\tilde{v}_i^l & \geq 0, \quad i = 1, ..., m. \\
\end{align*}
\] (2.7)

The efficiency of each component is obtained by solving this problem.

3 Proposed model to estimate the multi components efficiency

Recall the definitions of the previous section. The authors here proposed the following common set of weights model to obtain the efficiency scores of all components:

\[
\begin{align*}
\text{Max} & \quad \{ e_j^1, ..., e_j^n \} \\
\text{s.t.} & \quad e_j^j \leq 1, \quad j = 1, ..., n \\
& \quad e_j^l \leq 1, \quad j = 1, ..., n; \quad l = 1, ..., k
\end{align*}
\] (3.8)

or equivalently:
\[ \text{Max } \left\{ c_j^a \right\} = \frac{1}{\sum_{l=1}^{k} \sum_{r=1}^{s} u_r^l y_{rj} + \sum_{l=1}^{k} \sum_{r=1}^{t} \bar{u}_r^l \beta_r^l y_{rj}} 
\sum_{l=1}^{k} \sum_{i=1}^{t} \alpha_i^l x_{ij} + \sum_{l=1}^{k} \sum_{i=1}^{p} \bar{v}_i^l \bar{\alpha}_i^l x_{ij} \right\} \quad j = 1, \ldots, n \]

\[ \text{s.t. } \frac{1}{\sum_{l=1}^{k} \sum_{r=1}^{m} u_r^l y_{rj} + \sum_{l=1}^{k} \sum_{r=1}^{t} \bar{u}_r^l \beta_r^l y_{rj}} \leq 1, \quad j = 1, \ldots, n; \quad l = 1, \ldots, k \]

\[ \frac{1}{\sum_{i=1}^{k} \sum_{j=1}^{m} v_i^j x_{ij} + \sum_{i=1}^{k} \sum_{j=1}^{t} \bar{v}_i^j \bar{\alpha}_i^j x_{ij}} \leq 1, \quad i = 1, \ldots, p \]

\[ \sum_{l=1}^{k} \beta_r^l = 1, \quad r = 1, \ldots, t \]

\[ u_r \geq 0, \quad r = 1, \ldots, s. \]

\[ \bar{u}_r \geq 0, \quad r = 1, \ldots, t. \]

\[ v_i \geq 0, \quad i = 1, \ldots, m. \]

\[ \bar{v}_i \geq 0, \quad i = 1, \ldots, p. \]

\[ \alpha_i^j \geq 0, \quad l = 1, \ldots, k; \quad i = 1, \ldots, p \]

\[ \beta_r^l \geq 0, \quad l = 1, \ldots, k; \quad r = 1, \ldots, t. \]

(3.9)

Clearly, this problem is a fractional goal programming model. By adding the proposal of Liu and Peng [6], it can be written as follows:
\[ \text{Min } \sum_{j=1}^{n} z_j \]
\[ \sum_{l=1}^{r} \sum_{r=1}^{s} u_{r_l} y_{r_j} + \sum_{l=1}^{t} \sum_{r=1}^{p} u_{r_l}^l \beta_{r_l} y_{r_j} + z_j = 1, \quad j = 1, \ldots, n \]
\[ \sum_{l=1}^{r} \sum_{i=1}^{m} v_{l_i}^l x_{i_j} + \sum_{l=1}^{t} \sum_{i=1}^{p} \nu_{l_i}^l \alpha_{l_i}^l x_{i_j} \leq 1, \quad j = 1, \ldots, n \]
\[ \sum_{r=1}^{s} u_{r_l} y_{r_j} + \sum_{r=1}^{p} u_{r_l}^l \beta_{r_l} y_{r_j} \leq 1, \quad j = 1, \ldots, n; \quad l = 1, \ldots, k \]
\[ \sum_{i=1}^{k} v_{l_i}^l x_{i_j} + \sum_{i=1}^{p} \nu_{l_i}^l \alpha_{l_i}^l x_{i_j} \]
\[ \sum_{i=1}^{l} \alpha_{l_i}^i = 1, \quad i = 1, \ldots, p \]
\[ \sum_{l=1}^{t} \beta_{l_i}^l = 1, \quad r = 1, \ldots, t \]
\[ u_{r_l} \geq 0, \quad r = 1, \ldots, s. \]
\[ u_{r_l}^l \geq 0, \quad r = 1, \ldots, t. \]
\[ v_{l_i}^l \geq 0, \quad i = 1, \ldots, m. \]
\[ \nu_{l_i}^l \geq 0, \quad i = 1, \ldots, p. \]
\[ z_j \geq 0, \quad j = 1, \ldots, n. \]
\[ \alpha_{l_i}^i \geq 0, \quad l = 1, \ldots, k; \quad i = 1, \ldots, p \]
\[ \beta_{l_i}^l \geq 0, \quad l = 1, \ldots, k; \quad r = 1, \ldots, t \]

(3.10)
then we have

\[
\begin{align*}
M \text{ in } & \quad \sum_{l=1}^{k} z_l \\
\text{s.t. } & \quad \sum_{l=1}^{k} \sum_{r=1}^{s} u_{rl} y_{l,r} + \sum_{l=1}^{k} \sum_{r=1}^{s} a_{rl} \beta_i^l y_{l,r} - \sum_{l=1}^{k} \sum_{i=1}^{m} v_i^l x_{l,i} - \sum_{l=1}^{k} \sum_{i=1}^{p} \bar{v}_i \alpha_i^l x_{l,i} + z_j = 0, \quad j = 1, \ldots, n \\
& \quad \sum_{l=1}^{k} \sum_{r=1}^{s} u_{rl} y_{l,r} + \sum_{l=1}^{k} \sum_{r=1}^{s} a_{rl} \beta_i^l y_{l,r} - \sum_{l=1}^{k} \sum_{i=1}^{m} v_i^l x_{l,i} - \sum_{l=1}^{k} \sum_{i=1}^{p} \bar{v}_i \alpha_i^l x_{l,i} \leq 0, \quad j = 1, \ldots, n \\
& \quad \sum_{r=1}^{t} u_{rl}^l y_{l,r} + \sum_{r=1}^{t} a_{rl} \beta_i^l y_{l,r} - \sum_{r=1}^{t} v_i^l x_{l,i}^l - \sum_{r=1}^{t} \bar{v}_i \alpha_i^l x_{l,i}^l \leq 0, \quad j = 1, \ldots, n; \quad l = 1, \ldots, k \\
& \quad \sum_{l=1}^{k} \alpha_i^l = 1, \quad i = 1, \ldots, p \\
& \quad \sum_{l=1}^{k} \beta_i^l = 1, \quad r = 1, \ldots, t \\
& \quad u_{rl} \geq 0, \quad r = 1, \ldots, s. \\
& \quad a_{rl} \geq 0, \quad r = 1, \ldots, t. \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m. \\
& \quad \bar{v}_i \geq 0, \quad i = 1, \ldots, p. \\
& \quad z_j \geq 0, \quad j = 1, \ldots, n. \\
& \quad \alpha_i^l \geq 0, \quad l = 1, \ldots, k; \quad i = 1, \ldots, p \\
& \quad \beta_i^l \geq 0, \quad l = 1, \ldots, k; \quad r = 1, \ldots, t
\end{align*}
\]

Now, consider

\[
\begin{align*}
\overline{a}_{rl} &= \tilde{u}_{rl}, \quad l = 1, \ldots, k; \quad r = 1, \ldots, t \\
\overline{v}_i &= \tilde{v}_i, \quad l = 1, \ldots, k; \quad i = 1, \ldots, p \quad (3.12)
\end{align*}
\]

Therefore, the following linear programming model is obtained:

\[
\begin{align*}
M \text{ in } & \quad \sum_{j=1}^{n} z_j \\
\text{s.t. } & \quad \sum_{l=1}^{k} \sum_{r=1}^{s} u_{rl}^l y_{l,r} + \sum_{l=1}^{k} \sum_{r=1}^{s} a_{rl}^l \beta_i^l y_{l,r} - \sum_{l=1}^{k} \sum_{i=1}^{m} v_i^l x_{l,i}^l - \sum_{l=1}^{k} \sum_{i=1}^{p} \bar{v}_i \alpha_i^l x_{l,i}^l + z_j = 0, \quad j = 1, \ldots, n \\
& \quad \sum_{l=1}^{k} \sum_{r=1}^{s} u_{rl}^l y_{l,r} + \sum_{l=1}^{k} \sum_{r=1}^{s} a_{rl}^l \beta_i^l y_{l,r} - \sum_{l=1}^{k} \sum_{i=1}^{m} v_i^l x_{l,i}^l - \sum_{l=1}^{k} \sum_{i=1}^{p} \bar{v}_i \alpha_i^l x_{l,i}^l \leq 0, \quad j = 1, \ldots, n \\
& \quad \sum_{r=1}^{t} u_{rl}^l y_{l,r} + \sum_{r=1}^{t} a_{rl}^l \beta_i^l y_{l,r} - \sum_{r=1}^{t} v_i^l x_{l,i}^l - \sum_{r=1}^{t} \bar{v}_i \alpha_i^l x_{l,i}^l \leq 0, \quad j = 1, \ldots, n; \quad l = 1, \ldots, k \\
& \quad u_{rl} \geq 0, \quad r = 1, \ldots, s. \\
& \quad a_{rl}^l \geq 0, \quad r = 1, \ldots, t. \\
& \quad v_i \geq 0, \quad i = 1, \ldots, m. \\
& \quad \bar{v}_i \geq 0, \quad i = 1, \ldots, p. \\
& \quad z_j \geq 0, \quad j = 1, \ldots, n.
\end{align*}
\]

\[\quad (3.13)\]
Clearly, $DMU_j$ is efficient if and only if $z_j = 0$. The aggregated efficiency of $DMU_j$ is as follows:

$$e_j^r = \frac{\sum_{l=1}^{k} \sum_{r=1}^{m} u^r_l y^l_{r,j} + \sum_{l=1}^{k} \sum_{r=1}^{m} \tilde{u}^r_l y^l_{r,j}}{\sum_{l=1}^{k} \sum_{i=1}^{t} v^l_i x^l_{i,j} + \sum_{l=1}^{k} \sum_{i=1}^{t} \tilde{v}^l_i x^l_{i,j}} \quad j = 1, \ldots, n; \quad l = 1, \ldots, k$$  \hspace{1cm} (3.14)

and the efficiency score of each component is as follows:

$$e_j^l = \frac{\sum_{r=1}^{m} u^r_l y^l_{r,j} + \sum_{r=1}^{m} \tilde{u}^r_l y^l_{r,j}}{\sum_{i=1}^{t} v^l_i x^l_{i,j} + \sum_{i=1}^{t} \tilde{v}^l_i x^l_{i,j}} \quad j = 1, \ldots, n; \quad l = 1, \ldots, k$$  \hspace{1cm} (3.15)

It is obvious that the aggregated efficiency of each DMU is the convex combination of all its components’ efficiency scores.

4 Example

In this section we are going to apply our proposed method to a real data set. In order to do this, consider 19 Iranian car factories with the following data:

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Stocks (1000 million Rials)</th>
<th>Total Assets (1000 million Rials)</th>
<th>Sales (1000 million Rials)</th>
<th>Capital (1000 million Rials)</th>
<th>Total Equity (1000 million Rials)</th>
<th>Net Profit (1000 million Rials)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company1</td>
<td>100,000,000</td>
<td>492,726</td>
<td>848,854</td>
<td>100,000</td>
<td>1,676,683</td>
<td>43,000</td>
</tr>
<tr>
<td>Company2</td>
<td>40,000,000</td>
<td>321,486</td>
<td>298,067</td>
<td>40,000</td>
<td>1,159,972</td>
<td>48,949</td>
</tr>
<tr>
<td>Company3</td>
<td>972,000,000</td>
<td>10,660,537</td>
<td>11,462,010,000</td>
<td>972,000</td>
<td>1,909,307</td>
<td>840,107</td>
</tr>
<tr>
<td>Company4</td>
<td>3,200,000</td>
<td>10,186,199</td>
<td>2,414,483,000</td>
<td>3,200,000</td>
<td>5,636,671</td>
<td>1,259,910</td>
</tr>
<tr>
<td>Company5</td>
<td>1,200,000</td>
<td>6,043,419</td>
<td>4,796,789</td>
<td>1,200,000</td>
<td>2,244,373</td>
<td>912,000</td>
</tr>
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<td>142,000</td>
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<td>359,500</td>
<td>2,147,552</td>
<td>120,000</td>
<td>293,076</td>
<td>162,858</td>
</tr>
<tr>
<td>Company8</td>
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<td>829,380</td>
<td>687,994</td>
<td>72,000</td>
<td>129,203</td>
<td>5,040</td>
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<tr>
<td>Company9</td>
<td>10,000,000</td>
<td>193,570</td>
<td>135,061</td>
<td>10,000</td>
<td>36,841</td>
<td>1,300</td>
</tr>
<tr>
<td>Company10</td>
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<td>5,369,319</td>
<td>5,167,457</td>
<td>600,000</td>
<td>1,568,362</td>
<td>720,000</td>
</tr>
<tr>
<td>Company11</td>
<td>40,000,000</td>
<td>272,434</td>
<td>189,781,000</td>
<td>40,000</td>
<td>52,121</td>
<td>6,159</td>
</tr>
<tr>
<td>Company12</td>
<td>7,000,000</td>
<td>29,649,891</td>
<td>32,886,940,000</td>
<td>7,000,000</td>
<td>15,588,292</td>
<td>12,263,233</td>
</tr>
<tr>
<td>Company13</td>
<td>60,000,000</td>
<td>204,703</td>
<td>129,664</td>
<td>60,000</td>
<td>66,156</td>
<td>2,288</td>
</tr>
<tr>
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<td>34,000,000</td>
<td>167,676</td>
<td>172,681,000</td>
<td>34,000</td>
<td>55,498</td>
<td>35,747</td>
</tr>
<tr>
<td>Company15</td>
<td>600,000,000</td>
<td>12,355,705</td>
<td>6,034,301,000</td>
<td>600,000</td>
<td>879,072</td>
<td>684,072</td>
</tr>
<tr>
<td>Company16</td>
<td>13,000,000</td>
<td>148,175</td>
<td>218,298</td>
<td>13,000</td>
<td>21,059</td>
<td>5,200</td>
</tr>
<tr>
<td>Company17</td>
<td>170,100,000</td>
<td>1,257,490</td>
<td>167,962,000</td>
<td>170,100</td>
<td>414,424</td>
<td>93,555</td>
</tr>
<tr>
<td>Company18</td>
<td>5,000,000,000</td>
<td>64,766,600</td>
<td>43,633,916,000</td>
<td>5,000,000</td>
<td>6,802,398</td>
<td>4,164,647</td>
</tr>
<tr>
<td>Company19</td>
<td>150,000,000</td>
<td>820,490</td>
<td>1,600,077,000</td>
<td>150,000</td>
<td>163,426</td>
<td>366</td>
</tr>
</tbody>
</table>

where Stocks (the number of), Total Assets (1000 million Rials) and Capital (1000 million Rials) are inputs; Sales (1000 million Rials), Total Equity (Rials) and Net Profit(1000 million Rials) are suggested outputs. All of these car factories have two components: 1-Department of Production, 2-Department of Administration. Now, consider the following figure:
Fig. 2. A Car Factory with two components

Capital and stocks are exclusive inputs of the first and second components and total assets is the shared input. Also, sales and total equity are exclusive outputs of the components and net profit is the shared output.

By using model (2.7) the following efficiency scores are obtained:

Table 2
Efficiency Scores

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Aggregated Efficiency</th>
<th>First Component Efficiency</th>
<th>Second Component Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Company1</td>
<td>0.640574</td>
<td>0.000635</td>
<td>1</td>
</tr>
<tr>
<td>Company2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Company3</td>
<td>0.735246</td>
<td>1</td>
<td>0.709122</td>
</tr>
<tr>
<td>Company4</td>
<td>0.704606</td>
<td>0.014023</td>
<td>0.792339</td>
</tr>
<tr>
<td>Company5</td>
<td>0.709981</td>
<td>0.010820</td>
<td>0.853510</td>
</tr>
<tr>
<td>Company6</td>
<td>0.806255</td>
<td>0.001220</td>
<td>0.950728</td>
</tr>
<tr>
<td>Company7</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Company8</td>
<td>0.599226</td>
<td>1</td>
<td>0.374097</td>
</tr>
<tr>
<td>Company9</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Company10</td>
<td>0.939651</td>
<td>1</td>
<td>0.900295</td>
</tr>
<tr>
<td>Company11</td>
<td>0.467532</td>
<td>0.077381</td>
<td>0.550223</td>
</tr>
<tr>
<td>Company12</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Company13</td>
<td>0.444467</td>
<td>0.000253</td>
<td>0.507700</td>
</tr>
<tr>
<td>Company14</td>
<td>0.673767</td>
<td>1</td>
<td>0.639217</td>
</tr>
<tr>
<td>Company15</td>
<td>0.758871</td>
<td>1</td>
<td>0.713396</td>
</tr>
<tr>
<td>Company16</td>
<td>0.938249</td>
<td>1</td>
<td>0.000287</td>
</tr>
<tr>
<td>Company17</td>
<td>0.864048</td>
<td>0.170090</td>
<td>1</td>
</tr>
<tr>
<td>Company18</td>
<td>0.569334</td>
<td>1</td>
<td>0.501034</td>
</tr>
<tr>
<td>Company19</td>
<td>0.595755</td>
<td>1</td>
<td>0.000260</td>
</tr>
</tbody>
</table>

Now, consider company 4. Clearly it is inefficient because all of its components have efficiency scores less than one. Also, unit 1 is inefficient because the first component has an efficiency score less than unity. Of course company 2 is efficient because all of its components are efficient.

Again consider model (3.13). By using the data, the following results are obtained:
Table 3
Efficiency Scores in new model

<table>
<thead>
<tr>
<th>DMUs</th>
<th>$z_{1}$</th>
<th>Aggregated Efficiency</th>
<th>First Component Efficiency</th>
<th>Second Component Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0005</td>
<td>0.43798</td>
<td>0.41531</td>
<td>0.47998</td>
</tr>
<tr>
<td>2</td>
<td>0.0002</td>
<td>0.58042</td>
<td>0.45190</td>
<td>0.82791</td>
</tr>
<tr>
<td>3</td>
<td>0.0048</td>
<td>0.51939</td>
<td>0.52017</td>
<td>0.50327</td>
</tr>
<tr>
<td>4</td>
<td>0.0192</td>
<td>0.24498</td>
<td>0.09432</td>
<td>0.53696</td>
</tr>
<tr>
<td>5</td>
<td>0.0002</td>
<td>0.39421</td>
<td>0.27807</td>
<td>0.60886</td>
</tr>
<tr>
<td>6</td>
<td>0.0007</td>
<td>0.36285</td>
<td>0.24175</td>
<td>0.59067</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>0.0005</td>
<td>0.32865</td>
<td>0.34441</td>
<td>0.30286</td>
</tr>
<tr>
<td>9</td>
<td>0.0001</td>
<td>0.43844</td>
<td>0.41391</td>
<td>0.47492</td>
</tr>
<tr>
<td>10</td>
<td>0.0025</td>
<td>0.57693</td>
<td>0.48183</td>
<td>0.73813</td>
</tr>
<tr>
<td>11</td>
<td>0.0003</td>
<td>0.24001</td>
<td>0.20922</td>
<td>0.29470</td>
</tr>
<tr>
<td>12</td>
<td>0.0203</td>
<td>0.64886</td>
<td>0.46398</td>
<td>0.99730</td>
</tr>
<tr>
<td>13</td>
<td>0.0004</td>
<td>0.15790</td>
<td>0.09984</td>
<td>0.26973</td>
</tr>
<tr>
<td>14</td>
<td>0.0002</td>
<td>0.46298</td>
<td>0.36704</td>
<td>0.64074</td>
</tr>
<tr>
<td>15</td>
<td>0.0049</td>
<td>0.38333</td>
<td>0.40815</td>
<td>0.34684</td>
</tr>
<tr>
<td>16</td>
<td>0.0001</td>
<td>0.52869</td>
<td>0.64159</td>
<td>0.34375</td>
</tr>
<tr>
<td>17</td>
<td>0.0008</td>
<td>0.51118</td>
<td>0.46059</td>
<td>0.60002</td>
</tr>
<tr>
<td>18</td>
<td>0.0336</td>
<td>0.38618</td>
<td>0.39718</td>
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<td>19</td>
<td>0.0008</td>
<td>0.36047</td>
<td>0.43768</td>
<td>0.22940</td>
</tr>
</tbody>
</table>

Consider company 7; $z_{1} = 0$ so it is efficient, since all of its components have efficiency scores of one. Other units are inefficient because their derivation variables are positive. Also, their components have efficiency score of less than unity. The number of efficient units is decreased by using the common set of weights.

5 Conclusions

The DEA model presented here, can be used for the analysis of any real situation where a significant number of inputs and outputs are included and management views the production process as a multi-stage process. The provision of component efficiency facilitates managerial actions only on those components where the DMU is underperforming. The particular application area investigated is that involving the economical functions within Iranian car factories. Using the data from Iranian car factories, this study has measured the multi-component efficiency of the economical functions in 19 Iranian car factories. We used a "common set of weights" to propose a new model for measuring the efficiency score of multiple component units. One of the most important advantages of this model is that by solving only one linear programming problem, we can obtain the efficiency scores of all components and the aggregated efficiency of all DMUs. It can be observed that the number of efficient units is decreased by using the common set of weights.

References


