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Usnig LR-Fuzzy Numbers Data to Measure the Efficiency and the Malmquist Productivity Index in Data Envelopment Analysis , and Its Application in Insurance Organizations

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Abstract

In many real applications, the data of production processes can't be precisely measured. We develop some fuzzy versions of the classical DEA models (in particular, the CCR model) by using some ranking methods based on the comparison of cuts. Our approaches can be seen as an extension of the DEA methodology. The provides users and practitioners with models which represent some real life processes more appropriately. DEA- based Malmquist productivity index measures the productivity change over time. In this paper we provide an extension to the DEA- based Malmquist productivity index for all DMUs with fuzzy data.

Keywords : Data envelopment analysis (DEA), Efficiency, Malmquist productivity index, Fuzzy data.

1 Introduction

DEA models provide efficiency scores which assess the performance of the different DMUs in terms of either the use of several inputs or the production of certain outputs (or even simulation easily). Most of DEA efficiency scores vary in (0,1], the unity value being reserved

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to efficient units. In the particular case of the radial models, the CCR (Charnes, cooper and Rhodes [3]) and the BCC (Banker Charnes and Cooper [1]) models yield efficiency scores both in input and output orientation, although no oriented DEA efficiency scores can also be defined (see [4] for hyperbolic measures and [2] for directional measures). We can find several fuzzy approaches to the assessment of efficiency in DEA literature.The fuzzy DEA model by using Zimmermann's method [10]. Kao and Liu [7] developed a method to find the membership functions of the fuzzy efficiency scores when some observations are fuzzy numbers. The idea is based on the cuts and Zadeh's extension principle [9]. Entraining propose in [5] a DEA model with an interval efficiency consisting of efficiencies obtained from the pessimistic and the optimistic viewpoints.Their model which is able to deal with Fuzzy data, considers inefficiency intervals. In the present, paper we also use possibilities programming techniques to provide a new approach to the problem of the measurement of efficiency, but exploiting the use of the primal envelopment formulation of the DEA models instead of the dual multiplier one. The paper unfolds as follows: section 2 contains some results on fuzzy interval analysis that will be used in the paper. In section 3, we develop some fuzzy DEA models. To be specific, we are dealing with the input oriented CCR model, although the extension of the methodology developed here to other radial DEA models is straight forward. Using linear programming techniques, data envelopment analysis (DEA) (Charnes) Provides a suitable way to estimate a multiple inputs/ multiple outputs empirical efficient function as described by Farrell (1957). Fare et al (1992,1994 a) developed a DEA- based Malmquist productivity index which measured the productivity change over time. The Malmquist productivity index was first suggested by Malmquist [12] as a quantity index for use in the analysis of consumption of inputs. Fare et al combined ideas on the measurement of efficiency from Farell and the measurement of productivity from Cares, et al.(1982)to construct a Malmquist productivity index. This Index has proven itself to be a good tool for measuring the productivity change of DMUs. The original DEA-based Malmquist index assumes that inputs and outputs are measured by exact values on a ratio scale. In Section 6 contains our final conclusions.

2 Preliminaries

Inuiguchi, et al [6] and Lai and Hwang [8] refered to the linear programming with imprecise coefficients restricted by possibilities distributions as possibilities programming. In this section we are simply recalling how to perform the basic operation of Fuzzy intervals for ranking purposes. To be more precise, we deal with LR-fuzzy numbers whose definition is as follows.

Definition 2.1. A fuzzy number $\tilde{m}$ is said to be a LR-fuzzy number, $\tilde{m} = (\alpha^L, m^L, m^R, \alpha^R)_{L,R}$, if its membership function has the following form:

$$\mu_{\tilde{m}}(r) = \begin{cases} L(\frac{m^L - r}{\alpha^L}) & , \ r \leq m^L, \\ 1 & , \ m^L \leq r \leq m^R, \\ R(\frac{r - m^R}{\alpha^R}) & , \ r \geq m^R \end{cases} \tag{1}$$

Where L and R are reference functions, i.e, L, R:$[0, +\infty) \longrightarrow [0, 1]$ are strictly Dereasing in

$\sup(\tilde{m}) = \{r : \mu_{\tilde{m}}(r) > 0\}$and upper semi-continuous functions such that $L(0) = R(0) = 1$.
If $\sup(\tilde{m})$ is a bounded set, L and R are defined on [0,1] and satisfy $L(1) = R(1) = 0$.

In particular, for a given set of LR-fuzzy numbers $\tilde{a}_j = (\alpha_j^L, a_j^L, a_j^R, \alpha_j^R)_{L,R}, j = 1, ..., n$ and some scalars $x_j \geq 0, j = 1, ..., n$, we have that

$$\sum_{j=1}^{n} \tilde{a}_j x_j = (\sum_{j=1}^{n} \alpha_j^L x_j, \sum_{j=1}^{n} a_j^L x_j, \sum_{j=1}^{n} a_j^R x_j, \sum_{j=1}^{n} \alpha_j^R x_j)_{L,R}, \tag{2}$$

Where L and R are the common left and right reference functions, and $\sum_{j=1}^{n} \tilde{a}_j x_j$ denote the combination $\tilde{a}_1 x_1 \oplus \tilde{a}_2 x_2 \oplus ... \oplus \tilde{a}_n x_n$. Let us recall the definition of maximum of two fuzzy numbers.

Definition 2.2. Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers. Then, $\tilde{m} \bigvee \tilde{n}$ represents the fuzzy number having the following membership function:

$$\mu_{\tilde{m} \bigvee \tilde{n}}(r) = \sup_{r = s \bigvee t} \left\{ \mu_{\tilde{m}}(s) \bigvee \mu_{\tilde{n}}(t) \right\} \tag{3}$$

Definition 2.3. Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers. Then,

$$\tilde{m} \gtrsim \tilde{n} \Leftrightarrow \tilde{m} \bigvee \tilde{n} = \tilde{m} \tag{4}$$

Tanaka et al [13] have formulated FLP problems by using this order. In fact, Ramik and Rimanek provided an operative characterization of (2.4) in terms of the α-level sets:

Lemma 2.4. *Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers. Then $\tilde{m} \bigvee \tilde{n} = \tilde{m}$ if, and only if* $\forall h \in [0, 1]$ *the two statements below hold:*

$$\begin{array}{l} \inf \{s : \mu_{\tilde{m}}(s) \geq h\} \geq \inf \{t : \mu_{\tilde{n}}(t) \geq h\}, \\ \sup \{s : \mu_{\tilde{m}}(s) \geq h\} \geq \sup \{t : \mu_{\tilde{n}}(t) \geq h\}, \end{array} \tag{5}$$

In particular, for two LR-fuzzy numbers, $\tilde{m} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$ and $\tilde{n} = (n^L, n^R, \beta^L, \beta^R)_{L',R'}$ if

$$\begin{array}{ll} m^L - L^*(h)\alpha^L \geq n^L - L'^*(h)\beta^L & \forall h \in [0, 1], \\ m^R + R^*(h)\alpha^R \geq n^R + R'^*(h)\beta^R & \forall h \in [0, 1], \end{array} \tag{6}$$

Where

$$L^*(h) = \sup\{z : L(z) \geq h\}, \quad L'^*(h) = \sup\{z : L'(z) \geq h\},$$
$$R^*(h) = \sup\{z : R(z) \geq h\}, \quad R'^*(h) = \sup\{z : R'(z) \geq h\}, \tag{7}$$

however, if $\tilde{m} = (m^L, m^R, \alpha^L, \alpha^R)_{L,R}$ and $\tilde{n} = (n^L, n^R, \beta^L, \beta^R)_{L',R'}$ have bounded support and both $L = L'$ and $R = R'$, then (2.7) becomes

$$m^L \geq n^L, \quad m^L - \alpha^L \geq n^L - \beta^L,$$
$$m^R \geq n^R, \quad m^R + \alpha^R \geq n^R + \beta^R, \tag{8}$$

Definition 2.5. Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers and h a real number, $h \in [0,1]$. Then $(\tilde{m} \gtrsim^h \tilde{n})$ if, and only if, $\forall k \in [h,1]$ the following two statements hold:

$$\inf\{s : \mu_{\tilde{m}}(s) \geq k\} \geq \inf\{t : \mu_{\tilde{n}}(t) \geq k\},$$
$$\sup\{s : \mu_{\tilde{m}}(s) \geq k\} \geq \sup\{t : \mu_{\tilde{n}}(t) \geq k\}, \tag{9}$$

For LR-fuzzy numbers with bounded support, and using this ranking method for a given h, expression (2.9) becomes

$$m^L - L^*(k)\alpha^L \geq n^L - L'^*(k)\beta^L \quad \forall k \in [h,1],$$
$$m^R + R^*(k)\alpha^R \geq n^R + R'^*(k)\beta^R \quad \forall k \in [h,1], \tag{10}$$

Indeed when comparing $\tilde{m}$ and $\tilde{n}$ at a given possibility level h, it may happen that $(\tilde{m} \gtrsim^h \tilde{n})$ although $(\tilde{m} \gtrsim \tilde{n})$ does not hold.

2.1 Data envelopment analysis with fuzzy data

Consider that we are interested in evaluating the relative efficiency of (n) DMUs which use m inputs to produces outputs. Suppose that the data of inputs and outputs can not be precisely measured and, also that they can be expressed as LR-fuzzy number with bounded support:

$$\tilde{x}_{i,j} = (\alpha^L_{i,j}, x^L_{i,j}, x^R_{i,j}, \alpha^R_{i,j})_{L_{i,j},R_{i,j}}, \quad i = 1,...m,\ j = 1,...n,$$
$$\tilde{y}_{r,j} = (\beta^L_{r,j}, y^L_{r,j}, y^R_{r,j}, \beta^R_{r,j})_{L'_{r,j},R'_{r,j}}, \quad r = 1,...s,\ j = 1,...n. \tag{11}$$

Satisfying

$$\begin{array}{ll} L_{i1}=L_{i2}=...=L_{in}=L_i, & i=1,..,m,\\ L'_{r1}=L'_{r2}=...=L'_{rn}=L'_r, & r=1,..,s,\\ R_{i1}=R_{i2}=...=R_{in}=R_i, & i=1,..,m,\\ R'_{r1}=R'_{r2}=...=R'_{rn}=R'_r, & r=1,..,s. \end{array} \qquad (12)$$

Let us also assume that the input oriented CCR model is used to evaluate the relative efficiency of this set of DMUs. Then, the extended CCR model can be expressed as the following fuzzy Lp problem.

$(FCCR)$

$$\begin{array}{lll} \text{Min} & \theta & \\ \text{s.t.} & \sum_{j=1}^{n}\lambda_j\tilde{x}_{ij} \lesssim \theta\tilde{x}_{io}, & \text{i=1,...,m}\\ & \sum_{j=1}^{n}\lambda_j\tilde{y}_{rj} \gtrsim \tilde{y}_{ro}, & \text{r=1,...,s}\\ & \lambda_j \geq 0, & \text{j=1,...,n} \end{array} \qquad (13)$$

Since inputs and outputs are LR numbers, if, in particular, $\gtrsim$ is interpreted as in (2.4) and the linear combinations as in (2.2), then (2.13) can be transformed in

(q)

$$\begin{array}{lll} \text{Min} & \theta & \\ \text{s.t.} & \sum_{j=1}^{n}\lambda_j x^L_{ij} \leq \theta x^L_{io}, & \text{i=1,...,m}\\ & \sum_{j=1}^{n}\lambda_j x^R_{ij} \leq \theta x^R_{io}, & \text{i=1,...,m}\\ & \sum_{j=1}^{n}\lambda_j x^L_{ij} - \sum_{j=1}^{n}\lambda_j \alpha^L_{ij} \leq \theta x^L_{io} - \theta\alpha^L_{io}, & \text{i=1,...,m}\\ & \sum_{j=1}^{n}\lambda_j x^R_{ij} + \sum_{j=1}^{n}\lambda_j \alpha^R_{ij} \leq \theta x^R_{io} + \theta\alpha^R_{io}, & \text{i=1,...,m}\\ & \sum_{j=1}^{n}\lambda_j y^L_{rj} \geq y^L_{ro}, & \text{r=1,...,s}\\ & \sum_{j=1}^{n}\lambda_j y^R_{rj} \geq y^R_{ro}, & \text{r=1,...,s}\\ & \sum_{j=1}^{n}\lambda_j y^L_{rj} - \sum_{j=1}^{n}\lambda_j \beta^L_{rj} \geq y^L_{ro} - \beta^L_{ro}, & \text{r=1,...,s}\\ & \sum_{j=1}^{n}\lambda_j y^R_{rj} + \sum_{j=1}^{n}\lambda_j \beta^R_{rj} \geq y^R_{ro} + \beta^R_{ro}, & \text{r=1,...,s}\\ & \lambda_j \geq 0, & \text{j=1,...,n} \end{array} \qquad (14)$$

Hence, the optimal value of (2.14) provides an evaluation of the efficiency of a DMU in which all the possible values of the different variables for all the DMUs at all the possibility levels are considered. In this case we can use $\gtrsim^h$ for ranking (see [6]), then model (2.13) can be expressed as the following linear programming problem:

$$
(q)^h \quad
\begin{array}{lll}
\text{Min} & \theta & \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^L \leq \theta x_{io}^L, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^R \leq \theta x_{io}^R, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^L - L_i^*(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^L \leq \theta x_{io}^L - L_i^*(h)\theta\alpha_{io}^L, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^R + R_i^*(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^R \leq \theta x_{io}^R + R_i^*(h)\theta\alpha_{io}^R, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^L \geq y_{ro}^L, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^R \geq y_{ro}^R, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^L - L_r^{'*}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^L \geq y_{ro}^L - L_r^{'*}(h)\beta_{ro}^L, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^R + R_r^{'*}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^R \geq y_{ro}^R + R_r^{'*}(h)\beta_{ro}^R, & \text{r=1,...,s} \\
& \lambda_j \geq 0, & \text{j=1,...,n}
\end{array}
\tag{15}
$$

The optimal value of (2.15), $\theta^*(h)$, provides the efficiency score of a DMU at the h possibility level. As mentioned before, in practice we can solve this model for different values of h to observe how the efficiency scores of the DMUs change when the possibility level h varies. Notice that if h=0 then (2.15) coincides with (2.14).

3 Malmquist productivity index

Fare, et al. (1992) construeted the DEA-based Malmquist productivity index as the geometric mean of two Malmquist productivity indexes of Caves, et al. (1982), which are defined by a distance function D(.). Caves et al. assumed $D^k(k) = 1$ and their distance function does not reveal inefficiency. By allowing for inefficiency and modeling the technology frontier as piecewise linear, Fare, et al. decomposed their Malmquist productivity index into two components, one measuring the change in the efficiency and the other measuring the change in the frontier technology. The frontier technology determined by the efficient frontier is estimated using DEA for a set of DMUs. However, the frontier technology for a particular DMU under evaluation is only represented by a section of the DEA frontier or a facet.

The Malmquist productivity index calculation requires two single period and two mixed period measures. The two single period measures can be obtained by using the DCCR model (Charnes et al., 1978)

$$\begin{array}{lll} D_o^p(k) = & \text{Min} & \theta \\ & \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^p \leq \theta x_{io}^k, \quad \text{i=1,...,m} \\ & & \sum_{j=1}^{n} \lambda_j y_{rj}^p \geq y_{ro}^k, \quad \text{r=1,...,s} \\ & & \lambda_j \geq 0, \quad \text{j=1,...,n} \end{array} \tag{16}$$

where x_{io}^k is the $i-th$ input and y_{ro}^k is the $r-th$ output for DMU_o in time k and x_{ij}^p is the $i-th$ input and y_{rj}^p is the $r-th$ output for DMU_j in time p. The efficiency $(D_o^t(t))$ determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $t+1$ instead of t for the above method, we get $D_o^{t+1}(t+1)$, the technical efficiency score for DMU_o in time period $t+1$.

The first of the mixed period measures, which is defined as $D_o^t(t+1)$ for each DMU_o, is computed as the optimal value to the (3.16) linear programming problem, where $p=t$ and $k=t+1$. Similarly, the other mixed period measure, $D_o^{t+1}(t)$, which is needed in the computation of the Malmquist productivity index, is the optimal value to the (3.16) linear problem, where $p=t+1$ and $k=t$. Fare et al.'s input-oriented Malmquist productivity index, which measures the productivity change of a particular $DMU_o, o \in J=\{1,...,n\}$, in time $t+1$ and t is given as:

$$M_o = \left[\frac{D_o^t(t+1)}{D_o^t(t)} \times \frac{D_o^{t+1}(t+1)}{D_o^{t+1}(t)}\right]^{1/2} \tag{17}$$

It can be seen that the above measure actually is the geometric mean of two Caves et al.'s Malmquist productivity indexes. Thus, following Caves, et al.'s suit, Fare, et al. defined that $M_o > 1$ indicates the productivity gain; $M_o < 1$ indicates the productivity loss; and $M_o = 1$ means no change in the productivity from time t to $t+1$.

4 Malmquist productivity index with fuzzy data

In recent years, the fuzzy set theory has been proposed as a way to quantify imprecise and vague data in DEA models. fuzzy DEA models take the form of fuzzy linear programming model. The fuzzy CCR models cannot be solved by a standard LP solver like a crisp CCR model because coefficients in the fuzzy CCR model are fuzzy sets. With the fuzzy inputs and fuzzy outputs, the optimality conditions for the crisp DEA model need to be clarified and generalized. In this section, we are in purpose of evaluating the Malmquist productivity index for DMUs with fuzzy data. Therefore, assume that fuzzy numbers,

$$\begin{array}{ll} \tilde{x}_{i,j}^t = (\alpha_{i,j}^{tL}, x_{i,j}^{tL}, x_{i,j}^{tR}, \alpha_{i,j}^{tR})_{L_{i,j}^t, R_{i,j}^t}, & i=1,...m,\ j=1,...n, \\ \tilde{y}_{i,j}^t = (\beta_{i,j}^{tL}, y_{i,j}^{tL}, y_{i,j}^{tR}, \beta_{i,j}^{tR})_{L'^t_{i,j}, R'^t_{i,j}}, & i=1,...m,\ j=1,...n. \end{array} \tag{18}$$

are the $i-th$ input and the $r-th$ output for DMU_j in time t.
The two single period measures can be obtained by using the FCCR DEA model (we use $\gtrsim^h$ for ranking) :

$$\begin{array}{lll} \overline{D_o^p(k)^h} = & \text{Min} & \theta \\ & \text{s.t.} & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^p \lesssim^h \theta \tilde{x}_{io}^k, \quad \text{i=1,...,m} \\ & & \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^p \gtrsim^h \tilde{y}_{ro}^k, \quad \text{r=1,...,s} \\ & & \lambda_j \geq 0, \quad \text{j=1,...,n} \end{array} \tag{19}$$

The efficiency ($\overline{D_o^t(t)^h}$) determines the amount by which observed inputs can be proportionally reduced, while still producing the given output level. Using $(t+1)$ instead of (t) for the above method, we get $\overline{D_o^{t+1}(t+1)^h}$, the technical efficiency score for DMU_o in time period $(t+1)$. The first of the mixed period measures, which is defined as $\overline{D_o^t(t+1)^h}$ for each DMU_o, is computed as the optimal value to the(4.19) fuzzy linear programming problem, where $p=t$ and $k=t+1$. Similarly, the other mixed period measure, $\overline{D_o^{t+1}(t)^h}$, which is needed in the computation of the Malmquist productivity index, is the optimal value to the (4.19) fuzzy linear problem, where $p=t+1$ and $k=t$.

In this paper, we use $(q)^h$ this method to based Malmquist productivity index for analysis fuzzy data. However, we determine the $\overline{D_o^t(t)^h}$, $\overline{D_o^{t+1}(t+1)^h}$, $\overline{D_o^t(t+1)^h}$ and $\overline{D_o^{t+1}(t)^h}$ by solving corresponding linear programing problem as:

$(q^p(k))^h$

$$
\begin{array}{lll}
\overline{D_o^p(k)^h} = & \text{Min} & \theta \\
\text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}^{pL} \leq \theta x_{io}^{kL}, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{pR} \leq \theta x_{io}^{kR}, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{pL} - L_i^{p*}(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \leq \theta x_{io}^{kL} - L_i^{k*}(h)\theta\alpha_{io}^{kL}, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{pR} + R_i^{p*}(h) \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \leq \theta x_{io}^{kR} + R_i^{k*}(h)\theta\alpha_{io}^{kR}, & \text{i=1,...,m} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{pL} \geq y_{ro}^{kL}, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{pR} \geq y_{ro}^{kR}, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{pL} - L_r'^{p*}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^{pL} \geq y_{ro}^{kL} - L_r'^{k*}(h)\beta_{ro}^{kL}, & \text{r=1,...,s} \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{pR} + R_r'^{p*}(h) \sum_{j=1}^{n} \lambda_j \beta_{rj}^{pR} \geq y_{ro}^{kR} + R_r'^{k*}(h)\beta_{ro}^{kR}, & \text{r=1,...,s} \\
& \lambda_j \geq 0, & \text{j=1,...,n}
\end{array} \tag{20}
$$

Let the below definition represent the Malmquist productivity index, which measures the productivity change of a particular $DMU_o, o \in J = \{1, ..., n\}$, in time $t+1$ and t, at the (h) possibility level is given as

$$
\overline{M}_o^h = [\frac{\overline{D_o^t(t+1)^h}}{\overline{D_o^t(t)^h}} \times \frac{\overline{D_o^{t+1}(t+1)^h}}{\overline{D_o^{t+1}(t)^h}}]^{\frac{1}{2}} \tag{21}
$$

Now, we have several numbers $\overline{M}_o^h$ for DMUo, from the several (h) possibility levels.It can be seen that the above measure actually is the geometric mean of two Caves, et al.'s Malmquist productivity indexes at each (h) possibility levels. Thus, following Caves et al.'s suit, Fare, et al. define that $\overline{M}_o^h > 1$ indicates the productivity gain at (h) possibility level ; $\overline{M}_o^h < 1$ indicates the productivity loss at (h) possibility level; and $\overline{M}_o^h = 1$ means no change in the productivity from time t to $t+1$ in (h) possibility level. On the basis of what mentioned above, the productivity can be classified in three subsets as follows:

$$
\begin{array}{l}
M^{++} = \{j \in J \mid \overline{M}_j^h > 1, \forall h \in [0,1]\}, \\
M^{+} = \{j \in J \mid (\exists h \in [0,1]; \overline{M}_j^h \geq 1) \bigwedge (\exists h \in [0,1]; \overline{M}_j^h \leq 1)\}, \\
M^{--} = \{j \in J \mid \overline{M}_j^h < 1, \forall h \in [0,1]\}
\end{array} \tag{22}
$$

Definition 4.1. A fuzzy set $\tilde{P}$ is said to be a Progressive DMUs Fuzzy Set ,

$$
\tilde{P} = \{(DMU_j, \mu_{\tilde{P}}(DMU_j)) | j = 1, 2, ..., n\}
$$

, if its membership function has the following form:

$$\mu_{\tilde{P}}(DMUj) = \sup\{h|\overline{M}_j^k > 1, \forall k \in [0,h]\} \tag{23}$$

5 Methodology and examples

In this section, we employ the Malmquist index for evaluating 19 branches of Tehran Social Secarity Insurance Organization for two period times by using the methodology developed above.Each branch uses two inputs in order to produce two outputs. The lables of inputs and outputs are presented at table (1).

	Input	Output
1	The number of personals	The total number of insured persons
2	The total number of computers	The number of insured persons'agreements

Table1.The lables of inputs and outputs.

The total data is related to two chronological sections of 2003 and 2004(A-D). This section describes the data and results. Table 2, 3, 4 and 5 show inputs and outputs of those Insurance Organizations with L-R fuzzy data, $\tilde{m} = (\alpha^L, m^L, m^R, \alpha^R)_{L,R}$, at time periods 1 and 2.we cosider that :

$$\begin{array}{ll} \tilde{x}_{i,j} = (\alpha^L_{i,j}, x^L_{i,j}, x^R_{i,j}, \alpha^R_{i,j}), & i = 1, 2,\ j = 1, ...19, \\ \tilde{y}_{r,j} = (\beta^L_{r,j}, y^L_{r,j}, y^R_{r,j}, \beta^R_{r,j})\ , & r = 1, 2,\ j = 1, ...19. \end{array} \tag{24}$$

$$\begin{array}{ll} L_{1j}(h) = L_{2j}(h) = 1 - h, & j = 1, .., 19, \\ L'_{1j}(h) = L'_{2j}(h) = 1 - h, & j = 1, .., 19, \\ R_{1j}(h) = R_{2j}(h) = 1 - h, & j = 1, .., 19, \\ R'_{1j}(h) = R'_{2j}(h) = 1 - h, & j = 1, .., 19. \end{array} \tag{25}$$

AS mentioned before,

$$\begin{array}{ll} L^*(h) = \sup\{z : 1 - z \geq h\} = 1 - h, & L'^*(h) = \sup\{z : 1 - z \geq h\} = 1 - h, \\ R^*(h) = \sup\{z : 1 - z \geq h\} = 1 - h, & R'^*(h) = \sup\{z : 1 - z \geq h\} = 1 - h, \end{array} \tag{26}$$

	$\tilde{x_{1,j}}$	$\tilde{x_{2,j}}$
1	(1.41, 97.41, 99.41, 0.58)	(0.25,86.25,86.75,0.25)
2	(1.83,76.83,79.83,1.16)	(0.416,88.416,89.416, 0.58)
3	(0.75, 77.75, 79.25, 0.75)	(1, 86, 88, 1)
4	(0.66, 91.66, 93.16, 0.83)	(0.75, 93.75, 95.25, 0.75)
5	(0.83, 89.83, 91.33, 0.66)	(0, 83, 83, 0)
6	(0.5, 102.5, 104, 1)	(0, 97, 97, 0)
7	(0.66, 96.66, 98.66, 1.33)	(0.5, 90.5, 91.5, 0.5)
8	(1, 86, 88.5, 1.5)	(0, 92, 92, 0)
9	(1.25, 107.25, 110.25, 1.75)	(2.16, 86.16, 90.16, 1.83)
10	(0.91, 107.91, 109.91, 1.08)	(0, 95, 95, 0)
11	(1.58, 95.58, 99.08, 1.91)	(0, 78, 78, 0)
12	(0.41, 78.41, 78.91, 0.08)	(0, 89, 89, 0)
13	(0, 102, 102, 0)	(0.33 , 107.33, 109.33, 1.66)
14	(1.5, 83.5, 86.5, 1.5)	(0.58, 92.58, 93.58, 0.41)
15	(0.25, 89.25, 90.25, 0.75)	(0, 85, 85, 0)
16	(2.25, 86.25, 89.25, 0.75)	(0, 104, 104, 0)
17	(3.91, 97.91, 104.91, 3.08)	(0.33, 91.33, 91.83, 0.16)
18	(2, 99, 102, 1)	(0.08, 95.08, 95.58, 0.41)
19	(1, 83, 85.5, 1.5)	(0.083, 100.08,100.58,0.41)

Table 2. The LR- fuzzy Inputs 19 branches of Insurance Organization at time period (1)

	$\tilde{y_{1,j}}$	$\tilde{y_{2,j}}$
1	(370.33,56200.33,56944.33,373.66)	(3.4166,33.416,40.91, 4.08)
2	(30.25,36770.25,36826.25, 25.75)	(7.83, 7.83, 18.83, 3.16)
3	(221.08, 38225.08, 38614.58, 168.41)	(3.33, 14.33, 22.33, 4.66)
4	(108.08, 35577.08, 35851.08, 165.91)	(10.16, 20.16, 42.66, 12.33)
5	(471, 53398, 54343, 474)	(9.08, 18.08, 35.08, 7.91)
6	(1096.16, 71350.16, 75510.16, 3063.83)	(2.66, 9.66, 15.66, 3.33)
7	(1635.66, 34220.66, 36649.66, 793.33)	(23.08, 70.08, 111.08, 17.91)
8	(1063.83, 43963.83, 46148.83, 1121.16)	(3.16, 14.16, 22.16, 4.83)
9	(411.25, 85810.25, 86720.75, 499.25)	(7.58, 50.58, 77.58, 19.41)
10	(109.25, 47033.25, 47229.25, 86.75)	(8.66, 17.66, 31.16, 4.83)
11	(1915.16, 38567.16, 42390.16, 1907.83)	(48.75 ,129.75 ,210.25, 31.75)
12	(6.25, 39588.25, 39607.25, 12.75)	(4.66, 15.66, 25.66, 5.33)
13	(670.41, 56814.41, 58150.41, 665.58)	(6, 36, 49.5, 7.5)
14	(591.25, 88307.25, 89574.25, 675.75)	(3.83, 31.83, 39.33, 3.66)
15	(425.41, 48152.41, 49033.41, 455.58)	(3.66, 18.66, 26.16, 3.83)
16	(88.25, 53011.25, 53174.25, 74.75)	(2.5, 17.5, 24, 4)
17	(2342.75, 80892.75, 86173.25, 2937.75)	(3.83, 16.83, 22.83, 2.16)
18	(171.41, 46325.41, 46643.91, 147.08)	(1.58, 14.58, 18.58, 2.41)
19	(497.91, 28475.91, 30958.41, 1984.58)	(27.41, 56.41, 204.41, 120.58)

Table 3. The LR- fuzzy Out puts for 19 branches of Insurance Organization at time period (1)

	$\tilde{x_{1,j}}$	$\tilde{x_{2,j}}$
1	(0.91, 93.916, 95.91, 1.08)	(0.25, 84.25, 85.75, 1.25)
2	(1, 76, 78, 1)	(1, 92, 94, 1)
3	(0.75 ,75.75, 77.25, 0.75)	(0, 87, 87, 0)
4	(0.41, 92.416, 93.41 , 0.58)	(0, 93, 93, 0)
5	(1.16, 89.16, 91.16, 0.83)	(1.33, 84.33, 86.33, 0.66)
6	(0.66, 101.66, 103.66, 1.33)	(0, 97, 97, 0)
7	(0.25, 94.25, 94.75, 0.25)	(0.25, 90.25, 90.75, 0.25)
8	(1.66, 84.66, 87.66, 1.33)	(0.16, 92.16, 92.66, 0.33)
9	(0.41, 102.41, 104.41, 1.58)	(0, 92, 92, 0)
10	(0.25, 102.25, 102.75, 0.25)	(0.66, 95.66, 96.66, 0.33)
11	(1.25, 94.25, 95.75, 0.25)	(0, 79, 79, 0)
12	(1.08, 77.08, 78.58, 0.41)	(0, 91, 91, 0)
13	(0.83, 103.83, 105.83, 1.16)	(0.66, 103.66, 104.66, 0.33)
14	(1.25, 87.25, 89.25, 0.75)	(0, 95, 95, 0)
15	(1, 88, 90, 1)	(0.16, 85.16, 85.66, 0.33)
16	(0.75, 90.75, 92.25, 0.75)	(0, 104, 104, 0)
17	(1.66, 112.66, 115.66, 1.33)	(1.16, 93.16, 94.66, 0.33)
18	(1.16, 95.16, 97.66, 1.33)	(0, 98, 98 ,0)
19	(0.83, 85.83, 87.33, 0.66)	(0, 101, 101, 0)

Table 4. The LR- fuzzy Inputs for 19 branches of Insurance Organization at time period (2)

	$\tilde{y_{1,j}}$	$\tilde{y_{2,j}}$
1	(390.66, 58058.66, 58516.41, 576.83)	(8.58, 40.58, 60.08, 10.91)
2	(61.41, 36983.41, 39550.16, 67.08)	(3.91, 17.91, 28.41, 6.58)
3	(4539.16, 29899.16, 34623.91, 2505.33)	(5.83, 25.83, 39.33, 7.66)
4	(202.41, 36449.41, 41498.16, 185.58)	(10, 31, 50, 9)
5	(5009.16, 41380.16, 46987.5, 4846.33)	(6.25, 34.25, 44.75, 4.25)
6	(1368.83, 70439.83, 72244.33, 598.16)	(7.916 , 7.91, 24.41, 8.58)
7	(595.83, 38071.83, 39645.08, 435.66)	(15.91, 88.91, 115.41, 10.58)
8	(1047.58, 49141.58, 52412.41, 977.41)	(4, 16, 25.5, 5.5)
9	(1889.25, 86420.25, 90433.5, 2223.25)	(34.08, 34.08, 89.58, 21.41)
10	(1176, 48133, 50993.5 ,2124)	(5.58, 24.58, 35.08, 4.91)
11	(2209, 33763, 40131.91, 1684.5)	(21.33, 191.33, 240.33, 27.66)
12	(2139.58, 29151.58, 32064.91, 4159.91)	(2.5, 23.5, 29, 3)
13	(911.75, 60492.75, 63215.16, 773.75)	(8 ,39, 62, 15)
14	(4947.83, 85372.83, 91481.25, 1810.66)	(4.91, 40.91, 55.91, 10.08)
15	(1972.25, 41769.25, 46409.66, 3056.75)	(2.75 ,28.75, 34.25, 2.75)
16	(11993.83, 65592.83, 80853.83, 2668.16)	(5, 19, 31, 7)
17	(3368.83, 75921.83, 86140.66, 3267.16)	(4.66, 23.66, 35.16, 6.83)
18	(13383, 60280, 74386.91, 6850)	(5.16, 18.16, 28.66, 5.33)
19	(1667.08, 30522.08, 32389.25, 723.91)	(18.33, 41.33, 87.83, 28.16)

Table 5. The LR- fuzzy Out puts for 19 branches of Insurance Organization at time period (2)

	$\overline{M}^0$	$\overline{M}^{0.1}$	$\overline{M}^{0.2}$	$\overline{M}^{0.3}$	$\overline{M}^{0.4}$	$\overline{M}^{0.5}$	$\overline{M}^{0.6}$	$\overline{M}^{0.7}$	$\overline{M}^{0.8}$	$\overline{M}^{0.9}$	$\overline{M}^1$
1	1.12	1.11	1.1	1.07	1.06	1.07	1.07	1.06	1.05	1.03	1.03
2	2.43	1.9	1.6	1.53	1.44	1.37	1.31	1.25	1.18	1.14	1.12
3	0.88	0.9	0.9	0.96	0.99	1.01	1.02	1.03	1.04	1.02	1.02
4	1.14	1.12	1.11	1.1	1.09	1.09	1.08	1.07	1.06	1.08	1.11
5	2.04	2.57	2.12	1.7	1.5	1.38	1.3	1.19	1.08	0.97	0.84
6	0.92	0.91	0.89	0.88	0.86	0.84	0.82	0.8	0.81	0.85	0.92
7	1.55	1.51	1.44	1.39	1.34	1.3	1.25	1.2	1.15	1.1	1.06
8	1.28	1.19	1.29	4	2.41	1.93	1.56	1.33	1.2	1.13	1.1
9	0.52	0.31	0.23	0.56	0.68	0.78	0.83	0.86	0.87	0.88	0.92
10	1.17	1.18	1.17	1.15	1.13	1.11	1.09	1.06	1.04	1.01	1
11	2.51	2.26	2	1.72	1.48	1.33	1.22	1.14	1.08	1.03	0.98
12	1.17	1.06	1.01	0.97	0.97	0.98	0.98	0.97	0.95	0.9	0.85
13	1.06	1.05	1.03	1.01	1.01	1.03	1.05	1.06	1.07	1.07	1.05
14	1.79	1.6	1.43	1.27	1.17	1.11	1.07	1.03	1	0.98	0.98
15	1.03	1.01	0.99	0.96	0.93	0.91	0.9	0.89	0.88	0.88	0.87
16	0.95	0.97	1	1.05	1.09	1.13	1.16	1.19	1.21	1.23	1.27
17	10.02	1.57	1.21	1.07	1.01	0.97	0.94	0.92	0.92	0.92	0.91
18	1.49	1.55	1.59	1.63	1.66	1.69	1.72	1.73	1.67	1.57	1.46
19	0.97	0.93	0.89	0.84	0.8	0.76	0.73	0.69	0.66	0.61	0.57

Table 6. The Malmquist Productivity Indexs at the (h) possibility levels.

As it is apparent in the above table, we see that 9 branches are in M^{++} and 3 branches are in M^{--} 7 remaied branches are in M^{+}. It is viewed the most progress in the branch 17 at (0) possibility level and the most regress is in the branch 9 at (0.2)possibility level.The Progressive DMUs Fuzzy Set on approximation (0.1) of this example defined as follows:

$$\tilde{P} = \{(DMU_1, 1), (DMU_2, 1), (DMU_3, 0), (DMU_4, 1), (DMU_5, 0.8), (DMU_6, 0),$$

$$(DMU_7, 1), (DMU_8, 1), (DMU_9, 0), (DMU_{10}, 1), (DMU_{11}, 0.9), (DMU_{12}, 0.2), (DMU_{13}, 1),$$

$$(DMU_{14}, 0.8), (DMU_{15}, 0.1), (DMU_{16}, 0), (DMU_{17}, 0.4), (DMU_{18}, 0), (DMU_{19}, 0)\}$$

6 Conclusion

In this paper, we have developed use of the fuzzy linear programming to provide a new approach to the problem of assessing efficiency with DEA models. This may be especially appealing in real life. The purpose of this study was to develop the Malmquist productivity index for DMUs with fuzzy data. Since the level of inputs and outputs for DMU_o are not known exactly, we tried by using the concept of $(q)^h$ to develop a new approach of Malmquist productivity index and applied it to a numerical example.

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