



# Magnetohydrodynamic Flow in Horizontal Concentric Cylinders

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## Abstract

This article presents the exact solutions of the velocity and temperature fields for a steady fully developed magnetohydrodynamic flow of a viscous incompressible and electrically conducting fluid between two horizontal concentric cylinders. Using the velocity and temperature field expressions, we have calculated the entropy generation rate and irreversibility ratio. Our study focuses on the influence of the Hartmann number, Brinkman number, Peclet number and inner radius on the fluid temperature field, entropy generation rate and irreversibility ratio with the help of graphs and table.

*Keywords* : Free convection; Magnetohydrodynamics; Modified Bessel function; Entropy generation; Irreversibility ratio.

## 1 Introduction

THE study of hydromagnetic flow of an electrically conducting fluid in the presence of an external magnetic field has wide range of its applications in the various branches of science and technology, industries and geothermal power generation. Other potential applications of the study of transport phenomenon involving the annular geometry are the operation of magnetohydrodynamic generators, MHD pumps, plasma studies, nuclear reactor, the thermal recovery of oil, crystal formation, material manufacturing and geological formulation etc. Moreover, these studies

of forced convection flow are plausible impact by an imposed magnetic field. So many analyses have been done on the studies of MHD forced convective flow with various physical situations. We can observe such investigation in the following work of Shercliff [15], Sparrow and Chess [16], Gold [8], Cramer and Pai [6]. Recently, Azim et al. [2] have studied the MHD-conjugate free convection from an isothermal horizontal cylinder with stress work. Raju et al. [14] have studied the MHD convective flow through porous medium in a horizontal channel with insulated and impermeable bottom wall in the presence of viscous dissipation and Joule heating. Very recently, Kumar and Singh [10, 11, 12] have performed the influence of the Hall current on MHD natural convective flow between vertical Walls with taking different boundary conditions on induced magnetic field.

Some decades ago, we generally determine the efficiency of a system by the first law of thermodynamic. However, in recently years, so many researchers have identified that the entropy generation analysis is more appropriate and accu-

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rate via second law of thermodynamics than the first law of thermodynamics. Actually, the main basis of knowledge of entropy generation comes from the Clausius and Kelvins studies on the irreversibility aspects of the second law of thermodynamics. However, the entropy generation obtained by temperature differences has remained untreated by the classical thermodynamics. The entropy generation is closely associated with the thermodynamic irreversibility because it occurs in all the heat transfer processes. It is well known that most thermal processes are inherently irreversible. In a continuous entropy generation, exergy (useful energy or available energy for work) of a system destroys due to the irreversibility. This exergy loss is basically originated by the heat transfer occurring in different modes (conduction, convection and radiation) and which are very common in most thermal engineering systems. We can determine the performance of engineering processes and the thermal machines like power plants, heat engines, refrigerators, heat pumps and air conditioners with the help of entropy generation.

Bejan [4, 5] has very well presented the utilization of second law of thermodynamics in convective heat transfer. The expressions for the velocity, temperature distributions, entropy generation number and Bejan number to forced convection inside a cylindrical annular space with isoflux boundary conditions with help of first and second laws of thermodynamics have been obtained by Mahmud and Fraser [13]. Das and Jana [7] have performed the study of entropy generation due to hydromagnetic flow in a porous channel with the Navier slip. Jha et al. [9] have analyzed an exact study of MHD natural convection flow in a vertical parallel plate micro-channel. Baag et al. [3] have studied the entropy generation analysis for viscoelastic hydromagnetic flow over a stretching sheet embedded in a porous medium. Currently, the problem of magnetohydrodynamic flow of Sisko fluid near the axisymmetric stagnation point towards a stretching cylinder has solved numerically by Awais et al. [1].

In the present study, our study focuses on the influence of magnetic field on a steady fully developed hydromagnetic flow of a viscous incompressible and electrically conducting fluid between horizontal annular cylinders. The governing equations corresponding to the velocity and tempera-

ture fields have been obtained in exact form and further, the expression for the entropy generation rate and irreversibility ratio have also been obtained. Finally, we have presented the Hartmann number, Brinkman number, Peclet number and radius of inner cylinder on the velocity, temperature field, entropy generation number and irreversibility ratio by using the graphs and tables.

## 2 Mathematical Formulation

We consider a steady fully developed hydromagnetic flow of a viscous incompressible and electrically conducting fluid between two horizontal concentric cylinders. The radius of inner and outer cylinders are taken as  $a'$  and  $b'$  such that  $a' < b'$  respectively. We have used cylindrical polar coordinate system  $(r', \theta', z')$  with  $r'$  in the radial direction,  $z'$  lies along the central line of inner and outer cylinders. We have taken that the inner surface is kept at constant heat flux while the outer surface is adiabatic. A uniform magnetic field of strength  $\vec{B} = (B'_0, 0, 0)$  is applied in a direction perpendicular to the fluid flow. The physical model of the problem is given in Figure 1. The angular velocity of the fluid is considered as zero for the fully developed unidirectional flow. The magnetic Reynolds number of the flow is taken small so that the induced magnetic field can be neglected compared to the applied radial field.

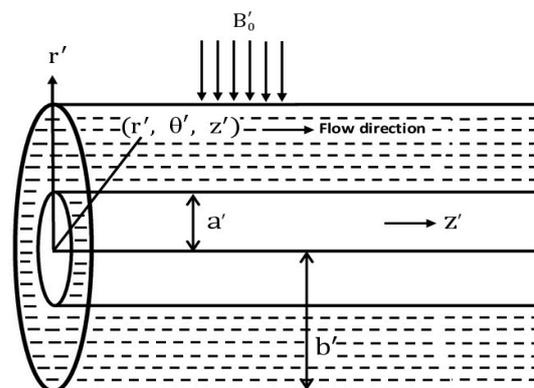


Figure 1: Physical Model

Thus, the momentum equation in the cylindrical polar coordinate system under these assump-

tions is given as:

$$\mu\left(\frac{d^2u'}{dr'^2} + \frac{1}{r'}\frac{du'}{dr'}\right) - \sigma B_0'^2 u' - \frac{\partial p}{\partial z'} = 0, \quad (2.1)$$

with boundary conditions:

$$u'(a') = 0, u'(b') = 0, \quad (2.2)$$

where  $\mu$  is the fluid viscosity,  $u'$  is the fluid velocity,  $\sigma$  is the fluid conductivity and  $\frac{\partial p}{\partial z'}$  is the applied pressure gradient.

We can obtain the solution of Eqn. (2.1). But, we should cast them in their most efficient form, non-dimensional form, thereby increasing the usefulness of whatever solution we find. For this purpose, we have used the appropriate non-dimensional variables for the flow, defined as follows:

$$\begin{aligned} r &= \frac{r'}{b'}, z = \frac{z'}{Peb'}, G = -\frac{\partial p}{\partial z'}, \\ u &= \frac{\mu u'}{Gb'^2}, a = \frac{a'}{b'}, Ha = B_0' b' \sqrt{\frac{\sigma}{\mu}}, \end{aligned} \quad (2.3)$$

where  $r$  is the dimensionless radial coordinate,  $z$  dimensionless axial coordinate,  $u$  the characteristic velocity and  $Ha$  the Hartmann number.

Using the Eq. (2.3), the momentum equation in the non-dimension form is given as follows:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - Ha^2u + 1 = 0, \quad (2.4)$$

with boundary conditions:

$$u(a) = 0, u(1) = 0. \quad (2.5)$$

The analytical solution of the Eq. (2.4) with boundary condition ( ) is obtained as follows:

$$u = CI_0(rHa) + DK_0(rHa) + \frac{1}{Ha^2}, \quad (2.6)$$

here C and D are constants which are defined in appendix-A,  $I_0$  and  $K_0$  are the modified Bessel functions of the first and second kinds of order zero, respectively.

The main basis of engineering purpose, for the formulating and solving convective heat transfer problems, is the second law of thermodynamics. In order to find steady state temperature distribution for the laminar flow of a viscous incompressible MHD forced convection flow between

the cylindrical annuli, the thermal energy equation including the effect of the viscous dissipation and joule heating for the considered model is given by:

$$\begin{aligned} \rho C_p u' \frac{\partial T'}{\partial z'} &= \kappa \left( \frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} \right) \\ &+ \mu \frac{du'}{dr'} + \sigma B_0'^2 u'^2, \end{aligned} \quad (2.7)$$

Intel condition:

$$T'(b', 0) = T_0'; \quad (2.8)$$

at the inner surface:

$$\frac{\partial T'}{\partial r'}(a', z') = -\frac{q}{\kappa}; \quad (2.9)$$

at the outer surface:

$$\frac{\partial T'}{\partial r'}(b', z') = 0; \quad (2.10)$$

where  $T_0'$  is the temperature at inlet wall.

Using non-dimensional variables given by Eq. (2.3), the energy equation (2.7) with boundary conditions Eqs. (2.8) - (2.9) in non-dimensional form is obtained as follows:

$$\begin{aligned} u \frac{\partial T}{\partial z} &= \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + Br \left( \frac{du}{dr} \right)^2 \\ &+ Br Ha^2 u^2, \end{aligned} \quad (2.11)$$

$$\begin{aligned} T(1, 0) &= 0; \frac{\partial T}{\partial r}(a, z) = -1; \\ \frac{\partial T}{\partial r}(1, z) &= 0; \end{aligned} \quad (2.12)$$

where additional parameters,  $Br = \frac{G^2 b'^3}{\mu q}$  is the Brinkman number and  $T = \frac{\kappa(T' - T_0')}{qb'}$  is the dimensionless temperature.

An analytical solution of the Eq. (2.11) with the boundary conditions (2.12) is obtained by the method of additive separating variables . Hence, we take

$$T(r, z) = R(r) + Z(z). \quad (2.13)$$

With the help of Eq. (2.13), energy equation (2.11) becomes as follows

$$\begin{aligned} u \frac{dZ}{dz} &= \left( \frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) + Br \left( \frac{du}{dr} \right)^2 \\ &+ Br Ha^2 u^2 = \lambda(Let). \end{aligned} \quad (2.14)$$

The solution of Eq. (??) is obtained as follows:

$$\begin{aligned} T(r, z) &= \lambda z + \lambda T_1(r) + Br T_2(r) \\ &+ Br Ha^2 T_3(r) + C_1 \ln(r) + C_2, \end{aligned} \quad (2.15)$$

where  $C_1$  and  $C_2$  are integration constants.

Using the inlet and boundary conditions, given by Eq. (2.12), Eq. (2.15) has given the following expression for  $T(r, z)$ :

$$T(r, z) = C_3z + T_4(r) + BrT_5(r) + BrHa^2T_6(r). \tag{2.16}$$

The constant appearing in above equations are defined in appendix-A

### 3 Entropy Generation Rate

Entropy generation plays an essential role in our understanding of many diverse phenomena ranging from the cosmology to biology. Their importance is manifest in areas of immediate practical interest such as the provision of global energy as well as in others of a more fundamental flavour such as the source of order and complexity in nature. They also form the basis of most modern formulations of both equilibrium and nonequilibrium thermodynamics. Today much progress is being made in our understanding of entropy generation in both fundamental aspects and application to concrete problems. Simply, we can say that entropy is the basic thermodynamic variable that is use to define and relate the thermal properties of the matter and the equilibrium state. The rate of destruction of useful work in an engineering system is directly proportional to the rate of entropy generation, and also the irreversibility of the process is measured by the entropy generation rate. According to the Bejan [4], the volumetric entropy generation rate is defined as:

$$E_G = \frac{\kappa}{T_0'^2} [(\frac{\partial T'}{\partial z'})^2 + (\frac{\partial T'}{\partial r'})^2] + \frac{\mu}{T_0'} (\frac{du'}{dr'})^2 + \frac{\sigma B_0'^2}{T_0'} u'^2. \tag{3.17}$$

In order to evaluate the irreversibility loss in the heat transfer, the entropy generation number may be defined as

$$N_s = \frac{\kappa T_0'^2}{q^2} E_G. \tag{3.18}$$

In terms of dimensionless velocity and temperature fields, the entropy generation number may

be defined as:

$$N_s = \frac{1}{Pe^2} (\frac{\partial T}{\partial z})^2 + (\frac{\partial T}{\partial r})^2 + \frac{Br}{\Omega} [(\frac{du}{dr})^2 + Ha^2u^2] = N_z + N_r + N_f, \tag{3.19}$$

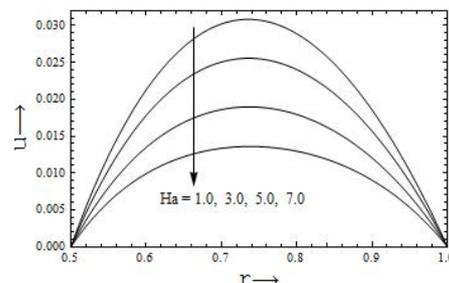
where  $\Omega = \frac{qb'}{\kappa T_0'}$ , while  $N_z$  and  $N_r$  are the entropy generations by heat transfer due to both axial and radial heat convection, respectively, and  $N_f$  is the entropy generation due to fluid friction.

In convection problem, both the fluid friction and heat transfer contribute to the rate of entropy generation. In order to judge the relative importance of the viscous effects to temperature gradients on entropy generation, a formula known as the irreversibility ratio is defined as

$$\phi = \frac{N_f}{(N_z + N_r)}. \tag{3.20}$$

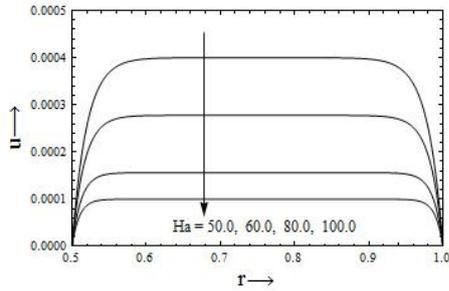
### 4 Results and Discussion

Heat and mass transfer from a circular cylinder has been a subject of many experiments because of its importance in heat exchanger design. The aim of this analysis is to investigate the effects of the Hartmann number (Ha), Brinkman number (Br), Peclet number (Pe) and radius of inner cylinder (a) on convective heat transfer flow of an electrically conducting viscous fluid confined in a horizontal circular annulus in the presence of constant heat flux at inner surface while the outer surface is adiabatic. Physical significance of these non-dimensional numbers is very important for analysis in such conditions.

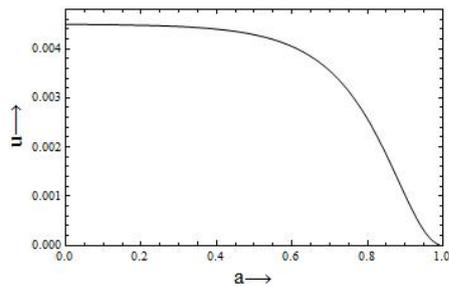


**Figure 2:** Effect of Hartmann number on velocity profiles at a=0.5 for low Ha

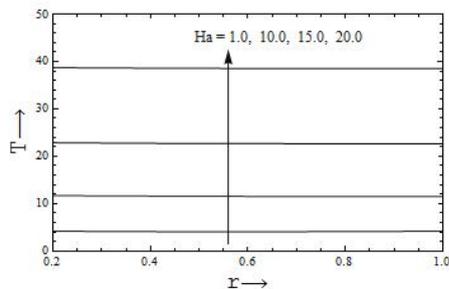
Figures 2, 3 and 4 show that the effect of the Hartmann number and radius of inner cylinder, on the velocity profiles. It is clear from Figures



**Figure 3:** Effect of Hartmann number on velocity profiles at  $a=0.5$  for high  $Ha$

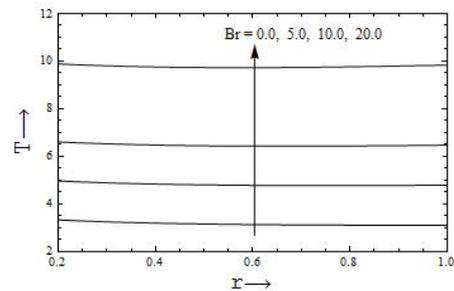


**Figure 4:** Effect of radius of inner cylinder on velocity profile at  $Ha=15.0$

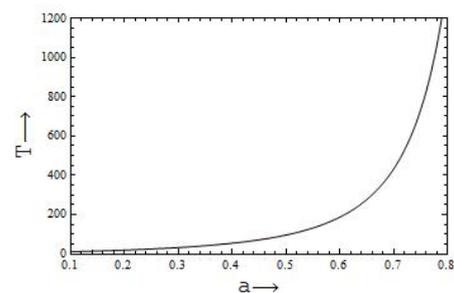


**Figure 5:** Effect of Hartmann number on temperature field profiles at  $a=0.2$ ,  $Br=2.0$  and  $z=0.2$ .

2 and 3 that the velocity profiles decrease with increasing the Hartmann number. This implies that the Hartmann number tends to retard the velocity because the Lorentz force (electromagnetic force) opposes the velocity. The maximum velocity lies in the middle of the annulus cylinder and the shape of velocity profiles is parabolic type in vertical upward direction. Also Figures 2 and 3 display that for smaller electromagnetic force velocity increases from surface to center smoothly while for larger electromagnetic force increases rapidly near the surfaces and around the center it is almost constant. Consequently a high velocity gradient is set up in the fluid in a direction normal to flow near the surfaces. Thus, a boundary layer establishes itself close to the surface with a high velocity gradient. It is observed from Figure 4 that with increasing the radius of inner cylinder the velocity first decreases slowly and then by decreasing rapidly it is tending to zero.

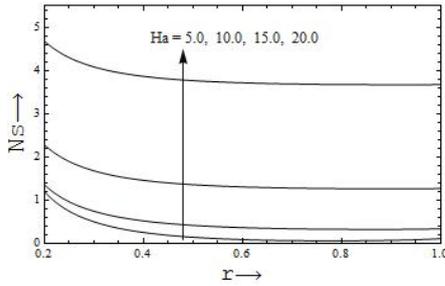


**Figure 6:** Effect of Brinkman number on temperature field profiles at  $a=0.2$ ,  $Ha=4.0$  and  $z=0.2$ .

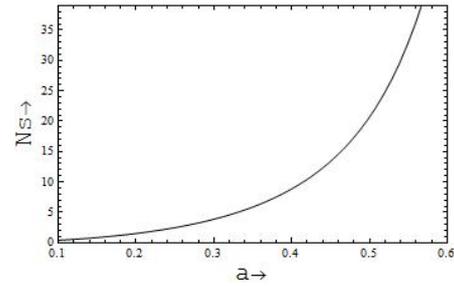


**Figure 7:** Effect of radius of inner cylinder on temperature field profiles at  $Ha=4.0$  and  $z=0.8$ .

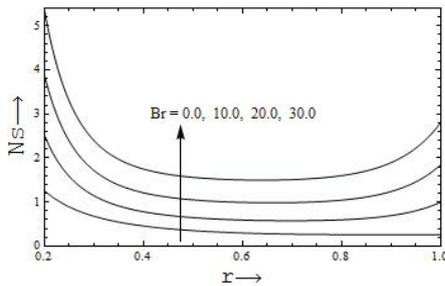
Figures 5-7 depict the temperature field profiles for different values of the Hartmann number, Brinkman number and radius of inner cylinder. It is observed from Figure 5 that as the Hartmann



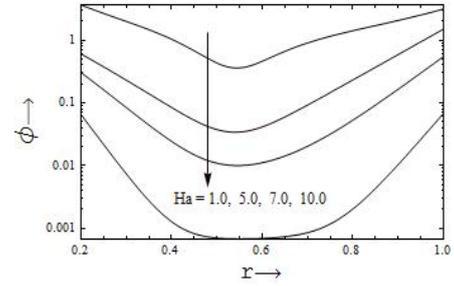
**Figure 8:** Effect of Hartmann number on the graphs of the entropy generation number at  $a=0.2$ ,  $Br=1.0$ ,  $\Omega = 0.5$ ,  $Pe=100$  and  $z=0.5$



**Figure 11:** Effect of radius of inner cylinder on the graphs of the entropy generation number at  $Pe=50$ ,  $Ha=10.0$ ,  $\Omega = 0.5$ ,  $Br=5.0$  and  $z=0.5$

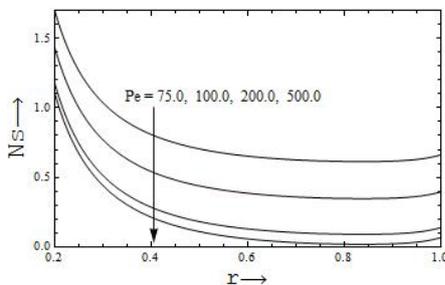


**Figure 9:** Effect of Brinkman number on the graphs of the entropy generation number at  $a=0.2$ ,  $Ha=6.0$ ,  $\Omega = 0.5$ ,  $Pe=50$  and  $z=0.5$



**Figure 12:** Effect of Hartmann number on the graphs of the irreversibility ratio at  $a=0.2$ ,  $Br=2.0$ ,  $\Omega = 0.2$ ,  $Pe=50$  and  $z=0.5$

number increases, there is an increase in the temperature field profiles due to presence of the electromagnetic force. From Figure 6, it is evident that the temperature field profiles leads to increase with increasing the effect of the Brinkman number. From Figure 7, it can be seen that as the radius of inner cylinder increases, i.e. inner boundary shifted towards outer boundary, the temperature field profile first increases slowly and after some instant it increases rapidly.

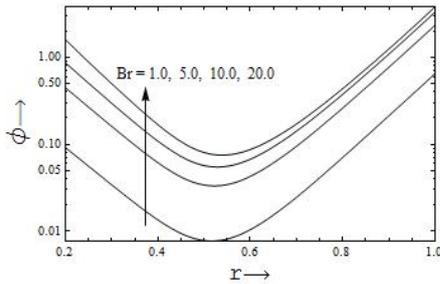


**Figure 10:** Effect of Peclet number on the graphs of the entropy generation number at  $a=0.2$ ,  $Ha=10.0$ ,  $\Omega = 0.5$ ,  $Br=3.0$  and  $z=0.3$

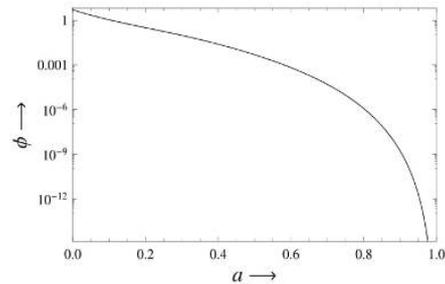
in energy conversion devices, engines or machines, which can only be driven by convertible energy. When a substance is heated or cooled, there is a change in the entropy and has a theoretical entropy minimization (maximum efficiency) while converting the energy to useful work. The effects of various parameters on the graphs of the entropy number are shown in Figures 8-11. It is found from Figure 8 and 9 that as the values of Hartmann number and Brinkman number increase, graphs of the entropy number increase in both cases of the Hartmann number and Brinkman number. Figure 10 shows that the effect of Peclet number is to decrease the profiles of entropy number because entropy decreases due to heat transfer only. Also, it clears from Figure 11 that as the inner radius increases i.e. inner boundary shifted towards to outer boundary, the entropy number rises slowly, and after some instant, it rapidly increases. From the comparative study of Figures 7 and 11, it is found that for the low energy entropy increases slowly but for high energy it increases rapidly.

Entropy is the measure of energy unavailable for useful work in a thermodynamic process, such as

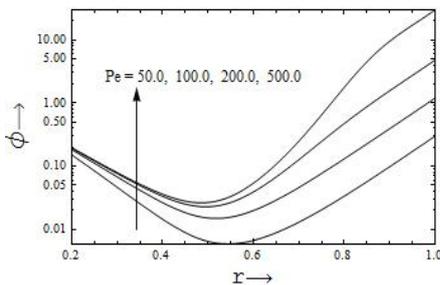
The effects of Hartmann number, Brinkman number, Peclet number and radius of inner annulus cylinder on profiles of the irreversibility ra-



**Figure 13:** Effect of Brinkman number on the graphs of the irreversibility ratio at  $a=0.2$ ,  $Ha=6.0$ ,  $\Omega = 0.5$ ,  $Pe=100$  and  $z=0.5$



**Figure 15:** Effect of radius of inner cylinder on the graphs of the irreversibility ratio at  $Pe=100$ ,  $Ha=4.0$ ,  $\Omega = 0.5$ ,  $Br=10.0$  and  $z=0.5$



**Figure 14:** Effect of Peclet number on the graphs of the irreversibility ratio at  $a=0.2$ ,  $Ha=6.0$ ,  $\Omega = 0.5$ ,  $Br=2.0$  and  $z=0.5$

tio are shown in Figures 12-15. It clearly shows that the impact of the Hartmann number is to decrease the graphs of the irreversibility ratio; therefore, it is observed from Figures 8 and 12 that as the electromagnetic force rises the entropy due to fluid friction very slowly increase in comparison to heat transfer while the profiles of the irreversibility ratio decrease with increasing Brinkman number and Peclet number. Also, it is observed from Figure 14 that the variation of entropy due to Peclet number is negligible near the inner boundary in comparison to the outer boundary. Figure 15 shows that the irreversibility ratio is to decrease with increasing the radius of the inner cylinder i.e. inner boundary shifted to the outer boundary.

Table 1 shows the effects of Hartmann number and radius of inner cylinder on the maximum velocity of the fluid and the minimum temperature with fixed values of Brinkman number and axial coordinate in non-dimensional form. It can be observed that the maximum velocity is shifting from the inner cylinder towards middle of the region with increasing the Hartmann number and inner radius. Also, we can see from table that the minimum temperature has shifted away from middle

of the region means towards the inner cylinder with increasing the effect of the Hartmann number and inner radius.

## 5 Conclusion

A theoretical analysis on a steady fully developed hydromagnetic flow of a viscous incompressible and electrically conducting fluid between two horizontal concentric cylinders under the effect of radial magnetic field has been presented. After obtaining an exact solution of the velocity, temperature field, entropy generation rate and irreversibility ratio, we have analyzed the influence of the Hartmann number, Brinkman number, Peclet number and radius of inner cylinder on these fields and the following conclusions have drawn:

1. It is observed that the increase in the Hartmann number leads to decrease the velocity and irreversibility ratio while to increase the temperature and entropy generation rate.
2. The effect of Brinkman number is to increase the temperature, entropy generation rate and irreversibility ratio.
3. The increase in the Peclet number leads to decrease the entropy generation rate whereas to increase the irreversibility ratio.
4. The impact of the radius of inner cylinder is to decrease the velocity and irreversibility ratio but to increase the temperature and entropy generation rate.
5. It is found from numerical calculation that the maximum velocity reduces while the

**Table 1:** Effect of Hartmann number on the maximum velocity and minimum temperature.

<i>a</i>	<i>Ha</i>	Max. velocity (Max point)	Max. velocity (%variation to center)	Min. temperature (Max point)	Min. temperature (%variation to center)
0.1	1.0	0.4646	15.5272	0.2683	51.2181
0.1	2.0	0.4673	15.0363	0.2535	53.9090
0.1	3.0	0.4712	14.3272	0.2493	54.6727
0.2	1.0	0.5465	8.9166	0.3679	38.6833
0.2	2.0	0.5524	7.9333	0.3613	39.7833
0.2	3.0	0.5536	7.7333	0.3592	40.1333
0.3	1.0	0.6166	5.1384	0.4485	31.0000
0.3	2.0	0.6176	4.9846	0.4436	31.7538
0.3	3.0	0.6187	4.8153	0.4402	32.2769

minimum temperature enhances with increasing Hartmann number and radius of inner cylinder.

### Nomenclature

*a'*: Radius of inner cylinder

*a*: Radius of inner cylinder in non-dimensional form

*b'*: Radius of outer cylinder

*Br*: Brinkman number

*C<sub>p</sub>*: Specific heat at constant pressure

*G*: Applied pressure gradient ( $-\frac{\delta p}{\delta z}$ )

*Ha*: Hartmann number

*I<sub>n</sub>(r)*: Modified Bessel function of first kind of order *n*

*K<sub>n</sub>(r)*: Modified Bessel function of second kind of order *n*

*P*: Pressure

*Pe*: Peclet number

*q*: Constant heat flux

*r'*: Radial coordinate

*r*: Radial coordinate in non-dimensional form

*T'*: Temperature of the fluid

*T*: Temperature of the fluid in non-dimensional form

*T<sub>0</sub>*: Inlet wall temperature

*u'*: Velocity of the fluid

*u*: Velocity in non-dimensional form

*z'*: Axial coordinate

*z*: Axial coordinate in non-dimensional form

### Greek symbols

$\kappa$ : Thermal conductivity of the fluid

$\mu$ : Viscosity of the fluid

$\Omega$ : Dimensionless constant heat flux ( $\frac{qb'}{\kappa T_0}$ )

$\rho$  Density of the fluid

### Appendix

$$C = \frac{[K_0(Ha) - K_0(aHa)]}{Ha^2[I_0(Ha)K_0(aHa) - I_0(aHa)K_0(Ha)]},$$

$$D = \frac{[I_0(Ha) - I_0(aHa)]}{Ha^2[I_0(Ha)K_0(aHa) - I_0(aHa)K_0(Ha)]},$$

$$T_1(r) = \left(\frac{1}{s^4}\right)f_0(r) + \frac{r^2}{4Ha^2},$$

$$T_2(r) = \frac{1}{2}\left[\left(\frac{r}{s}\right)^2\{(f_0(r))^2 - (f_1(r))^2 - f_{00}(r)\} - \left(\frac{r}{s^3}\right)\{f_{01}(r) - g_{501}(r)\} + \left(\frac{r^3}{s}\right)g_{01}(r) - \left(\frac{1}{s^4}\right)\{(f_0(r))^2 - 2f_{00}(r)\}\right],$$

$$T_3(r) = \frac{1}{2}\left[\left(\frac{r}{Ha^2}\right)^2\{(f_1(r))^2 - (f_0(r))^2\} + \left(\frac{r}{Ha^5}\right)\{f_{01}(r) - h_{01}(r)\} - \left(\frac{4}{Ha^6}\right)f_0(r) - \left(\frac{r}{2Ha}\right)^2\right],$$

$$T_4(r) = C_{01}(T_1(r) - E_1) + D_1 \ln\left(\frac{r}{c_0}\right),$$

$$T_5(r) = C_{02}\{T_1(r) - E_1\} + T_2(r) + D_2 \ln\left(\frac{r}{c_0}\right) - E_2,$$

$$T_6(r) = C_{03}\{T_1(r) - E_1\} + T_3(r) + D_3 \ln\left(\frac{r}{c_0}\right) - E_3,$$

$$f_0(r) = CI_0(rHa) + DK_0(rHa),$$

$$f_1(r) = CI_1(rHa) + DK_1(rHa),$$

$$f_{00}(r) = CD\{I_0(rHa)K_0(rHa) +$$

$$I_1(rHa)K_1(rHa)\},$$

$$\frac{f_{01}(r)}{D^2 K_0(rHa)K_1(rHa)} = \frac{\{C^2 I_0(rHa)I_1(rHa)\}}{-}$$

$$\frac{g_{01}(r)}{I_1(rHa)K_0(rHa)} = \frac{CD\{I_0(rHa)K_1(rHa)\}}{+}$$

$$\frac{\bar{g}_{01}(r)}{I_1(rHa)K_0(rHa)} = \frac{CD\{I_0(rHa)K_1(rHa)\}}{-}$$

$$\frac{g_{201}(r)}{I_1(rHa)K_1(rHa)} = \frac{CD\{2I_0(rHa)K_0(rHa)\}}{-}$$

$$g_{501}(r) = g_{01}(r) + 2h_{01}(r),$$

$$h_{01}(r) = 2CDI_0(rHa)K_1(rHa),$$

$$h_{11}(r) = 2CDI_1(rHa)K_1(rHa),$$

$$A_1 = \frac{1}{(Ha)^3} f_0(a) + \frac{a}{2(Ha)^2},$$

$$B_1 = \frac{1}{(Ha)^3} f_1(1) + \frac{1}{2(Ha)^2},$$

$$A_2 = \frac{a}{2(Ha)^2} \{(f_0(a))^2 - (f_1(a))^2 - 2g_{201}(a)\} - \frac{1}{(Ha)^3} \{f_0(a)f_1(a)\} - \frac{a^2}{(Ha)} \bar{g}_{01}(a) - \frac{1}{a(Ha)^4} h_{11}(a),$$

$$B_2 = \frac{1}{2(Ha)^2} \{(f_0(1))^2 - (f_1(1))^2 - 2g_{201}(1)\} - \frac{1}{(Ha)^3} \{f_0(1)f_1(1)\} - \frac{1}{(Ha)} \bar{g}_{01}(1) - \frac{1}{(Ha)^4} h_{11}(1),$$

$$A_3 = \frac{a}{2(Ha)^4} \{(f_1(a))^2 - (f_0(a))^2\} - \frac{2}{(Ha)^5} \{f_1(a)\} - \frac{a}{2(Ha)^4},$$

$$B_3 = \frac{1}{2(Ha)^4} \{(f_1(1))^2 - (f_0(1))^2\} - \frac{2}{(Ha)^5} \{f_1(1)\} - \frac{1}{2(Ha)^4},$$

$$\lambda = C_{01} + BrC_{02} + Ha^2 BrC_{03},$$

$$c_0 = 1,$$

$$C_{01} = \frac{a}{(B_1 - aA_1)},$$

$$C_{02} = \frac{aA_2 - B_2}{(B_1 - aA_1)},$$

$$C_{03} = \frac{aA_3 - B_3}{(B_1 - aA_1)},$$

$$C_1 = D_1 + BrD_2 + Ha^2 BrD_3,$$

$$C_2 = -\lambda E_1 - BrE_2 + Ha^2 BrE_3 - C_1 \ln(c_0),$$

$$C_3 = C_{01} + C_{02} + C_{03},$$

$$D_1 = \frac{aB_1}{(aA_1 - B_1)},$$

$$D_2 = \frac{a(A_2 B_1 - A_1 B_2)}{(aA_1 - B_1)},$$

$$D_3 = \frac{a(A_3 B_1 - A_1 B_3)}{(aA_1 - B_1)},$$

$$E_1 = T_1(c_0),$$

$$E_2 = T_2(c_0),$$

$$E_3 = T_3(c_0).$$

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