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Research Article

The Use of Fuzzy Variational Iteration Method For Solving Second-Order Fuzzy Abel-Volterra Integro-Differential Equations

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Abstract

In this paper, fuzzy variational iteration method (FVIM) is proposed to solve the second- order fuzzy Abel-Volterra integro-differential equations. The existence and uniqueness of the solution and convergence of the proposed method are proved in details is investigated to verify convergence results and to illustrate the efficiently of the method.

Keywords : Fuzzy integro-differential equations; Abel and Volterra integral equations; Fuzzy-valued function; H-difference; Generalized differentiability; Fuzzy variational iteration method (FVIM).

1 Introduction

THe fuzzy integral equations and the fuzzy integro-differential equations are very useful for solving many problems in several applied fields like mathematical economics, electrical engineering, medicine and biology and optimal control theory. Since these equations usually can not be solved explicitly, so it is required to obtain approximate solutions. There are numerous numerical methods which have been focusing on the solution of these equations (1.1)-(3.19).

In this work, we develope the fuzzy variational iteration method to solve the second- order fuzzy Abel-Volterra integro-differential equation of the second kind as follows:

$$\widetilde{u}''(x) \ominus^g \mu \int_a^x k(x,t)\frac{1}{\sqrt{x-t}}\,\widetilde{u}(t)\,dt \ominus^g \widetilde{f}(x) = \widetilde{0}. \qquad (1.1)$$

With fuzzy initial conditions:

$$\widetilde{u}^r(a) = \widetilde{b_r}, \quad r = 0, 1. \qquad (1.2)$$

Where a and μ are crisp constant values and $k(x,t)$ is function that has derivatives on an interval $a \leq t \leq x \leq b$ and b_r are fuzzy constant values and $\widetilde{f}(x)$ is fuzzy function. The second-order Abel-Volterra integro-differential equations occur in various physical applications including heat transfer and viscoelasticity [13].

Here is an outline of the paper. In section 2, the basic notations and definitions in fuzzy calculus are briefly presented. Section 3 describes how to find an approximate solution of the given second-order fuzzy Abel-Volterra integro-differential equations by using proposed method.

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The existence and uniqueness of the solution and convergence of the proposed methods are proved in Section 4 respectively. Finally in section 5, we apply the proposed method by an example to show the simplicity and efficiency of the method, and a brief conclusion is given in Section 6.

2 Basic concepts

Here basic definitions of a fuzzy number are given as follows, [9, 14, 15, 16, 19, 23, 27]

Definition 2.1 *An arbitrary fuzzy number $\widetilde{u}$ in the parametric form is represented by an ordered pair of functions $(\underline{u}, \overline{u})$ which satisfy the following requirements:*

(i) $\overline{u} : r \to \underline{u}(r) \in \mathbb{R}$ *is a bounded left-continuous non-decreasing function over* $[0, 1]$,

(ii) $\underline{u} : r \to \overline{u}(r) \in \mathbb{R}$ *is a bounded left-continuous non-increasing function over* $[0, 1]$,

(iii) $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1.$

Definition 2.2 *For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E$, we use the distance (Hausdorff metric)* $D(u(r), v(r)) = \max\{\sup_{r\in[0,1]}|\underline{u}(r) - \underline{v}(r)|, \sup |\overline{u}(r) - \overline{v}(r)|\}$, *and it is shown that (E , D) is a complete metric space and the following properties are well known:*

$$D(\widetilde{u} + \widetilde{w}, \widetilde{v} + \widetilde{w}) = D(\widetilde{u}, \widetilde{v}), \forall\, \widetilde{u}, \widetilde{v} \in E,$$
$$D(k\widetilde{u}, k\widetilde{v}) = | k \mid D(\widetilde{u}, \widetilde{v}), \forall\, k \in \mathbb{R}, \widetilde{u}, \widetilde{v} \in E,$$
$$D(\widetilde{u} + \widetilde{v}, \widetilde{w} + \widetilde{e}) \leq D(\widetilde{u}, \widetilde{w}) + D(\widetilde{v}, \widetilde{e}),$$
$$\forall\, \widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{e} \in E.$$

Definition 2.3 *Consider x, $y \in E$. If there exists $z \in E$ such that $x = y + z$ then z is called the H- difference of x and y, and is denoted by $x \ominus y$.*

Proposition 2.1 *If $f : (a, b) \to E$ is a continuous fuzzy-valued function then $g(x) = \int_a^x f(t)\, dt$ is differentiable, with derivative $g'(x) = f(x)$.*

Definition 2.4 *Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. We say that f is generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that:*

i) *for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:*

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

or

ii) *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:*

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

iii) *for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0), \exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold:*

$$\lim_{h\to 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

or

iv) *for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h), \exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold:*

$$\lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\to 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

Definition 2.5 *Let $f : (a, b) \to E$. We say f is (i)-differentiable on (a, b) if f is differentiable in the sense (i) of Definition (2.4) and similarly for (ii), (iii) and (iv) differentiability.*

Definition 2.6 *A triangular fuzzy number is defined as a fuzzy set in E, that is specified by an ordered triple $u = (a, b, c) \in R^3$ with $a \leq b \leq c$ such that $u(r) = [\underline{u}(r), \overline{u}(r)]$ are the endpoints of r-level sets for all $r \in [0, 1]$, where $\underline{u}(r) = a+(b-a)r$ and $\overline{u}(r) = c - (c - b)r$. Here, $\underline{u}(0) = a, \overline{u}(0) = c, \underline{u}(1) = \overline{u}(1) = b$, which is denoted by $u(1)$. The set of triangular fuzzy numbers will be denoted by E.*

Definition 2.7 *The mapping $f : T \to E$ for some interval T is called a fuzzy process. Therefore, its r-level set can be written as follows:*

$$f(t)(r) = [\underline{f}(t, r), \overline{f}(t, r)], \quad t \in T, \quad r \in [0, 1].$$

Definition 2.8 *Let $f : T \to E$ be Hukuhara differentiable and denote $f(t)(r) = [\underline{f}(t, r), \overline{f}(t, r)]$. Then, the boundary function $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable (or Seikkala differentiable) and*

$$(f^{'}(t))(r) = [\underline{f}^{'}(t, r), \overline{f}^{'}(t, r)],\ t \in T,\ r \in [0, 1].$$

If f is (ii)-differentiable then

$$f^{'}(t)(r) = [\overline{f}^{'}(t, r), \underline{f}^{'}(t, r)],\ t \in T,\ r \in [0, 1].$$

Definition 2.9 *A fuzzy number $\tilde{A}$ is of LR-type if there exist shape functions L(for left), R(for right) and scalar $\alpha \geq 0, \beta \geq 0$ with*

$$\tilde{\mu}_A(x) = \begin{cases} L(\frac{a-x}{\alpha}) & x \leq a \\ R(\frac{x-b}{\beta}) & x \geq a \end{cases} \tag{2.3}$$

the mean value of $\tilde{A}$, a is a real number, and α, β are called the left and right spreads, respectively. $\tilde{A}$ is denoted by (a, α, β).

Definition 2.10 *Let $\tilde{M} = (m, \alpha, \beta)_{LR}$ and $\tilde{N} = (n, \gamma, \delta)_{LR}$ and $\lambda \in \mathbb{R}^+$. Then,*
$(1) : \lambda\tilde{M} = (\lambda m, \lambda\alpha, \lambda\beta)_{LR}$
$(2) : -\lambda\tilde{M} = (-\lambda m, \lambda\beta, \lambda\alpha)_{LR}$
$(3) : \tilde{M} \oplus \tilde{N} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$

$$(4) : \tilde{M} \odot \tilde{N} \simeq \tag{2.4}$$

$$\begin{cases} (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR},\ \tilde{M}, \tilde{N} > 0 \\ (mn, n\alpha - m\delta, n\beta - m\gamma)_{LR},\ \tilde{M} > 0, \tilde{N} < 0 \\ (mn, -n\beta - m\delta, -n\alpha - m\gamma)_{LR}, : \tilde{M}, \tilde{N} < 0 \end{cases} \tag{2.5}$$

Definition 2.11 *The generalized Hukuhara difference of two intervals, A and B, (gh-difference) is defined as follows*

$$A \ominus^g B = C \Leftrightarrow \begin{cases} (a), & A = B + C \\ or\ (b), & B = A + (-1)C. \end{cases}$$

This difference has many interesting new properties, for example $A \ominus^g A = (0)$. Also, the gh-difference of two intervals $A = [a, b]$ and $B = [c, d]$ always exists and it is equal to

$$A \ominus^{gh} B = [\min\{a - c, b - d\}, \max\{a - c, b - d\}].$$

3 Description of the method

In this section we are going to solve the second-order fuzzy Abel-Volterra integro-differential equation with fuzzy initial conditions by using fuzzy variational iteration method [7].

We consider the second-order fuzzy Abel-Volterra integro-differential equation with fuzzy initial conditions as follows:

$$L(\widetilde{u}(t)) \ominus^g N(\widetilde{u}(t)) \ominus^g g(t) = \widetilde{0}, \quad 0 \leq t \leq T, T \in \mathbb{R}. \tag{3.6}$$

Where the linear operator L is defined as $L = \frac{d^2}{dt^2}$, $N = \mu \int_a^x k(x, t) \frac{1}{\sqrt{x-t}}\ \widetilde{u}(t)\ dt$ is a nonlinear operator, $g(t) = t$ is a known fuzzy function and $\widetilde{0}$ is singleton fuzzy zero with membership function as follows:

$$\mu_{\widetilde{0}}(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

With fuzzy initial condition:

$$u^{(k)}(0) = \widetilde{c}_k, \quad k = 0, 1. \tag{3.7}$$

where $\widetilde{c}_k$ are fuzzy constant values.

In this case, a correction functional can be constructed as follows:

$$\widetilde{u}_{n+1}(t) = \widetilde{u}_n(t) + \int_a^t \lambda(\tau)\{L(\widetilde{u}_n(\tau)) \ominus^g N(\widetilde{u}_n(\tau)) \ominus^g g(\tau)\}d\tau, \quad n \geq 0, \tag{3.8}$$

where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here the function $\widetilde{u}_n(\tau)$ is a restricted variations which means $\delta\, \widetilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $\widetilde{u}_n(t)$, $n \geq 0$ of the solution $\widetilde{u}(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $\widetilde{u}_0$. The zeroth approximations $\widetilde{u}_0$ may be selected any function that just satisfies at least the initial and boundary conditions. With λ determined, then several approximation $\widetilde{u}_n(t)$, $n \geq 1$ follow immediately. Consequently, the exact solution may be obtained by

$$\widetilde{u}(t) = \lim_{n\to\infty} \widetilde{u}_n(t). \tag{3.9}$$

Case (1):
$\widetilde{u}^{(i)}(t)$ is (1)-differentiable for any i (1, 2), in this case we have,

$$\widetilde{u}_{k+1}(t) = \widetilde{u}_k(t) + \int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) \ominus^g N\widetilde{u}_k(\tau) \ominus^g g(\tau)]\, d\tau. \tag{3.10}$$

$$\delta\widetilde{u}_{k+1}(t) = \delta\widetilde{u}_k(t) + \delta \int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) \ominus^g N\widetilde{u}_k(\tau) \ominus^g g(\tau)]\, d\tau. \tag{3.11}$$

We apply restricted variations to nonlinear term $N\widetilde{u}$ $(\delta N\widetilde{u} = \widetilde{0})$, so, we can write Eq.(3.11) as follows:

$$\delta\widetilde{u}_{k+1}(t) = \delta\widetilde{u}_k(t) + \delta \int_0^t [\lambda(\tau)(L\widetilde{u}_k(\tau) \ominus^g g(\tau)]\, d\tau. \tag{3.12}$$

We can write,

$$\delta\underline{u}_{k+1} = \delta\underline{u}_k + \int_0^t \lambda\underline{u}_k^{(2)}(\tau)d\tau,$$

$$\delta\overline{u}_{k+1} = \delta\overline{u}_k + \int_0^t \lambda\overline{u}_k^{(2)}(\tau)d\tau.$$

$$\delta\underline{u}_{k+1} = \delta\overline{u}_{k+1} = 0.$$

$$\begin{aligned}\int_0^t \lambda\underline{u}_k^{(2)}(\tau)d\tau =& \lambda\underline{u}_k^{(1)} - \int_0^t \lambda^{(1)}\underline{u}_k^{(1)}(\tau)\, d\tau \\ =& \lambda\underline{u}_k^{(1)} - (\lambda^{(1)}\underline{u}_k \\ & - \int_0^t \lambda^{(2)}\, \underline{u}_k(\tau)\, d\tau) \\ =& \lambda\underline{u}_k^{(1)} - \lambda^{(1)}\underline{u}_k \\ & + \int_0^t \lambda^{(2)}\underline{u}_k(\tau)d\tau.\end{aligned}$$

$$\begin{aligned}\int_0^t \lambda\overline{u}_k^{(2)}(\tau)d\tau =& \lambda\overline{u}_k^{(1)} - \int_0^t \lambda^{(1)}\overline{u}_k^{(1)}(\tau)\, d\tau \\ =& \lambda\overline{u}_k^{(1)} - (\lambda^{(1)}\overline{u}_k \\ & - \int_0^t \lambda^{(2)}\overline{u}_k(\tau)d\tau) \\ =& \lambda\overline{u}_k^{(1)} - \lambda^{(1)}\overline{u}_k \\ & + \int_0^t \lambda^{(2)}\overline{u}_k(\tau)d\tau.\end{aligned}$$

Finally, we can write

$$\begin{aligned}0 = \delta\underline{u}_{k+1} =& \delta\underline{u}_k + (\lambda\delta\underline{u}_k^{(1)} + \lambda^{(1)}\delta\underline{u}_k) \\ & + \int_0^t \lambda^{(2)}\delta\underline{u}_k(\tau)\, d\tau. \\ 0 = \delta\overline{u}_{k+1} =& \delta\overline{u}_k + (\lambda\delta\overline{u}_k^{(1)} + \lambda^{(1)}\delta\overline{u}_k) \\ & + \int_0^t \lambda^{(2)}\delta\overline{u}_k(\tau)\, d\tau.\end{aligned}$$

So, we have

$$\begin{cases} 1 + \lambda^{(1)} = 0, \\ \lambda^{(2)} = 0, \\ \lambda = \lambda^{(1)} = 0. \end{cases}$$

Finally, we obtain λ as follows

$$\lambda = \tau - t, \quad 0 < t < \tau < T. \tag{3.13}$$

Therefore, substituting (3.13) into functional (3.10), we obtain the following iteration formula,

$$\widetilde{u}_{k+1}(t) = \widetilde{u}_k(t) + \int_0^t [(\tau - t)(L\widetilde{u}_k(\tau) \ominus^g N\widetilde{u}_k(\tau) \ominus^g g(\tau)]\, d\tau. \tag{3.14}$$

Now, define the operator $A[\widetilde{u}]$ as,

$$A[\widetilde{u}] = \int_0^t [(\tau - t)(L\widetilde{u}_k(\tau) \ominus^g N\widetilde{u}_k(\tau) \ominus^g g(\tau)]\, d\tau,$$

and define the components $\widetilde{v}_k,\ k = 0,1,2,\ldots$ as,

$$\begin{aligned}&\widetilde{v}_0 = \widetilde{u}_o,\\ &\widetilde{v}_1 = A[\widetilde{v}_0],\\ &.\\ &.\\ &\widetilde{v}_{k+1} = A[\widetilde{v}_0 + \widetilde{v}_1 + \cdots + \widetilde{v}_k].\end{aligned}$$

We have $\widetilde{u}(t) = \lim_{k\to\infty} \widetilde{u}_k(t) = \sum_{k=0}^{\infty} \widetilde{v}_k(t)$, therefore, we can write recursive relations as follows:

$$\begin{aligned}&\widetilde{v}_0(t) = \widetilde{c}_0 + \widetilde{c}_1 t,\\ &\widetilde{v}_{k+1}(t) = \int_0^t [(\tau - t)(\frac{d^2}{d\tau^2}[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau)\\ &\ominus^g N[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau) \ominus^g g(\tau)]\ d\tau. \qquad (3.15)\end{aligned}$$

Case (2): $\widetilde{u}^{(i)}(t)$ is (2)-differentiable for any i $(1,2)$, in this case we have,

$$\begin{aligned}&\delta\underline{u}_{k+1} = \delta\underline{u}_k + \int_0^t \lambda\overline{u}_k^{(2)}(\tau)d\tau,\\ &\delta\overline{u}_{k+1} = \delta\overline{u}_k + \int_0^t \lambda\underline{u}_k^{(2)}(\tau)d\tau.\\ &\delta\underline{u}_{k+1} = \delta\overline{u}_{k+1} = 0.\end{aligned}$$

$$\begin{aligned}\int_0^t \lambda\underline{u}_k^{(2)}(\tau)d\tau =&\lambda\underline{u}_k^{(1)} - \int_0^t \lambda^{(1)}\underline{u}_k^{(1)}(\tau)\ d\tau\\ =&\lambda\underline{u}_k^{(1)} - (\lambda^{(1)}\underline{u}_k\\ &- \int_0^t \lambda^{(2)}\underline{u}_k(\tau)\ d\tau)\\ =&\lambda\underline{u}_k^{(1)} - \lambda^{(1)}\underline{u}_k\\ &+ \int_0^t \lambda^{(2)}\underline{u}_k(\tau)d\tau.\end{aligned}$$

$$\begin{aligned}\int_0^t \lambda\overline{u}_k^{(2)}(\tau)d\tau =&\lambda\overline{u}_k^{(1)} - \int_0^t \lambda^{(1)}\ \overline{u}_k^{(1)}(\tau)d\tau\\ =&\lambda\overline{u}_k^{(1)} - (\lambda^{(1)}\overline{u}_k\\ &- \int_0^t \lambda^{(2)}\ \overline{u}_k(\tau)\ d\tau)\\ =&\lambda\overline{u}_k^{(1)} - \lambda^{(1)}\overline{u}_k\\ &+ \int_0^t \lambda^{(2)}\overline{u}_k(\tau)d\tau.\end{aligned}$$

Finally, we can write

$$\begin{aligned}&0 = \delta\underline{u}_{k+1} = \delta\underline{u}_k +\\ &(\lambda\delta\overline{u}_k^{(1)} + \lambda^{(1)}\delta\overline{u}_k) + \int_0^t \lambda^{(2)}\delta\overline{u}_k(\tau)\ d\tau.\end{aligned}$$

$$\begin{aligned}&0 = \delta\overline{u}_{k+1} = \delta\overline{u}_k +\\ &(\lambda\delta\underline{u}_k^{(1)} + \lambda^{(1)}\delta\underline{u}_k) + \int_0^t \lambda^{(2)}\delta\underline{u}_k(\tau)\ d\tau.\end{aligned}$$

So, we have

$$\begin{cases} 1 + \lambda^{(1)} = 0,\\ \lambda^{(2)} = 0,\\ \lambda = \lambda^{(1)} = 0.\end{cases}$$

Finally, we obtain λ as follows

$$\lambda = \tau - t, \quad 0 < t < \tau < T. \qquad (3.16)$$

Therefore, we can write recursive relations as follows:

$$\begin{aligned}&\widetilde{v}_0(t) = \widetilde{c}_0 \ominus (-1)\widetilde{c}_1 t,\\ &\widetilde{v}_{k+1}(t) = \int_0^t [(\tau - t)(\frac{d^2}{d\tau^2}[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau)\\ &\ominus^g N[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau) \ominus^g g(\tau)]\ d\tau. \qquad (3.17)\end{aligned}$$

Case (3):

$\widetilde{u}^{(i)}(t)$ is (1)-differentiable for some i,$(1,2)$ and for another is (2)-differentiable. In this case let:

$$\begin{aligned}&P = \{1 \le i \le 2 \mid \widetilde{u}^{(i)}(t) \text{ is } (1) - \text{differentiable}\},\\ &P' = \{1 \le i \le 2 \mid \widetilde{u}^{(i)}(t) \text{ is } (2) - \text{differentiable}\}.\end{aligned}$$

λ in this case is similar to the previous cases.

$$\lambda = \tau - t, \quad 0 < t < \tau < T. \qquad (3.18)$$

If $u^{(i)}(t) \in P$ then $\widetilde{s}_1 = \widetilde{c}_1 t$ and if $u^{(i)}(t) \in P'$ then $\widetilde{s}_1 = \ominus\widetilde{c}_1 t$.

Therefore, we can write recursive relations as follows:

$$\begin{aligned}&\widetilde{v}_0(t) = \widetilde{c}_0 + \widetilde{s}_1,\\ &\widetilde{v}_{k+1}(t) = \int_0^t [(\tau - t)(\frac{d^2}{d\tau^2}[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau)\\ &\ominus^g N[\widetilde{v}_0 + \cdots + \widetilde{v}_k](\tau) \ominus^g g(\tau)]\ d\tau. \qquad (3.19)\end{aligned}$$

Table 1: Numerical results for Example 5.1

r	$(\underline{v}n = 14, t = 0.3)$	$(\overline{v}n = 14, t = 0.3)$
0.0	0.3039702	0.6433234
0.1	0.3136721	0.6221077
0.2	0.3338308	0.6166108
0.3	0.3435612	0.5913441
0.4	0.3563807	0.5827567
0.5	0.3723432	0.5672941
0.6	0.3828568	0.5587114
0.7	0.4035348	0.5357707
0.8	0.4154875	0.5248223
0.9	0.4317526	0.5072443
1.0	0.4454315	0.4454315

Table 2: Numerical results for Example 5.1

r	$(\underline{v}, n = 16, t = 0.3)$	$(\overline{v}, n = 16, t = 0.3)$
0.0	0.3742606	0.7125412
0.1	0.3829503	0.6944317
0.2	0.3954443	0.6872789
0.3	0.4131756	0.6669733
0.4	0.4377873	0.6588403
0.5	0.4452548	0.6365866
0.6	0.4661098	0.6106643
0.7	0.4727766	0.6058915
0.8	0.5068306	0.5837114
0.9	0.5265325	0.5681856
1.0	0.5379755	0.5379755

Table 3: Numerical results for Example 5.1

r	$(\underline{v}, n = 15, t = 0.3)$	$(\overline{v}, n = 15, t = 0.3)$
0.0	0.4625667	0.7723409
0.1	0.4755794	0.7844202
0.2	0.4833572	0.7770843
0.3	0.5019982	0.7537559
0.4	0.5280251	0.7423187
0.5	0.5398431	0.7273561
0.6	0.5568436	0.7023817
0.7	0.5666382	0.6988303
0.8	0.5960981	0.6717429
0.9	0.6122504	0.6586058
1.0	0.6288901	0.6288901

4 Existence and convergence analysis

In this section we are going to prove the convergence and the maximum absolute truncation error of the proposed method.

Theorem 4.1 *The series solution* $\widetilde{u}(t) = \sum_{k=0}^{\infty} \widetilde{v}_k(t)$ *obtained from the relation* (3.15) *using FVIM converges to the exact solution of the problems (1,2) if* $\exists 0 < \gamma < 1$ *such that* $D(\widetilde{v}_{k+1}, \widetilde{0}) \leq \gamma D(\widetilde{v}_k, \widetilde{0})$.

Proof. Define the sequence $\{\widetilde{s}_n\}_{n=0}^{\infty}$ as,

$$\begin{aligned}&\widetilde{s}_0=\widetilde{v}_0,\\&\widetilde{s}_1=\widetilde{v}_0+\widetilde{v}_1,\\&\vdots\\&\widetilde{s}_n=\widetilde{v}_0+\widetilde{v}_1+\cdots+\widetilde{v}_n,\end{aligned}$$

and we show that $\{\widetilde{s}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space. According to the property (1) from Hausdorff metric we can write,

$$\begin{aligned}D(\widetilde{s}_{n+1},\widetilde{s}_n)=&D(\widetilde{v}_{n+1},\widetilde{0})\\&\leq\gamma D(\widetilde{v}_n,\widetilde{0})\\&\leq\gamma^2D(\widetilde{v}_{n-1},\widetilde{0})\\&\leq\cdots\leq\gamma^{n+1}D(\widetilde{v}_0,\widetilde{0}).\end{aligned}$$

For every $n, J\in N$, $n\geq j$, we have,

$$\begin{aligned}D(\widetilde{s}_n,\widetilde{s}_j)\leq&D(\widetilde{s}_n,\widetilde{s}_{n-1})+D(\widetilde{s}_{n-1},\widetilde{s}_{n-2})\\&+\cdots+D(\widetilde{s}_{j+1},\widetilde{s}_j)\\\leq&\gamma^nD(\widetilde{v}_0,\widetilde{0})+\gamma^{n-1}D(\widetilde{v}_0,\widetilde{0})\\&+\cdots+\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0})\\=&\frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0}),\end{aligned}$$

and since $0<\gamma<1$, we get,

$$\lim_{n,j\to\infty}D(\widetilde{s}_n,\widetilde{s}_j)=0.$$

Therefore, $\{\widetilde{s}_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space.

Theorem 4.2 *The maximum absolute truncation error of the series solution $\widetilde{u}(t)=\sum_{k=0}^{\infty}\widetilde{v}_k(t)$ to problems (1,2) by using FVIM is estimated to be*

$$E_j(t)=D(\widetilde{u}(t),\widetilde{u}_j(t))\leq\frac{1}{1-\gamma}\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0}).$$

Proof. We have,

$$\begin{aligned}D(\widetilde{s}_n,\widetilde{s}_j)\leq&D(\widetilde{s}_n,\widetilde{s}_{n-1})+D(\widetilde{s}_{n-1},\widetilde{s}_{n-2})\\&+\cdots+D(\widetilde{s}_{j+1},\widetilde{s}_j)\\\leq&\gamma^nD(\widetilde{v}_0,\widetilde{0})+\gamma^{n-1}D(\widetilde{v}_0,\widetilde{0})\\&+\cdots+\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0})\\=&\frac{1-\gamma^{n-j}}{1-\gamma}\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0}),\end{aligned}$$

for $n\geq j$, then $\lim_{n\to\infty}\widetilde{s}_n=\widetilde{u}(t)$. So,

$$D(\widetilde{u}(t),\sum_{k=0}^{j}\widetilde{v}_k)\leq\frac{1-\gamma^{n-1}}{1-\gamma}\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0}).$$

Also, since $0<\gamma<1$ we have $(1-\gamma^{n-j})<1$. Therefore the above inequality becomes,

$$D(\widetilde{u}(t),\sum_{k=0}^{j}\widetilde{v}_k)\leq\frac{1}{1-\gamma}\gamma^{j+1}D(\widetilde{v}_0,\widetilde{0}).$$

5 Numerical examples

In this section, we solve the second-order fuzzy Abel-Volterra integro-differential equation by using the FVIM. The program has been provided with Mathematica 6 according to the following algorithm where ε is a given positive value.

Algorithm :
Step 1. Set $n\leftarrow 0$.
Step 2. Calculate the recursive relations (3.15) or (3.17) or (3.19).
Step 3. If $D(\widetilde{v}_{n+1},\widetilde{v}_n)<\varepsilon$ then go to step 4, else $n\leftarrow n+1$ and go to step 2.
Step 4. Print $\sum_{i=0}^{\infty}\widetilde{v}_i(t)$ as the approximate of the exact solution.

Example 5.1 *Consider the fuzzy Abel-Volterra integro-differential equation as follows:*

$$u''(x)\ominus^g\widetilde{f}(x)\ominus^g\frac{1}{7}\int_0^x\frac{(s^2+t^3)}{\sqrt{x-t}}\widetilde{u}(t)\,dt=\widetilde{0},$$

where,

$$\begin{aligned}&\widetilde{u}(0)=(0.02,0.04,0.06),\\&\widetilde{u}'(0)=(0.03,0.07,0.09).\\&\widetilde{f}(x)=(x^3,x^3+5,x^3+7).\end{aligned}$$

$\varepsilon=10^{-5}$.
Case (1):
$\gamma=0.58307$.
Table 1 shows that, the approximation solution of the fuzzy Abel-Volterra integro-differential equation is convergent with 14 iterations by using FVIM when $\widetilde{u}'$ and $\widetilde{u}''$ are (i)-differentiable.
Case (2):

$\gamma = 0.56416$. Table 2 shows that, the approximation solution of the fuzzy Abel-Volterra integro-differential equation is convergent with 16 iterations by using FVIM when $\widetilde{u}'$ and $\widetilde{u}''$ are (ii)-differentiable.

Case (3):

$\gamma = 0.56875$.

Table 3 shows that, the approximation solution of the fuzzy Abel-Volterra integro-differential equation is convergent with 15 iterations by using FVIM when $\widetilde{u}'$ is (i)-differentiable and $\widetilde{u}''$ is (ii)-differentiable.

6 Conclusion

The VIM gives several successive approximations through using the iteration of the correction functional without any transformation and hence the procedure is direct and straightforward. The VIM proved to be easy to use and provides an efficient method for handling nonlinear problems. In this work, we presented the fuzzy variational iteration method to solve the second-order fuzzy Abel-Volterra integro-differential equation. This method has been successfully employed to obtain the approximate solution of this equation under generalized H-differentiability. We can use this method to solve another nonlinear fuzzy problems, for example fuzzy partial differential equations and fuzzy integral equations.

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