Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 10, No. 2, 2018 Article ID IJIM-001061, 12 pages
Research Article

An Efficient Numerical Algorithm For Solving Linear Differential Equations of Arbitrary Order And Coefficients

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Received Date: 2017-04-27 Revised Date: 2017-09-07 Accepted Date: 2017-10-22

Abstract

Referring to one of the recent works of the authors, presented in [13], for numerical solution of linear differential equations, an alternative scheme is proposed in this article to considerably improve the accuracy and efficiency. For this purpose, triangular functions as a set of orthogonal functions are used. By using a special representation of the vector forms of triangular functions and the related operational matrix of integration, solving the differential equation reduces to solve a linear system of algebraic equations. The formulation of the method is quite general, such that any arbitrary linear differential equation may be solved by it. Moreover, the algorithm does not include any integration and, instead, uses just sampling of functions, that results in a lower computational complexity. Also, the formulation of this approach needs no modification when a singularity occurs in the coefficients of differential equation. Some problems are numerically solved by the proposed method to illustrate that it is much more accurate and applicable than the prior method in [13].

Keywords : Linear differential equation; Numerical algorithm; Triangular functions; Vector forms; Operational matrix of integration.

1 Introduction

The differential equations beside the other forms of functional equations such as integral and integro-differential equations are widely used for modeling of many problems in physical science and engineering. Such models often have no analytical solution and, therefore, obtaining an approximate solution for them requires a suitable numerical method [5, 15, 14, 23, 12, 16, 8, 20, 18, 21, 17, 1, 2, 3, 4, 19].

An interesting numerical method for solving ordinary linear differential equations has been presented in [13]. It uses vector forms of block-pulse functions (BPFs) [22, 9] for setting up an algebraic equations system and, finally, computing the approximate solution. Although the mentioned method shows a good efficiency (especially, in view of generality), it has some drawbacks regarding the accuracy and singularity. It is the main aim of this article to present a suitable approach to overcome the disadvantages.

This article proposes a numerical method for solving linear ordinary differential equations of arbitrary order and coefficients. For this purpose, a special representation of the vector forms of triangular functions (TFs) [10] are used to ap-

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proximate the solution, its derivatives, and the equation coefficients. By using the related TFs operational matrix of integration, the TFs coefficients vector of the solution and that of its various derivatives are expressed in terms of the TFs coefficients vector of the highest order derivative and the initial conditions vectors, that results in a linear system of algebraic equations. Solving this system gives the TFs coefficients vector of the highest order derivative and, accordingly, an approximate solution for the differential equation is obtained. The main advantages of the proposed method are as follows:

[•]The formulation of the method is quite general, without limitation or restriction. Therefore, it can be used for numerically solving every linear ordinary differential equation of arbitrary order and coefficients. The accuracy of the method is high (considerably higher than that of the BPFs method). The algorithm does not include any integration and, instead, uses just sampling of functions, that results in a lower computational complexity. This is due to the use of orthogonal TFs set, as it uses piecewise linear approximation technique where the coefficients are samples of the approximated function. The algorithm of BPFs method, for a normal discretization size of the problem, includes a great number of integrations for setting up the algebraic equations system. The formulation of this approach needs no modification when a singularity occurs in the coefficients of differential equation. Since the method uses no integration, then there is no need to pass through the singular point, necessarily. In fact, the sampling points may easily be set up such that none of them coincides with the singular point. It should be mentioned that the formulation of BPFs method needs some modification in such a case. The algorithm is simple and clear to use and can be implemented easily.

The organization of this article is as follows. A brief review on TFs and their vector forms is provided in section 2. A special representation of TFs, introduced in [5], is surveyed in section 3. Section 4 presents the numerical method for solving arbitrary linear ordinary differential equations by using the special representation of TFs vector forms and the related operational matrix of integration. Some test problems are numerically solved in section 5 by the proposed method and the related numerical results are given. There will be extensive varieties of orders, coefficients, types, and solutions associated with the test problems to illustrate the generality and computational efficiency of the proposed method. The obtained results are also compared with those of the method presented in [13] to confirm the superiority of the proposed method in this article over the BPFs method in view of accuracy and flexibility. Finally, conclusions will be given in section 6.

2 Review of triangular functions [10]

2.1 Definition

Two m-sets of TFs are defined over the interval $[0, H)$ as [10]

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = H/m$, and $T1_i$ as the ith left-handed TF and $T2_i$ as the ith right-handed TF.

Here, we assume that $H = 1$, so TFs are defined over $[0, 1)$, and $h = 1/m$.

From the definition of TFs, it is clear that they are disjoint, orthogonal, and complete [10]. Also, we can write

$$\varphi_i(t) = T1_i(t) + T2_i(t), \quad i = 0, 1, \ldots, m-1, \tag{2.2}$$

where $\varphi_i(t)$ is the ith BPF defined as

$$\varphi_i(t) = \begin{cases} 1, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

where $i = 0, 1, ..., m-1$.

2.2 Vector forms

Consider the first m terms of left-handed TFs and the first m terms of right-handed TFs and write them concisely as m-vectors:

$$\begin{aligned}\mathbf{T1}(t) &= [T1_0(t), T1_1(t), ..., T1_{m-1}(t)]^T,\\ \mathbf{T2}(t) &= [T2_0(t), T2_1(t), ..., T2_{m-1}(t)]^T,\end{aligned} \tag{2.4}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

2.3 TFs expansion

The expansion of a function $f(t)$ over $[0, 1)$ with respect to TFs, may be compactly written as

$$\begin{aligned}f(t) &\simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t)\\ &= \mathbf{c}^T\mathbf{T1}(t) + \mathbf{d}^T\mathbf{T2}(t),\end{aligned} \tag{2.5}$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \dots, m-1$. So, approximating a known function by TFs needs no integration to evaluate the coefficients.

2.4 Operational matrix of integration

Expressing $\int_0^s \mathbf{T1}(\tau)\tau$ and $\int_0^s \mathbf{T2}(\tau)\tau$ in terms of TFs follows [10]:

$$\begin{aligned}\int_0^s \mathbf{T1}(\tau)\tau &\simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s),\\ \int_0^s \mathbf{T2}(\tau)\tau &\simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s),\end{aligned} \tag{2.6}$$

where $P1_{m\times m}$ and $P2_{m\times m}$ are called operational matrices of integration in TFs domain and represented as follows:

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1\\ 0 & 0 & 1 & \dots & 1\\ 0 & 0 & 0 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0\end{pmatrix},$$
$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1\\ 0 & 1 & 1 & \dots & 1\\ 0 & 0 & 1 & \dots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}. \tag{2.7}$$

So, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned}&\int_0^s f(\tau)\tau \simeq \int_0^s \left[\mathbf{c}^T\mathbf{T1}(\tau) + \mathbf{d}^T\mathbf{T2}(\tau)\right]\tau\\ &\simeq (\mathbf{c}+\mathbf{d})^T P1\mathbf{T1}(s) + (\mathbf{c}+\mathbf{d})^T P2\mathbf{T2}(s).\end{aligned} \tag{2.8}$$

3 A special representation of TFs vector forms and other properties [5]

In this section, we survey a special representation of TFs vector forms that has originally been introduced in [5]. Then, some characteristics of TFs are presented based on this representation.

3.1 Definition and expansion

Let $\mathbf{T}(t)$ be a $2m$-vector defined as [5]

$$\mathbf{T}(t) = \begin{pmatrix}\mathbf{T1}(t)\\ \mathbf{T2}(t)\end{pmatrix}, \quad 0 \leqslant t < 1, \tag{3.9}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ have been defined in (2.4). Now, the expansion of $f(t)$ with respect to TFs can be written as

$$\begin{aligned}f(t) &\simeq F1^T\mathbf{T1}(t) + F2^T\mathbf{T2}(t)\\ &= F^T\mathbf{T}(t)\\ &= \mathbf{T}^T(t)F,\end{aligned} \tag{3.10}$$

where $F1$ and $F2$ are TFs coefficients with $F1_i = f(ih)$ and $F2_i = f((i+1)h)$, for $i = 0, 1, \dots, m-$

1. Also, $2m$-vector F is defined as

$$F = \begin{pmatrix} F1 \\ \\ F2 \end{pmatrix}. \quad (3.11)$$

Now, assume that $k(s,t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$k(s,t) \simeq \mathbf{T}^T(s)\ K\ \mathbf{T}(t), \quad (3.12)$$

where $\mathbf{T}(s)$ and $\mathbf{T}(t)$ are $2m_1$- and $2m_2$-dimensional TFs respectively, and K is a $2m_1 \times 2m_2$ TFs coefficient matrix. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} (K11)_{m\times m} & (K12)_{m\times m} \\ \\ (K21)_{m\times m} & (K22)_{m\times m} \end{pmatrix}, \quad (3.13)$$

where $K11$, $K12$, $K21$, and $K22$ can be computed by sampling of function $k(s,t)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, \ldots, m$. Therefore,

$$\begin{aligned} (K11)_{i,j} &= k(s_i,t_j), & i &= 0,1,\ldots,m-1, \\ && j &= 0,1,\ldots,m-1, \\ (K12)_{i,j} &= k(s_i,t_j), & i &= 0,1,\ldots,m-1, \\ && j &= 1,2,\ldots,m, \\ (K21)_{i,j} &= k(s_i,t_j), & i &= 1,2,\ldots,m, \\ && j &= 0,1,\ldots,m-1, \\ (K22)_{i,j} &= k(s_i,t_j), & i &= 1,2,\ldots,m, \\ && j &= 1,2,\ldots,m. \end{aligned} \quad (3.14)$$

3.2 Product properties

Let X be a $2m$-vector which can be written as $X^T = (X1^T \quad X2^T)$ such that $X1$ and $X2$ are m-vectors. Now, it can be concluded that [5]

$$\mathbf{T}(t)\mathbf{T}^T(t)X \simeq \tilde{X}\mathbf{T}(t), \quad (3.15)$$

where $\tilde{X} = diag(X)$ is a $2m\times 2m$ diagonal matrix.

Now, let B be a $2m \times 2m$ matrix. We have [5]

$$\mathbf{T}^T(t)B\mathbf{T}(t) \simeq \hat{B}^T\mathbf{T}(t), \quad (3.16)$$

in which $\hat{B}$ is a $2m$-vector with elements equal to the diagonal entries of matrix B. Moreover, it is concluded that [5]

$$\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)\ t \simeq D, \quad (3.17)$$

where D is a $2m \times 2m$ matrix defined as

$$D = \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix}. \quad (3.18)$$

3.3 Operational matrix

Expressing $\int_0^s \mathbf{T}(\tau)\tau$ in terms of $\mathbf{T}(s)$, and from Eqs. (2.6), we can write [5]

$$\int_0^s \mathbf{T}(\tau)\tau \simeq P\mathbf{T}(s), \quad (3.19)$$

where $P_{2m\times 2m}$, operational matrix of $\mathbf{T}(s)$, is

$$P = \begin{pmatrix} P1 & P2 \\ \\ P1 & P2 \end{pmatrix}, \quad (3.20)$$

in which $P1$ and $P2$ are given by (2.7).

Now, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned} \int_0^s f(\tau)\tau &\simeq \int_0^s F^T\mathbf{T}(\tau)\tau \\ &\simeq F^T P\mathbf{T}(s). \end{aligned} \quad (3.21)$$

4 Numerical algorithm for solving arbitrary linear differential equations

Here, by using the mentioned representation of TFs vector forms and properties, we propose an effective numerical algorithm for solving linear differential equations of arbitrary order and coefficients.

Let us consider a general ordinary linear differential equation, with arbitrary coefficients, of arbitrary order n as follows:

$$\begin{aligned} x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + a_{n-2}(t)x^{(n-2)}(t) \\ + \cdots + a_1(t)x'(t) + a_0(t)x(t) = b(t), \end{aligned} \quad (4.22)$$

with the following initial conditions:

$$\begin{cases} x(t_0) = \alpha_0, \\ x'(t_0) = \alpha_1, \\ \vdots \\ x^{(n-1)}(t_0) = \alpha_{n-1}, \end{cases} \tag{4.23}$$

where x is the unknown function, with respect to variable t, to be determined; $x^{(k)}$, $k = 1, 2, \ldots, n$, is the kth derivative of x with respect to t; coefficients a_k, $k = 0, 1, \ldots, n-1$, and b are functions of t; and α_k, $k = 0, 1, \ldots, n-1$, is a scalar. Also, without loss of generality, it is supposed that $t_0 = 0$.

Approximating the functions x; $x^{(k)}$, $k = 1, 2, \ldots, n$; a_k, $k = 0, 1, \ldots, n-1$; and b with respect to TFs, using Eq. (3.10), gives

$$\begin{aligned} x(t) &\simeq X_0^T \mathbf{T}(t) = \mathbf{T}^T(t) X_0, \\ x^{(k)}(t) &\simeq X_k^T \mathbf{T}(t) = \mathbf{T}^T(t) X_k, \\ a_k(t) &\simeq A_k^T \mathbf{T}(t) = \mathbf{T}^T(t) A_k, \\ b(t) &\simeq B^T \mathbf{T}(t) = \mathbf{T}^T(t) B, \end{aligned} \tag{4.24}$$

where the m-vectors X_0, X_k, A_k, and B are TFs coefficients of x, $x^{(k)}$, a_k, and b, respectively.

Substituting Eqs. (4.24) into Eq. (4.22) gives

$$\begin{aligned} X_n^T \mathbf{T}(t) &+ A_{n-1}^T \mathbf{T}(t) \mathbf{T}^T(t) X_{n-1} \\ &+ A_{n-2}^T \mathbf{T}(t) \mathbf{T}^T(t) X_{n-2} + \cdots \\ &+ A_1^T \mathbf{T}(t) \mathbf{T}^T(t) X_1 \\ &+ A_0^T \mathbf{T}(t) \mathbf{T}^T(t) X_0 \simeq B^T \mathbf{T}(t). \end{aligned} \tag{4.25}$$

Using Eq. (3.15) follows

$$\begin{aligned} X_n^T \mathbf{T}(t) &+ A_{n-1}^T \tilde{X}_{n-1} \mathbf{T}(t) + A_{n-2}^T \tilde{X}_{n-2} \mathbf{T}(t) \\ &+ \cdots + A_1^T \tilde{X}_1 \mathbf{T}(t) + A_0^T \tilde{X}_0 \mathbf{T}(t) \simeq B^T \mathbf{T}(t), \end{aligned} \tag{4.26}$$

or

$$\begin{aligned} X_n^T &+ A_{n-1}^T \tilde{X}_{n-1} + A_{n-2}^T \tilde{X}_{n-2} \\ &+ \cdots + A_1^T \tilde{X}_1 + A_0^T \tilde{X}_0 \simeq B^T. \end{aligned} \tag{4.27}$$

Transposition of both sides of Eq. (4.27) yields

$$\begin{aligned} X_n &+ \tilde{X}_{n-1} A_{n-1} + \tilde{X}_{n-2} A_{n-2} \\ &+ \cdots + \tilde{X}_1 A_1 + \tilde{X}_0 A_0 \simeq B, \end{aligned} \tag{4.28}$$

because $\tilde{X}_k^T = \tilde{X}_k$, $k = 0, 1, \ldots, n-1$. Therefore, by considering $\tilde{X}_k A_k = \tilde{A}_k X_k$, $k = 0, 1, \ldots, n-1$, we get

$$\begin{aligned} X_n &+ \tilde{A}_{n-1} X_{n-1} + \tilde{A}_{n-2} X_{n-2} \\ &+ \cdots + \tilde{A}_1 X_1 + \tilde{A}_0 X_0 \simeq B. \end{aligned} \tag{4.29}$$

On the other hand we have

$$\begin{cases} \int_0^t x^{(n)}(\tau)\tau = x^{(n-1)}(t) - \alpha_{n-1}, \\ \int_0^t x^{(n-1)}(\tau)\tau = x^{(n-2)}(t) - \alpha_{n-2}, \\ \int_0^t x^{(n-2)}(\tau)\tau = x^{(n-3)}(t) - \alpha_{n-3}, \\ \vdots \\ \int_0^t x'(\tau)\tau = x(t) - \alpha_0, \end{cases} \tag{4.30}$$

which results in

$$\begin{cases} P^T X_n \simeq X_{n-1} - \vec{\alpha}_{n-1}, \\ P^T X_{n-1} \simeq X_{n-2} - \vec{\alpha}_{n-2}, \\ P^T X_{n-2} \simeq X_{n-3} - \vec{\alpha}_{n-3}, \\ \vdots \\ P^T X_1 \simeq X_0 - \vec{\alpha}_0, \end{cases} \tag{4.31}$$

or

$$\begin{cases} X_{n-1} \simeq P^T X_n + \vec{\alpha}_{n-1}, \\ X_{n-2} \simeq P^T X_{n-1} + \vec{\alpha}_{n-2}, \\ X_{n-3} \simeq P^T X_{n-2} + \vec{\alpha}_{n-3}, \\ \vdots \\ X_0 \simeq P^T X_1 + \vec{\alpha}_0, \end{cases} \tag{4.32}$$

in which P is the operational matrix of integration in Eq. (3.19) and $\vec{\alpha}_k$, $k = 0, 1, \ldots, n-1$, is an m-vector with elements equal to α_k. Equations (4.32) may be rewritten as

$$\begin{cases} X_{n-1} \simeq P^T X_n + \vec{\alpha}_{n-1}, \\ X_{n-2} \simeq (P^T)^2 X_n + P^T \vec{\alpha}_{n-1} + \vec{\alpha}_{n-2}, \\ X_{n-3} \simeq (P^T)^2 X_{n-1} + P^T \vec{\alpha}_{n-2} + \vec{\alpha}_{n-3}, \\ \vdots \\ X_0 \simeq (P^T)^2 X_2 + P^T \vec{\alpha}_1 + \vec{\alpha}_0. \end{cases} \tag{4.33}$$

After successive substitutions we finally obtain

$$\begin{cases} X_{n-1} \simeq P^T X_n + \vec{\alpha}_{n-1}, \\ X_{n-2} \simeq (P^T)^2 X_n + P^T \vec{\alpha}_{n-1} + \vec{\alpha}_{n-2}, \\ X_{n-3} \simeq (P^T)^3 X_n + (P^T)^2 \vec{\alpha}_{n-1} \\ \qquad + P^T \vec{\alpha}_{n-2} + \vec{\alpha}_{n-3}, \\ \vdots \\ X_0 \simeq (P^T)^n X_n + (P^T)^{n-1} \vec{\alpha}_{n-1} \\ \qquad + (P^T)^{n-2} \vec{\alpha}_{n-2} + \cdots + P^T \vec{\alpha}_1 + \vec{\alpha}_0. \end{cases} \tag{4.34}$$

Now, we substitute Eqs. (4.34) into Eq. (4.29) and get

$$\begin{aligned} & X_n + \tilde{A}_{n-1}(P^T X_n + \vec{\alpha}_{n-1}) \\ & \quad + \tilde{A}_{n-2}((P^T)^2 X_n + P^T \vec{\alpha}_{n-1} + \vec{\alpha}_{n-2}) \\ & \quad + \tilde{A}_{n-3}((P^T)^3 X_n + (P^T)^2 \vec{\alpha}_{n-1} \\ & \quad + P^T \vec{\alpha}_{n-2} + \vec{\alpha}_{n-3}) \\ & \quad + \cdots \\ & \quad + \tilde{A}_0((P^T)^n X_n + (P^T)^{n-1} \vec{\alpha}_{n-1} \\ & \quad + (P^T)^{n-2} \vec{\alpha}_{n-2} + \cdots + P^T \vec{\alpha}_1 + \vec{\alpha}_0) \\ & \quad \simeq B, \end{aligned} \tag{4.35}$$

or

$$\begin{aligned} & X_n + \Big[(\tilde{A}_{n-1} P^T) X_n + \tilde{A}_{n-1} \vec{\alpha}_{n-1}\Big] \\ & \quad + \Big[(\tilde{A}_{n-2}(P^T)^2) X_n + \tilde{A}_{n-2} P^T \vec{\alpha}_{n-1} \\ & \quad + \tilde{A}_{n-2} \vec{\alpha}_{n-2}\Big] \\ & \quad + \Big[(\tilde{A}_{n-3}(P^T)^3) X_n + \tilde{A}_{n-3}(P^T)^2 \vec{\alpha}_{n-1} \\ & \quad + \tilde{A}_{n-3} P^T \vec{\alpha}_{n-2} + \tilde{A}_{n-3} \vec{\alpha}_{n-3}\Big] \\ & \quad + \cdots \\ & \quad + \Big[(\tilde{A}_0 (P^T)^n) X_n + \tilde{A}_0 (P^T)^{n-1} \vec{\alpha}_{n-1} \\ & \quad + \tilde{A}_0 (P^T)^{n-2} \vec{\alpha}_{n-2} + \cdots + \tilde{A}_0 P^T \vec{\alpha}_1 \\ & \quad + \tilde{A}_0 \vec{\alpha}_0\Big] \\ & \quad \simeq B. \end{aligned} \tag{4.36}$$

Equation (4.36) may be rewritten as

$$\begin{aligned} & \Big[I + \tilde{A}_{n-1} P^T + \tilde{A}_{n-2}(P^T)^2 \\ & \quad + \tilde{A}_{n-3}(P^T)^3 + \cdots + \tilde{A}_0 (P^T)^n\Big] X_n \\ & \quad \simeq B - \Big[(\tilde{A}_{n-1} \vec{\alpha}_{n-1}) \\ & \quad + (\tilde{A}_{n-2} P^T \vec{\alpha}_{n-1} + \tilde{A}_{n-2} \vec{\alpha}_{n-2}) \\ & \quad + (\tilde{A}_{n-3}(P^T)^2 \vec{\alpha}_{n-1} + \tilde{A}_{n-3} P^T \vec{\alpha}_{n-2} \\ & \quad + \tilde{A}_{n-3} \vec{\alpha}_{n-3}) \\ & \quad + \cdots \\ & \quad + (\tilde{A}_0 (P^T)^{n-1} \vec{\alpha}_{n-1} + \tilde{A}_0 (P^T)^{n-2} \vec{\alpha}_{n-2} \\ & \quad + \cdots + \tilde{A}_0 P^T \vec{\alpha}_1 + \tilde{A}_0 \vec{\alpha}_0)\Big]. \end{aligned} \tag{4.37}$$

Now, we replace $\simeq$ with $=$, and write Eq. (4.37) in a simpler form as

$$G X_n = W, \tag{4.38}$$

in which

$$G = I + \sum_{r=0}^{n-1} \tilde{A}_r (P^T)^{n-r}, \tag{4.39}$$

and

$$W = B - \sum_{k=0}^{n-1} \sum_{r=k}^{n-1} \tilde{A}_k (P^T)^{r-k} \vec{\alpha}_r. \tag{4.40}$$

Equation (4.38) is a linear system of m algebraic equations with respect to m unknowns $x_{n_0}, x_{n_1}, \ldots, x_{n_{m-1}}$, components of X_n. Solution of this system gives vector X_n. Then, form Eqs. (4.34) we have

$$\begin{aligned} X_0 \simeq & (P^T)^n X_n + (P^T)^{n-1} \vec{\alpha}_{n-1} \\ & + (P^T)^{n-2} \vec{\alpha}_{n-2} + \cdots + P^T \vec{\alpha}_1 + \vec{\alpha}_0. \end{aligned} \tag{4.41}$$

Substituting the determined X_n into Eq. (4.41) gives the unknown vector X_0. Hence, an approximate solution for differential equation (4.22) is obtained as

$$x(t) \simeq X_0^T \mathbf{T}(t). \tag{4.42}$$

Remark 4.1 *The method proposed in this section can be used to obtain the numerical solution* *of Eq. (4.22) in any arbitrary bounded real interval. For this purpose, we assume that TFs are defined over arbitrary bounded interval* $[\alpha, \beta)$*, where* $\alpha, \beta \in \mathbb{R}$*. It is clear that all the properties and relations presented as to the TFs can be easily generalized over this interval provided that* $T1_i$ *and* $T2_i$*,* $i = 0, 1, \ldots, m-1$*, are defined as*

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih-\alpha}{h}, & \alpha + ih \leqslant t < \alpha + (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$

$$T2_i(t) = \begin{cases} \frac{t-ih-\alpha}{h}, & \alpha + ih \leqslant t < \alpha + (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{4.43}$$

in which $h = (\beta - \alpha)/m$. By using the above generalization, the formulation proposed in the current section can be applied in solution of the differential equation in any arbitrary bounded real interval $[\alpha, \beta)$ without needing any modification to the formulas of the presented method.

5 Test problems and numerical results

Some test problems are numerically solved here by the proposed method in this article and the related numerical results are compared with those of the method proposed in [13]. There are extensive varieties of orders, coefficients, types, and solutions associated with the test problems given here to illustrate the generality and computational efficiency of the proposed method for the solution of arbitrary linear ordinary differential equations.

The approximate results obtained by both methods for each test problem are calculated at ten points t_i in the related interval $[\alpha, \beta)$ such that $t_i = \alpha + ih'$, where $i = 0, 1, \ldots, 9$ and $h' = (\beta - \alpha)/10$. Moreover, all the results are given in the form of mean-absolute error. If, for a given m, we obtain the approximate solution at ten points t_i, then we can consider the mean-absolute error, related to this value of m, as follows:

$$E_m = \frac{1}{10} \sum_{i=0}^{9} |x(t_i) - x_m(t_i)|, \tag{5.44}$$

where E is the mean-absolute error, and x and x_m stand for the exact and approximate solutions, respectively.

In general, the numerical results obtained by both methods in solution of the considered test problems show the superiority of the proposed method in this article over the BPFs method [13] in view of accuracy and flexibility. This will be illustrated in tables 1–5 related to the test problems.

It should be mentioned that all the computations associated with both methods have been performed using MATLAB software.

Example 5.1 *Numerical solution of Bessel's equation [13]*

The well-known Bessel's equation as a linear homogeneous second-order ordinary differential equation is given by [7, 11]

$$t^2 x''(t) + t x'(t) + (t^2 - \nu^2) x(t) = 0, \tag{5.45}$$

or

$$x''(t) + \frac{1}{t} x'(t) + (1 - \frac{\nu^2}{t^2}) x(t) = 0, \tag{5.46}$$

where ν is a real constant. As a second-order differential equation, Bessel's equation has two independent solutions. If $\nu = \ell$ is an integer, one solution defines $J_\ell(t)$ as the Bessel function of the first kind of order ℓ, and another solution defines $Y_\ell(t)$ referred to as the Bessel function of the second kind of order ℓ. The Bessel functions play an important role in physical and engineering problems; for instance, the Bessel function of the first kind appears in the solution of electromagnetic wave equation inside circular waveguides [11]. We apply both methods for numerically solving Bessel's equation to obtain its approximate solutions. The initial conditions are set such that the equation has a unique solution $J_\ell(t)$. The exact values of J_ℓ have been extracted by MATLAB software. However, a series solution is available as [11]

$$J_\ell(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\ell)!} \left(\frac{t}{2}\right)^{2m+\ell}. \tag{5.47}$$

Moreover

$$\begin{aligned} J_{-\ell}(t) &= (-1)^\ell J_\ell(t), \\ J_\ell(-t) &= (-1)^\ell J_\ell(t). \end{aligned} \tag{5.48}$$

Table 1: Mean-absolute errors for test problem 5.1 (results for J_0).

m	Proposed method in this article	Presented method in [13]
2	2.6 e−2	4.2 e−2
4	1.1 e−2	2.1 e−2
8	3.9 e−3	7.9 e−3
16	1.3 e−3	3.8 e−3
32	3.9 e−4	1.8 e−3
64	1.2 e−4	7.9 e−4
128	3.3 e−5	3.7 e−4
256	9.4 e−6	1.9 e−4
512	2.6 e−6	9.6 e−5
1024	7.3 e−7	4.8 e−5

Table 2: Mean-absolute errors for test problem 5.2 (results for P_1).

m	Proposed method in this article	Presented method in [13]
2	0	2.5 e−1
4	0	1.2 e−1
8	0	4.9 e−2
16	0	2.0 e−2
32	0	8.6 e−3
64	0	4.2 e−3
128	0	2.1 e−3
256	0	1.0 e−3
512	0	5.1 e−4
1024	0	2.5 e−4

Table 3: Mean-absolute errors for test problem 5.3.

m	Proposed method in this article (without needing modification)	Presented method in [13] (with modification)
2	3.3 e−1	1.8 e−1
4	8.1 e−2	1.0 e−1
8	2.0 e−2	4.8 e−2
16	5.1 e−3	2.6 e−2
32	1.3 e−3	1.3 e−2
64	3.2 e−4	6.5 e−3
128	8.0 e−5	3.1 e−3
256	2.0 e−5	1.6 e−3
512	5.0 e−6	7.9 e−4
1024	1.2 e−6	4.1 e−4

The following relations may be used to obtain the derivatives of Bessel functions with respect to t for the required initial conditions. Letting $U_\nu(t)$ denote an arbitrary solution to Bessel's equation, we have [11]

$$\begin{aligned} U'_\nu(t) &= U_{\nu-1} - \frac{\nu}{t}U_\nu, \\ U'_\nu(t) &= -U_{\nu+1} + \frac{\nu}{t}U_\nu. \end{aligned} \tag{5.49}$$

The mean-absolute errors associated with both methods in solution of Bessel's equation in interval $[0, 1)$, for $\ell = 0$, are given in Table 1.

Table 4: Mean-absolute errors for test problem 5.4.

m	Proposed method in this article	Presented method in [13]
2	2.2 e−2	1.3 e−1
4	5.7 e−3	6.5 e−2
8	1.4 e−3	3.2 e−2
16	3.6 e−4	1.6 e−2
32	9.0 e−5	8.1 e−3
64	2.3 e−5	4.1 e−3
128	5.7 e−6	2.0 e−3
256	1.4 e−6	1.0 e−3
512	3.5 e−7	5.1 e−4
1024	8.8 e−8	2.5 e−4

Table 5: Mean-absolute errors for test problem 5.5.

m	Proposed method in this article	Presented method in [13]
2	1.5 e−2	3.8 e−2
4	3.9 e−3	1.8 e−2
8	9.8 e−4	9.3 e−3
16	2.5 e−4	4.1 e−3
32	6.2 e−5	2.1 e−3
64	1.5 e−5	1.0 e−3
128	3.9 e−6	5.4 e−4
256	9.6 e−7	2.5 e−4
512	2.4 e−7	1.3 e−4
1024	6.0 e−8	6.3 e−5

Example 5.2 *Numerical solution of Legendre differential equation [13]*

The Legendre differential equation as a linear homogeneous second-order ordinary differential equation is given by [6]

$$(1-t^2)x''(t) - 2tx'(t) + \nu(\nu+1)x(t) = 0, \quad (5.50)$$

or

$$x''(t) - \frac{2t}{1-t^2}x'(t) + \frac{\nu(\nu+1)}{1-t^2}x(t) = 0. \quad (5.51)$$

where ν is a real constant. One solution to the Legendre differential equation defines the Legendre function of the first kind $P_\nu(t)$. If $\nu = \ell$ is an integer, solution of the Legendre differential equation results in function $P_\ell(t)$ known as the Legendre polynomial of degree ℓ. Another solution to this equation gives $Q_\ell(t)$ referred to as the Legendre function of the second kind. Like the Bessel functions, the Legendre functions too have various applications in physical and engineering problems; e.g., associated Legendre functions appear in the solution of Helmholtz equation in spherical coordinates [11]. The exact values of the first six polynomials P_ℓ may be calculated through the following analytical relations [6]:

$$\begin{aligned} P_0(t) &= 1, \\ P_1(t) &= t, \\ P_2(t) &= \frac{1}{2}(3t^2-1), \\ P_3(t) &= \frac{1}{2}(5t^3-3t), \\ P_4(t) &= \frac{1}{8}(35t^4-30t^2+3), \\ P_5(t) &= \frac{1}{8}(63t^5-70t^3+15t). \end{aligned} \quad (5.52)$$

We apply both methods in solving the Legendre differential equation and set the initial conditions such that the equation has a unique solution $P_\ell(t)$. The mean-absolute errors associated with both methods in solution of this equation in interval $[1,2)$, for $\ell = 1$, are given in Table 2.

Example 5.3 *Numerical solution of a third-order inhomogeneous differential equation with singular coefficients [13]*

We survey in this problem the flexibility of the proposed method in solution of differential equations with singular coefficients. For this purpose, let us consider the following third-order inhomogeneous linear differential equation:

$$\begin{aligned} x'''(t) &- \frac{t}{t^2-0.64}\ln(t^2+0.64)x''(t) \\ &+ t^2\sin\left(\frac{1}{t-0.8}\right)x'(t) \\ &+ \cos(\pi t^2)x(t) = b(t), \end{aligned} \tag{5.53}$$

with exact solution $x(t) = t^3 + \sin(\pi t)$ and right side $b(t)$ as

$$\begin{aligned} b(t) &= 6 - \pi^3\cos(\pi t) \\ &- \frac{t}{t^2-0.64}\ln(t^2+0.64)[6t - \pi^2\sin(\pi t)] \\ &+ t^2[3t^2 + \pi\cos(\pi t)]\sin\left(\frac{1}{t-0.8}\right) \\ &+ \cos(\pi t^2)[t^3 + \sin(\pi t)]. \end{aligned} \tag{5.54}$$

Coefficient $a_1 = t^2\sin\left(\frac{1}{t-0.8}\right)$ has an essential singularity and coefficient $a_2 = -\frac{t}{t^2-0.64}\ln(t^2+0.64)$ has a pole-type singularity at $t = 0.8$ in interval $[0, 1)$.

In such a case, the BPFs method is unable to obtain an appropriate solution by its original formulation, and it needs some modification [13]. However, the proposed method in this article can give a reasonable solution without any modification.

The mean-absolute errors associated with both methods in solution of test problem 5.3, in interval $[0, 1)$, are shown in Table 3. As mentioned above, the BPFs method has been implemented via modification.

Example 5.4 *Numerical solution of a high-order inhomogeneous differential equation with both complex solution and complex coefficients [13]*

We show in this problem that the proposed method is applicable in solving differential equations with complex solution and/or complex coefficients. Let us consider an inhomogeneous linear differential equation of order 15 *as*

$$\begin{aligned} x^{(15)}(t) &+ (t^3 - jt^2 + 1)x^{(10)}(t) \\ &+ (t + j)H_0^{(2)}(t)x^{(5)}(t) \\ &+ jt\sin(t^2 + jt)x(t) = b(t), \end{aligned} \tag{5.55}$$

where $H_0^{(2)}$ is Hankel function of the second kind of zero-order, j is imaginary unit and $j^2 = -1$. Assuming complex exact solution $x(t) = \exp(jt)$ for Eq. (5.55), the right side will be

$$\begin{aligned} b(t) = -\exp(jt)\Big\{ & t^3 + jt^2 \\ & -j[H_0^{(2)}(t) + \sin(t^2 + jt)]t \\ & +H_0^{(2)}(t) + j + 1\Big\}. \end{aligned} \tag{5.56}$$

Both methods are applied in solving Eq. (5.55) to obtain its approximate solutions in interval $[3, 4)$. The mean-absolute errors are given in Table 4.

Example 5.5 *Numerical solution of a very high-order inhomogeneous differential equation [13]*

We survey here the efficiency of the proposed method for numerical solution of very high-order differential equations. For this purpose, we consider the following inhomogeneous linear differential equation of order 35*:*

$$\begin{aligned} x^{(35)}(t) &+ \tan(\sqrt{|t|})x^{(20)}(t) \\ &+ t^2\sin(t^2)x^{(11)}(t) \\ &+ \cos(\sqrt{t^4+1})x(t) = b(t), \end{aligned} \tag{5.57}$$

with exact solution $x(t) = \exp(t) + \sin(t)$ and right side $b(t)$ as

$$\begin{aligned} b(t) = \exp(t)\Big[& 1 + \tan(\sqrt{|t|}) + t^2\sin(t^2) \\ & + \cos(\sqrt{t^4+1})\Big] \\ & + \sin(t)\Big[\tan(\sqrt{|t|}) + \cos(\sqrt{t^4+1})\Big] \\ & - \cos(t)\Big[1 + t^2\sin(t^2)\Big]. \end{aligned} \tag{5.58}$$

The mean-absolute errors associated with both methods in solution of this problem in interval $[-5, -4)$, are shown in Table 5.

6 Conclusion

A numerical approach for solving arbitrary linear ordinary differential equations was proposed in this article by using a special representation of TFs vector forms and the related operational matrix of integration. Some test problems were numerically solved by the method to illustrate its computational efficiency and to show that it is applicable in solving various types of ordinary linear differential equations. In comparison with the BPFs method, we saw that the proposed method is more accurate and flexible and has no limitation.

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