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Po-S-Dense Monomorphism

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Abstract

In this paper we take $\mathcal{A}$ to be the category **Pos-S** of S-posets, for a posemigroup S, $\mathcal{M}_{pd}$ to be the class of partially ordered sequantially-dense monomorphisms and study the categorical properties, such as limits and colimits, of this class. These properties are usually needed to study the homological notions, such as injectivity, of S-posets. Also we show that it is actually equivalent to C^{pd}-density resulting from a closure operator.

Keywords : Po-S-Dense; Semigroup; Limit; Colimit.

1 Introduction

Throughout this paper S denotes a nonempty posemigroup and $\mathcal{M}_{pd}$ stands for the class of po-s-dense monomorphisms of S-posets. To study mathematical notions in a category $\mathcal{A}$, such as injectivity, tensor products, flatness, with respect to a class $\mathcal{M}$ of its (mono)morphisms, one should know some of the categorical properties of the pair $(\mathcal{A}, \mathcal{M})$. In this paper we take $\mathcal{A}$ to be the category **Pos-S** and $\mathcal{M}_{pd}$ to be a particular interesting class of monomorphisms, to be called *partially ordered-s-dense(po-s-dense)* monomorphisms, and investigate its categorical properties.

A study of S-posets from a category-theoretic standpoint forms the content of [8], and extends the results found in [6]. For more information on various properties of S-posets, see also [5].

In the rest of this section we give some preliminaries about S-acts, posets, and S-posets needed in the sequel.

Let S be a semigroup. Recall that a *(right) S-act* is a set A equipped with a map $\lambda : A\times S \to A$, called its *action*, such that, denoting $\lambda(a, s)$ by as, we have $a(st) = (as)t$, for all $a \in A$, $s, t \in S$ and, if S is a monoid with the identity element 1, $a1 = a$. The category of all S-acts, with action-preserving maps between them, is denoted by **Act-**S. An *S-act congruence* θ on A is an equivalence relation with the property that $a\theta a'$, $a, a' \in A$, implies that $as\theta a's$, for all $s \in S$. A quotient S-act is the set A/θ with the natural action, $[a]s = [as]$, which makes the canonical map $\gamma : A \to A/\theta$, $a \mapsto [a]$, an S-act map. For more information about S-acts, see [10].

A semigroup S is said to be a *posemigroup* if it is also a poset whose partial order is compatible with the binary operation.

For a posemigroup S, a *(right) S-poset* is a poset A which is also an S-act whose action is monotone in both arguments. An *S-poset map* (*morphism*) is an action preserving monotone map between S-posets. Note that each poset P can be made into an S-poset with trivial action: $ps = p$, for every $p \in P, s \in S$.

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Let A be an S-poset. An *S-poset congruence* on A is an S-act congruence θ with the property that the S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map. For a binary relation R on A, define the relation $\leq_R$ on A by

$$a \leq_R a' \text{ if and only if}$$

$$a \leq a_1 R a'_1 \leq ... \leq a_n R a'_n \leq a'$$

for some $a_1, a'_1, ..., a_n, a'_n \in A$. Then an S-act congruence θ on A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by $[a] \leq [b]$ if and only if $a \leq_\theta b$. Also the S-poset congruence $\theta(H)$ on A generated by $H \subseteq A \times A$ can be characterized as follows:

$a\theta(H)a'$ if and only if $a = a'$, or there exist $s_1, s_2, ..., s_n, t_1, t_2, ..., t_m \in S^1$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, ..., s_nd_n \leq a';$$
$$a' \leq t_1p_1, t_1q_1 \leq t_2p_2, t_2q_2 \leq t_3p_3, ..., t_mq_m \leq a,$$

where $(c_i, d_i), (p_j, q_j) \in H \bigcup H^{-1}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$.

Moreover, the order relation on $A/\theta(H)$ can be defined by: $[a] \leq [a']$ if and only if $a \leq a'$, or there exist $s_1, s_2, ..., s_n \in S^1$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, ..., s_nd_n \leq a',$$

where $(c_i, d_i) \in H \bigcup H^{-1}$ for $i = 1, 2, ..., n$.

Recall that the *product* of a family of S-posets is their cartesian product, with componentwise action and order. The *coproduct* is their disjoint union, with natural action and componentwise order. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. Also for a family $(A_\alpha)_{\alpha\in I}$ of S-posets each with a unique fixed element 0, the *direct sum* $\bigoplus A_\alpha$ is defined to be the sub S-poset of the product $\prod A_\alpha$ consisting of all $(a_\alpha)_{\alpha\in I}$ such that $a_\alpha = 0$ for all $\alpha \in I$ except a finite number of indices.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \stackrel{g}{\to} & B \end{array}$$

in **Pos-S** is the sub S-poset $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps. Notice that for the case where g is an inclusion, P can be taken as $f^{-1}(C)$.

All colimits in **Pos-S** exist and are calculated as in **Set** with the natural action of S on them. In particular, $\emptyset$ with the empty action of S on it, is the initial object of **Pos-S**. Also, the *coproduct* of S-posets A, B is their disjoint union $A \sqcup B = (A\times\{1\}) \bigcup (B\times\{2\})$ with the obvious action, and coproduct injections are defined naturally.

The pushout of a given diagram

$$\begin{array}{ccc} A & \stackrel{g}{\to} & C \\ f\downarrow & & \\ B & & \end{array}$$

in **Pos-S** is the factor act $Q = (B \sqcup C)/\theta$ where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_Bf(a), u_Cg(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $q_1 = \pi u_C : C \to (B \sqcup C)/\theta$, $q_2 = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Pos-S** are constructed analogously.

Let **I** be a small category and $\mathcal{A} : \mathbf{I} \to$ **Pos-S** be a diagram in **Pos-S** determining the acts A_α, for $\alpha \in I = Obj\mathbf{I}$, and S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor\mathbf{I}$. Recall that the limit of this diagram is $\varprojlim A_\alpha := \bigcap_{\alpha\in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha\in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta}p_\alpha(a) = p_\beta(a)\}$ and p_α, p_β are the α, βth projection maps of the product. The limit S-maps are $q_\alpha : \varprojlim A_\alpha \to A_\alpha$. Also the limit has the universal property which is, if $\{f_\alpha : A \to A_\alpha\}$ is a family of morphisms such that $g_{\alpha\beta}f_\alpha(a) = f_\beta(a)$, then there is a morphism $f : A \to \varprojlim A_\alpha$ such that $q_\alpha f = f_\alpha$.

Remind that a directed system of S-posets and S-maps is a family $(B_\alpha)_{\alpha\in I}$ of S-posets indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha\leq\beta\in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma}g_{\alpha\beta} = g_{\alpha\gamma}$, also $g_{\alpha\alpha} = id$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta\in I})$ in **Pos-S** is given as $\varinjlim B_\alpha = \coprod_\alpha B_\alpha/\rho$ where the congruence ρ is given by $b_\alpha\rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$, in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \varinjlim B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \varinjlim B_\alpha$

is the natural S-map. Also directed colimit has a dual universal property of limit.

2 C^{pd}-Closure operator

In this section, we introduce and briefly study a closure operator, so called C^{pd}-*Closure operator*. For a sub S-poset A of B let us denote $A\downarrow = \{b \in B \mid \exists a \in A, b \leq a\}$ and $Sub(B)$, the set of all sub S-posets of B. First recall the following definition of C^{pd}-closure operator.

Definition 2.1 A family $C^{pd} = (C_B^{pd})_{B\in \mathbf{Pos-S}}$, with $C_B^{pd} : sub(B) \to Sub(B)$, is defined as

$$C_B^{pd}(A) = \{b \in B : bS \subseteq A\downarrow\}.$$

It is easy to show that C^{pd} is a closure operator on **Pos-S** in the sense of [7]. This means that $C_B^{pd}(A)$ is a sub S-poset of B and,

(i) $A \subseteq C_B^{pd}(A)$,

(ii) $A_1 \subseteq A_2 \subseteq B$ implies $C_B^{pd}(A_1) \subseteq C_B^{pd}(A_2)$,

(iii) for every homomorphism $f : B \to D$ and each sub S-poset A of B, $f(C_B^{pd}(A)) \subseteq C_D^{pd}(f(A))$.

We just prove (iii). Let $f : B \to C$ be a homomorphism and $b \in C_B^{pd}(A)$. For every $s \in S$, there exists $a \in A$ such that $bs \leq a$. Then $f(b)s = f(bs) \leq f(a) \in f(A)$ and hence $f(b)S \subseteq f(A)\downarrow$ which deduced that $f(b) \in C_B^{pd}(f(A))$.

Dikranjan and Tholen in [7] state some properties of a closure operator in general. Here we are going to investigate the satisfaction of those properties for the closure operator C^{pd}.

Definition 2.2 *The closure operator C^{pd} is said to be:*

(1) *idempotent(if $C_B^{pd}(A) = C_B^{pd}(C_B^{pd}(A))$).*

(2) *hereditary(if for $A_1 \subseteq A_2 \subseteq B$, $C_{A_2}^{pd}(A_1) = C_B^{pd}(A_1)\bigcap A_2$).*

(3) *weakly hereditary(if for every $A \subseteq B$, $C_{C_B^{pd}(A)}^{pd}(A) = C_B^{pd}(A)$).*

(4) *grounded(if $C_B^{pd}(\emptyset) = \emptyset$).*

(5) *additive(if $C_B^{pd}(A\bigcup C) = C_B^{pd}(A)\bigcup C_B^{pd}(C)$).*

(6) *productive(if for every family of sub S-posets A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B^{pd}(A) = \prod_i C_{B_i}^{pd}(A_i)$).*

(7) *fully additive(if $C_B^{pd}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} C_B^{pd}(A_i)$).*

(8) *discrete(if $C_B^{pd}(A) = A$ for every S-poset B and $A \subseteq B$).*

(9) *trivial(if $C_B^{pd}(A) = B$ for every B and $A \subseteq B$).*

(10) *minimal(if for $C \subseteq A \subseteq B$ one has $C_B^{pd}(A) = A\bigcup C_B^{pd}(C)$).*

Theorem 2.1 *The closure operator C^{pd} is hereditary, weakly hereditary, grounded and productive.*

Proof. It is easy to check that the closure operator C^{pd} is hereditary, weakly hereditary and grounded. We just prove productivity. Let $b \in C_B^{pd}(A)$, $b = \{b_i\}$. For every $s \in S, bs \in A\downarrow$, then for each $i \in I$, $b_is \in A_i\downarrow$ and hence for each $i \in I, b_i \in C_{B_i}^{pd}(A_i)$ which implies that $b \in \prod C_{B_i}^{pd}(A_i)$. The converse is obvious.

Theorem 2.2 The closure operator C^{pd} is idempotent if and only if $S \subseteq S^2\downarrow$.

Proof. ($\Rightarrow$) It is clear that $C_{S^1}^{pd}(S) = S^1$ and $S \subseteq C_{S^1}^{pd}(S^2)$. Since C^{pd} is idempotent, $S^1 = C_{S^1}^{pd}(S) \subseteq C_{S^1}^{pd}(C_{S^1}^{pd}(S^2)) = C_{S^1}^{pd}(S^2) \subseteq S^1$. Thus $1 \in C_{S^1}^{pd}(S^2)$, which implies $S \subseteq S^2\downarrow$.

($\Leftarrow$) By definition of the closure operator, for each $A \subseteq B$ we see that $C_B^{pd}(A) \subseteq C_B^{pd}(C_B^{pd}(A))$. Conversely, let $b \in C_B^{pd}(C_B^{pd}(A))$. So $bS \subseteq C_B^{pd}(A)\downarrow$. Thus for each $s \in S$, $bs \leq b'_s$ for some $b'_s \in C_B^{pd}(A)$. Since $S \subseteq S^2\downarrow$, then $s \leq tt' \in S^2$ and hence $bs \leq btt' \leq b'_tt'$, which $b'_t \in C_B^{pd}(A)$. Thus $b'_tt' \leq a$, for some $a \in A$. Therefore $bs \leq a$ and hence $bs \subseteq A\downarrow$ which implies that $b \in C_B^{pd}(A)$.

Note that the condition $S \subseteq S^2\downarrow$ is equal to $S = S^2\downarrow$.

In the following theorem let us denote, $DSub(B)$, the set of all down closed sub S-posets of an S-poset B.

Theorem 2.3 *The closure operator C^{pd} is additive if and only if for every element b in an S-poset B, bS is join prime in the lattice $DSub(B)$.*

Proof. let C^{pd} be additive. Let $x \in B$ and $xS \subseteq A\bigcup C$, where A and C are down closed sub S-posets of B. Then, by monotonicity and additivity,

$$C_B^{pd}(xS) \subseteq C_B^{pd}(A\bigcup C) = C_B^{pd}(A)\bigcup C_B^{pd}(C).$$

Now, since $x \in C_B^{pd}(xS)$, $x \in C_B^{pd}(A)$ or $x \in C_B^{pd}(C)$. Thus, $xS \subseteq A\downarrow = A$ or $xS \subseteq C\downarrow = C$, proving that xS is join prime in $Sub(B)$.

Conversely, suppose that A and D are sub S-posets of an S-poset B. By definition of the closure operator, $C_B^{pd}(A)\bigcup C_B^{pd}(C) \subseteq C_B^{pd}(A\bigcup C)$. Consider $x \in C_B^{pd}(A\bigcup C)$. So $xS \subseteq (A\bigcup C)\downarrow = (A\downarrow)\bigcup(B\downarrow)$. Since xS is join prime, $xS \subseteq A\downarrow$ or $xS \subseteq C\downarrow$. Thus, $x \in C_B^{pd}(A)\bigcup C_B^{pd}(C)$. This shows that each C_B^{pd}, and hence C^{pd}, is additive.

Corollary 2.1 *If S is cyclic as an S-poset (in particular,has a left identity element), then C^{pd} is additive.*

Proof. Let A and C be down closed sub S-posets of B and $bS \subseteq A\bigcup C$, for $b \in B$. Then there exist right ideals I and J of S such that $bI \subseteq A$ and $bJ \subseteq C$. Since S is cyclic as an S-poset, one can easily seen $bS \subseteq A$ or $bS \subseteq C$.

Now we show that some properties of the closure operator C^{pd} are not satisfied in general.

Lemma 2.1 *The closure operator C^{pd} is not necessarily fully additive.*

Proof. Let $S = (\mathbb{N}, min)$, $B = \mathbb{N}^\infty$ and $A = \mathbb{N}$ all endowd with ordinary relation on $\mathbb{N}$ as posets. Consider $A_n = \{m \in \mathbb{N} \mid m \leq n\}$ for each $n \in \mathbb{N}$. It is easy to check that $C_{\mathbb{N}^\infty}^{pd}(A_n) = A_n$ and hence $\bigcup C_{\mathbb{N}^\infty}^{pd}(A_n) = \bigcup(A_n) = \mathbb{N}$, but $C_{\mathbb{N}^\infty}^{pd}(\bigcup A_n) = C_{\mathbb{N}^\infty}^{pd}(\mathbb{N}) = \mathbb{N}^\infty$.

Lemma 2.2 *For every semigroup S, the closure operator C^{pd} is not discrete nor trivial nor minimal.*

Proof. Let $0 \in A$ be a fixed element of a nonempty S-poset A. Adjoin two elements θ, ω to A by the actions $\omega s = \omega$ and $\theta s = 0$ and $a < \theta < \omega$, for each $a \in A$. Consider $B = A\bigcup\{\theta, \omega\}$. It is clear that $C_B^{pd}(A) = A\bigcup\{\theta\}$. This shows that C^{pd} is neither discrete nor trivial. Also, it is not minimal, because of, adjoining two elements θ, ω to a nonempty S-poset C by the actions $\omega s = \theta$ and orders $\theta s = \theta$, and $c < \theta < \omega$, for each $c \in C$. Taking $A = C\bigcup\{\theta\}$, $B = C\bigcup\{\theta, \omega\}$, we get $C \subset A \subset B$, and $C_B^{pd}(A) = B$ while $C_B^{pd}(C) = C$.

Theorem 2.4 (i) *The closure C^{pd} is discrete if and only if S has a left identity element and every sub S-poset is down closed.*

(ii) *The closure C^{pd} is trivial if and only if S is the emptyset.*

Proof. (i) Let C^{pd} be a discrete closure operator and the nonempty semigroup S do not have a left identity. Consider $t_0 \in S$ and adjoin an element x to S defined by $xs = t_0 s$ for each $s \in S$. It is clear that $C_{S^x}^{pd}(S) = S^x$ and by the hypothesis we have $C_{S^x}^{pd}(S) = S$. So $S^x = S$ which is a contradiction. Now let A be a sub S-poset of B. It is clear that $C_{A\downarrow}^{pd}(A) = A\downarrow$ and since C^{pd} is discrete, $C_{A\downarrow}^{pd}(A) = A$. So $A = A\downarrow$, which completes the proof. The converse is obvious.

(ii) Let The closure operator C^{pd} is trivial and $S \neq \emptyset$. Consider $B = \{a, b\}$ is an S-poset, whose elements are fixed element and $a < b$ and $A = \{a\}$ is a proper sub S-poset of B. Then $C_B^{pd}(A) = A \neq B$ which is a contradiction. Thus S is the emptyset.

Conversly, let $S = \emptyset$. Then it is clear that $C_B^{pd}(A) = B$.

3 Categorical properties of po-s-dense monomorphisms

In this section we investigate the categorical and algebraic properties of the category **Pos-S** with respect to the class $\mathcal{M}_{pd}$ of po-s-dense monomorphisms in the following three subsections.

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