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Po-S-Dense Monomorphism

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Abstract

In this paper we take $\mathcal{A}$ to be the category **Pos-S** of S-posets, for a posemigroup S, $\mathcal{M}_{pd}$ to be the class of partially ordered sequantially-dense monomorphisms and study the categorical properties, such as limits and colimits, of this class. These properties are usually needed to study the homological notions, such as injectivity, of S-posets. Also we show that it is actually equivalent to C^{pd}-density resulting from a closure operator.

Keywords : Po-S-Dense; Semigroup; Limit; Colimit.

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1 Introduction

Throughout this paper S denotes a nonempty posemigroup and $\mathcal{M}_{pd}$ stands for the class of po-s-dense monomorphisms of S-posets. To study mathematical notions in a category $\mathcal{A}$, such as injectivity, tensor products, flatness, with respect to a class $\mathcal{M}$ of its (mono)morphisms, one should know some of the categorical properties of the pair ($\mathcal{A}$, $\mathcal{M}$). In this paper we take $\mathcal{A}$ to be the category **Pos-S** and $\mathcal{M}_{pd}$ to be a particular interesting class of monomorphisms, to be called *partially ordered-s-dense(po-s-dense)* monomorphisms, and investigate its categorical properties.

A study of S-posets from a category-theoretic standpoint forms the content of [8], and extends the results found in [6]. For more information on various properties of S-posets, see also [5].

In the rest of this section we give some preliminaries about S-acts, posets, and S-posets needed in the sequel.

Let S be a semigroup. Recall that a (*right*) S-*act* is a set A equipped with a map $\lambda : A\times S \to A$, called its *action*, such that, denoting $\lambda(a, s)$ by as, we have $a(st) = (as)t$, for all $a \in A$, $s, t \in S$ and, if S is a monoid with the identity element 1, $a1 = a$. The category of all S-acts, with action-preserving maps between them, is denoted by **Act**-S. An S-*act congruence* θ on A is an equivalence relation with the property that $a\theta a'$, $a, a' \in A$, implies that $as\theta a's$, for all $s \in S$. A quotient S-act is the set A/θ with the natural action, $[a]s = [as]$, which makes the canonical map $\gamma : A \to A/\theta$, $a \mapsto [a]$, an S-act map. For more information about S-acts, see [10].

A semigroup S is said to be a *posemigroup* if it is also a poset whose partial order is compatible with the binary operation.

For a posemigroup S, a (*right*) S-*poset* is a poset A which is also an S-act whose action is monotone in both arguments. An S-*poset map* (*morphism*) is an action preserving monotone map between S-posets. Note that each poset P can be made into an S-poset with trivial action: $ps = p$, for every $p \in P, s \in S$.

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Let A be an S-poset. An *S-poset congruence* on A is an S-act congruence θ with the property that the S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map. For a binary relation R on A, define the relation $\leq_R$ on A by

$$a \leq_R a' \text{ if and only if}$$

$$a \leq a_1 R a'_1 \leq ... \leq a_n R a'_n \leq a'$$

for some $a_1, a'_1, ..., a_n, a'_n \in A$. Then an S-act congruence θ on A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by $[a] \leq [b]$ if and only if $a \leq_\theta b$. Also the S-poset congruence $\theta(H)$ on A generated by $H \subseteq A \times A$ can be characterized as follows:

$a\theta(H)a'$ if and only if $a = a'$, or there exist $s_1, s_2, ..., s_n, t_1, t_2, ..., t_m \in S^1$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, ..., s_nd_n \leq a';$$
$$a' \leq t_1p_1, t_1q_1 \leq t_2p_2, t_2q_2 \leq t_3p_3, ..., t_mq_m \leq a,$$

where $(c_i, d_i), (p_j, q_j) \in H \bigcup H^{-1}$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$.

Moreover, the order relation on $A/\theta(H)$ can be defined by: $[a] \leq [a']$ if and only if $a \leq a'$, or there exist $s_1, s_2, ..., s_n \in S^1$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, ..., s_nd_n \leq a',$$

where $(c_i, d_i) \in H \bigcup H^{-1}$ for $i = 1, 2, ..., n$.

Recall that the *product* of a family of S-posets is their cartesian product, with componentwise action and order. The *coproduct* is their disjoint union, with natural action and componentwise order. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. Also for a family $(A_\alpha)_{\alpha\in I}$ of S-posets each with a unique fixed element 0, the *direct sum* $\bigoplus A_\alpha$ is defined to be the sub S-poset of the product $\prod A_\alpha$ consisting of all $(a_\alpha)_{\alpha\in I}$ such that $a_\alpha = 0$ for all $\alpha \in I$ except a finite number of indices.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \overset{g}{\to} & B \end{array}$$

in **Pos-S** is the sub S-poset $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps. Notice that for the case where g is an inclusion, P can be taken as $f^{-1}(C)$.

All colimits in **Pos-S** exist and are calculated as in **Set** with the natural action of S on them. In particular, $\emptyset$ with the empty action of S on it, is the initial object of **Pos-S**. Also, the *coproduct* of S-posets A, B is their disjoint union $A \sqcup B = (A\times\{1\})\bigcup(B\times\{2\})$ with the obvious action, and coproduct injections are defined naturally.

The pushout of a given diagram

$$\begin{array}{ccc} A & \overset{g}{\to} & C \\ f\downarrow & & \\ B & & \end{array}$$

in **Pos-S** is the factor act $Q = (B \sqcup C)/\theta$ where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $q_1 = \pi u_C : C \to (B \sqcup C)/\theta$, $q_2 = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Pos-S** are constructed analogously.

Let $\mathbf{I}$ be a small category and $\mathcal{A} : \mathbf{I} \to$ **Pos-S** be a diagram in **Pos-S** determining the acts A_α, for $\alpha \in I = Obj\mathbf{I}$, and S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\alpha \to \beta$ in $Mor\mathbf{I}$. Recall that the limit of this diagram is $\underleftarrow{lim}A_\alpha := \bigcap_{\alpha\in I} E_\alpha$, where $E_\alpha = \{a = (a_\alpha)_{\alpha\in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta}p_\alpha(a) = p_\beta(a)\}$ and p_α, p_β are the α, βth projection maps of the product. The limit S-maps are $q_\alpha : \underleftarrow{lim}A_\alpha \to A_\alpha$. Also the limit has the universal property which is, if $\{f_\alpha : A \to A_\alpha\}$ is a family of morphisms such that $g_{\alpha\beta}f_\alpha(a) = f_\beta(a)$, then there is a morphism $f : A \to \underleftarrow{lim}A_\alpha$ such that $q_\alpha f = f_\alpha$.

Remind that a directed system of S-posets and S-maps is a family $(B_\alpha)_{\alpha\in I}$ of S-posets indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha\leq\beta\in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma}g_{\alpha\beta} = g_{\alpha\gamma}$, also $g_{\alpha\alpha} = id$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta\in I})$ in **Pos-S** is given as $\underrightarrow{lim}B_\alpha = \coprod_\alpha B_\alpha/\rho$ where the congruence ρ is given by $b_\alpha\rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$, in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \underrightarrow{lim}B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \underrightarrow{lim}B_\alpha$

is the natural S-map. Also directed colimit has a dual universal property of limit.

2 C^{pd}-Closure operator

In this section, we introduce and briefly study a closure operator, so called C^{pd}-*Closure operator*. For a sub S-poset A of B let us denote $A\downarrow=\{b\in B \mid \exists a\in A, b\leq a\}$ and $Sub(B)$, the set of all sub S-posets of B. First recall the following definition of C^{pd}-closure operator.

Definition 2.1 A family $C^{pd}=(C^{pd}_B)_{B\in\mathbf{Pos-S}}$, with $C^{pd}_B: sub(B)\to Sub(B)$, is defined as

$$C^{pd}_B(A)=\{b\in B : bS\subseteq A\downarrow\}.$$

It is easy to show that C^{pd} is a closure operator on **Pos-S** in the sense of [7]. This means that $C^{pd}_B(A)$ is a sub S-poset of B and,

(i) $A\subseteq C^{pd}_B(A)$,

(ii) $A_1\subseteq A_2\subseteq B$ implies $C^{pd}_B(A_1)\subseteq C^{pd}_B(A_2)$,

(iii) for every homomorphism $f:B\to D$ and each sub S-poset A of B, $f(C^{pd}_B(A))\subseteq C^{pd}_D(f(A))$.

We just prove (iii). Let $f:B\to C$ be a homomorphism and $b\in C^{pd}_B(A)$. For every $s\in S$, there exists $a\in A$ such that $bs\leq a$. Then $f(b)s=f(bs)\leq f(a)\in f(A)$ and hence $f(b)S\subseteq f(A)\downarrow$ which deduced that $f(b)\in C^{pd}_B(f(A))$.

Dikranjan and Tholen in [7] state some properties of a closure operator in general. Here we are going to investigate the satisfaction of those properties for the closure operator C^{pd}.

Definition 2.2 *The closure operator C^{pd} is said to be:*

(1) *idempotent(if $C^{pd}_B(A)=C^{pd}_B(C^{pd}_B(A))$).*

(2) *hereditary(if for $A_1\subseteq A_2\subseteq B$, $C^{pd}_{A_2}(A_1)=C^{pd}_B(A_1)\bigcap A_2$).*

(3) *weakly hereditary(if for every $A\subseteq B$, $C^{pd}_{C^{pd}_B(A)}(A)=C^{pd}_B(A)$).*

(4) *grounded(if $C^{pd}_B(\emptyset)=\emptyset$).*

(5) *additive(if $C^{pd}_B(A\bigcup C)=C^{pd}_B(A)\bigcup C^{pd}_B(C)$).*

(6) *productive(if for every family of sub S-posets A_i of B_i, taking $A=\prod_i A_i$ and $B=\prod_i B_i$, $C^{pd}_B(A)=\prod_i C^{pd}_{B_i}(A_i)$).*

(7) *fully additive(if $C^{pd}_B(\bigcup_{i\in I}A_i)=\bigcup_{i\in I}C^{pd}_B(A_i)$).*

(8) *discrete(if $C^{pd}_B(A)=A$ for every S-poset B and $A\subseteq B$).*

(9) *trivial(if $C^{pd}_B(A)=B$ for every B and $A\subseteq B$).*

(10) *minimal(if for $C\subseteq A\subseteq B$ one has $C^{pd}_B(A)=A\bigcup C^{pd}_B(C)$).*

Theorem 2.1 *The closure operator C^{pd} is hereditary, weakly hereditary, grounded and productive.*

Proof. It is easy to check that the closure operator C^{pd} is hereditary, weakly hereditary and grounded. We just prove productivity. Let $b\in C^{pd}_B(A)$, $b=\{b_i\}$. For every $s\in S, bs\in A\downarrow$, then for each $i\in I$, $b_is\in A_i\downarrow$ and hence for each $i\in I, b_i\in C^{pd}_{B_i}(A_i)$ which implies that $b\in\prod C^{pd}_{B_i}(A_i)$. The converse is obvious.

Theorem 2.2 The closure operator C^{pd} is idempotent if and only if $S\subseteq S^2\downarrow$.

Proof. ($\Rightarrow$) It is clear that $C^{pd}_{S^1}(S)=S^1$ and $S\subseteq C^{pd}_{S^1}(S^2)$. Since C^{pd} is idempotent, $S^1=C^{pd}_{S^1}(S)\subseteq C^{pd}_{S^1}(C^{pd}_{S^1}(S^2))=C^{pd}_{S^1}(S^2)\subseteq S^1$. Thus $1\in C^{pd}_{S^1}(S^2)$, which implies $S\subseteq S^2\downarrow$.

($\Leftarrow$) By definition of the closure operator, for each $A\subseteq B$ we see that $C^{pd}_B(A)\subseteq C^{pd}_B(C^{pd}_B(A))$. Conversely, let $b\in C^{pd}_B(C^{pd}_B(A))$. So $bS\subseteq C^{pd}_B(A)\downarrow$. Thus for each $s\in S$, $bs\leq b'_s$ for some $b'_s\in C^{pd}_B(A)$. Since $S\subseteq S^2\downarrow$, then $s\leq tt'\in S^2$ and hence $bs\leq btt'\leq b'_tt'$, which $b'_t\in C^{pd}_B(A)$. Thus $b'_tt'\leq a$, for some $a\in A$. Therefore $bs\leq a$ and hence $bs\subseteq A\downarrow$ which implies that $b\in C^{pd}_B(A)$.

Note that the condition $S\subseteq S^2\downarrow$ is equal to $S=S^2\downarrow$.

In the following theorem let us denote, $DSub(B)$, the set of all down closed sub S-posets of an S-poset B.

Theorem 2.3 *The closure operator C^{pd} is additive if and only if for every element b in an S-poset B, bS is join prime in the lattice $DSub(B)$.*

Proof. let C^{pd} be additive. Let $x \in B$ and $xS \subseteq A \bigcup C$, where A and C are down closed sub S-posets of B. Then, by monotonicity and additivity,

$$C_B^{pd}(xS) \subseteq C_B^{pd}(A \bigcup C) = C_B^{pd}(A) \bigcup C_B^{pd}(C).$$

Now, since $x \in C_B^{pd}(xS)$, $x \in C_B^{pd}(A)$ or $x \in C_B^{pd}(C)$. Thus, $xS \subseteq A \downarrow = A$ or $xS \subseteq C \downarrow = C$, proving that xS is join prime in $Sub(B)$.

Conversely, suppose that A and D are sub S-posets of an S-poset B. By definition of the closure operator, $C_B^{pd}(A) \bigcup C_B^{pd}(C) \subseteq C_B^{pd}(A \bigcup C)$. Consider $x \in C_B^{pd}(A \bigcup C)$. So $xS \subseteq (A \bigcup C) \downarrow = (A \downarrow) \bigcup (B \downarrow)$. Since xS is join prime, $xS \subseteq A \downarrow$ or $xS \subseteq C \downarrow$. Thus, $x \in C_B^{pd}(A) \bigcup C_B^{pd}(C)$. This shows that each C_B^{pd}, and hence C^{pd}, is additive.

Corollary 2.1 *If S is cyclic as an S-poset (in particular,has a left identity element), then C^{pd} is additive.*

Proof. Let A and C be down closed sub S-posets of B and $bS \subseteq A \bigcup C$, for $b \in B$. Then there exist right ideals I and J of S such that $bI \subseteq A$ and $bJ \subseteq C$. Since S is cyclic as an S-poset, one can easily seen $bS \subseteq A$ or $bS \subseteq C$.

Now we show that some properties of the closure operator C^{pd} are not satisfied in general.

Lemma 2.1 *The closure operator C^{pd} is not necessarily fully additive.*

Proof. Let $S = (\mathbb{N}, min)$, $B = \mathbb{N}^\infty$ and $A = \mathbb{N}$ all endowd with ordinary relation on $\mathbb{N}$ as posets. Consider $A_n = \{m \in \mathbb{N} \mid m \leq n\}$ for each $n \in \mathbb{N}$. It is easy to check that $C_{\mathbb{N}^\infty}^{pd}(A_n) = A_n$ and hence $\bigcup C_{\mathbb{N}^\infty}^{pd}(A_n) = \bigcup(A_n) = \mathbb{N}$, but $C_{\mathbb{N}^\infty}^{pd}(\bigcup A_n) = C_{\mathbb{N}^\infty}^{pd}(\mathbb{N}) = \mathbb{N}^\infty$.

Lemma 2.2 *For every semigroup S, the closure operator C^{pd} is not discrete nor trivial nor minimal.*

Proof. Let $0 \in A$ be a fixed element of a nonempty S-poset A. Adjoin two elements θ, ω to A by the actions $\omega s = \omega$ and $\theta s = 0$ and $a < \theta < \omega$, for each $a \in A$. Consider $B = A \bigcup \{\theta, \omega\}$. It is clear that $C_B^{pd}(A) = A \bigcup \{\theta\}$. This shows that C^{pd} is neither discrete nor trivial. Also, it is not minimal, because of, adjoining two elements θ, ω to a nonempty S-poset C by the actions $\omega s = \theta$ and orders $\theta s = \theta$, and $c < \theta < \omega$, for each $c \in C$. Taking $A = C \bigcup \{\theta\}$, $B = C \bigcup \{\theta, \omega\}$, we get $C \subset A \subset B$, and $C_B^{pd}(A) = B$ while $C_B^{pd}(C) = C$.

Theorem 2.4 (i) *The closure C^{pd} is discrete if and only if S has a left identity element and every sub S-poset is down closed.*

(ii) *The closure C^{pd} is trivial if and only if S is the emptyset.*

Proof. (i) Let C^{pd} be a discrete closure operator and the nonempty semigroup S do not have a left identity. Consider $t_0 \in S$ and adjoin an element x to S defined by $xs = t_0 s$ for each $s \in S$. It is clear that $C_{S^x}^{pd}(S) = S^x$ and by the hypothesis we have $C_{S^x}^{pd}(S) = S$. So $S^x = S$ which is a contradiction. Now let A be a sub S-poset of B. It is clear that $C_{A\downarrow}^{pd}(A) = A \downarrow$ and since C^{pd} is discrete, $C_{A\downarrow}^{pd}(A) = A$. So $A = A \downarrow$, which completes the proof. The converse is obvious.

(ii) Let The closure operator C^{pd} is trivial and $S \neq \emptyset$. Consider $B = \{a, b\}$ is an S-poset, whose elements are fixed element and $a < b$ and $A = \{a\}$ is a proper sub S-poset of B. Then $C_B^{pd}(A) = A \neq B$ which is a contradiction. Thus S is the emptyset.

Conversly, let $S = \emptyset$. Then it is clear that $C_B^{pd}(A) = B$.

3 Categorical properties of po-s-dense monomorphisms

In this section we investigate the categorical and algebraic properties of the category **Pos-S** with respect to the class $\mathcal{M}_{pd}$ of po-s-dense monomorphisms in the following three subsections.

3.1 Composition Property

In this subsection we investigate some properties of the class $\mathcal{M}_{pd}$ of po-s-dense monomorphisms which are mostly related to the composition of po-s-dense monomorphisms. These properties and the ones given in the next two subsections are what normally used to study injectivity with respect to a class of monomorphisms, see [2].

The class $\mathcal{M}_{pd}$ is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.

Definition 3.1 *An S-poset A is said to be partially ordered-s-dense(or simply po-s-dense) sub S-poset of B, if for every $b \in B, bS \subseteq A \downarrow$. In other word $C_B^{pd}(A) = B$. A monomorphism $f : A \to B$ is called po-s-dense if $f(A)$ is a po-s-dense sub S-poset of B.*

The proof of the next proposition is straightforward and is omitted.

Proposition 3.1 *Let $A \overset{f}{\to} B \overset{g}{\to} C$ be two monomorphisms and gf be a po-s-dense monomorphism. Then both f and g are po-s-dense monomorphisms.*

The class $\mathcal{M}_{pd}$ is said to be composition closed, if the composition of two po-s-dense monomorphisms is also a po-s-dense monomorphism. The following lemma shows that the class $\mathcal{M}_{pd}$ is not always closed under composition.

Lemma 3.1 *The class $\mathcal{M}_{pd}$ is closed under composition if and only if the closure operator C^{pd} is idempotent.*

Proof. Suppose that the closure operator C^{pd} is idempotent and $f : A \to B$ and $g : B \to C$ are two po-s-dense monomorphisms. So we have $C = C_C^{pd}(g(B)) = C_C^{pd}(g(C_B^{pd}(f(A))) \subseteq C_C^{pd}((C_C^{pd}(gf(A))) = C_C^{pd}(gf(A)) \subseteq C$. Thus $C_C^{pd}(gf(A)) = C$, means that gf is po-s-dense monomorphism.
For the converse, let the composition of po-s-dense monomorphisms be po-s-dense monomorphism. For every sub s-poset A of B, A is po-s-dense sub S-poset of $C_B^{pd}(A)$, in view of Theorem 2.1. Thus $A \to C_B^{pd}(A) \to C_B^{pd}(C_B^{pd}(A))$ are po-s-dense monomorphisms. So A is po-s-dense sub S-poset of $C_B^{pd}(C_B^{pd}(A))$ and hence $C^{pd}_{C_B^{pd}(C_B^{pd}(A))}(A) = C_B^{pd}(C_B^{pd}(A))$. Now by Theorem 2.1, since C^{pd} is hereditary, $C^{pd}_{C_B^{pd}(C_B^{pd}(A))}(A) = C_B^{pd}(A)$. So C^{pd} is idempotent.

As a clear and important deduction of Theorem 2.2 and Lemma 3.1, we have the following corollary.

Corollary 3.1 *The class $\mathcal{M}_{pd}$ is composition closed if and only if $S \subseteq S^2 \downarrow$.*

3.2 Limits of po-s-dense monomorphisms

In this subsection we will investigate the behaviour of po-s-dense monomorphisms with respect to limits. First recall that, the class $\mathcal{M}_{pd}$ is said to be closed under products(coproduct, direct sum), if for every family of po-s-dense monomorphisms $\{f_i : A_i \to B_i\}$, $\prod f_i : \prod A_i \to \prod B_i (\coprod f_i, \oplus f_i)$ is po-s-dense monomorphism.

Proposition 3.2 (i) *The class $\mathcal{M}_{pd}$ is closed under products.*

(ii) *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a family of po-s-dense monomorphisms and A be a complete upward directed S-poset. Then their product homomorphism $f : A \to \prod_{\alpha \in I} B_\alpha$ is also an po-s-dense monomorphism.*

Proof. (i) The proof is straightforward.

(ii) Let $\{b_i\} \in \prod_{i \in I} B_i$ and $s \in S$. For each $i \in I$, there exists $a_i \in A$ such that $b_i s \leq f_i(a_i)$. Since A is a complete upward directed set, there is an element $a \in A$, such that $a_i \leq a$. So for every $i \in I$, $b_i s \leq f_i(a)$ and hence $\{b_i\}s \leq f(a)$.

Proposition 3.3 *The class $\mathcal{M}_{pd}$ is closed under direct sums.*

Theorem 3.1 *Consuder the following pullback diagram*

$$\begin{array}{ccc} f^{-1}(C) & \overset{\tau}{\hookrightarrow} & A \\ \bar{f}\downarrow & & \downarrow f \\ C & \overset{\iota}{\hookrightarrow} & B \end{array}$$

in which C is down closed S-poset, ι is inclusion and $C \subseteq f(A)\downarrow$. If C is po-s-dense in B, then $\bar{f}$ and τ are po-s-dense monomorphisms.

Proof. We have to show that $Im(\bar{f})$ is po-s-dense in B. Let $b \in B$ and $s \in S$. Since C is down closed and po-s-dense in B, there exists $c \in C$ such that $bs \in C$ and hence there exists $a' \in A$ such that $bs \leq f(a')$. Thus $\bar{f}(a') = \iota\bar{f}(a') = f(a') \geq bs$, which it is deduced that $bs \in Im(\bar{f})\downarrow$. Now let $a \in A$ and $s \in S$. So $f(as) = f(a)s \in C\downarrow = C$, which implies $as \in f^{-1}(C) \subseteq f^{-1}(C)\downarrow$.

Remark 3.1 *Pullbacks does not transfer po-s-dense monomorphisms. Let S be a semigroup and S_x be the S-poset obtained by adjoining a fixed element x to S, with $x \leq s$ for each $s \in S$. Consider the coproduct $S \coprod S$ as an S-poset defined by $(s_1, i) \leq (s_2, j)$ if and only if $i = j$ and $s_1 \leq s_2$ in S. The following diagram*

$$\begin{array}{ccc} S \times \{1\} & \stackrel{\tau}{\hookrightarrow} & S \coprod S \\ \downarrow & & \downarrow f \\ S & \stackrel{\gamma}{\hookrightarrow} & S_x \end{array}$$

, which γ is an inclusion map and f is a homomorphism defined by $f(s, 1) = s$ and $f(s, 2) = x$, is a pullback diagram. It is clear that γ is po-s-dense, but τ is not po-s-dense monomorphism.

3.3 Colimits of po-s-dense monomorphisms

This subsection is devoted to the study of po-s-dense monomorphisms with respect to colimits.

Proposition 3.4 *$\mathcal{M}_{pd}$ is closed under coproducts.*

Proof. Consider the diagram

$$\begin{array}{ccc} A_i & \stackrel{f_i}{\to} & B_i \\ u_i \downarrow & & \downarrow u'_i \\ \coprod_{i \in I} A_i & \stackrel{f}{\to} & \coprod_{i \in I} B_i \end{array}$$

in which $\{f_i : A_i \to B_i : i \in I\}$ is a family of po-s-dense monomorphisms. We have to show that $f : \coprod_{i \in I} A_i \to \coprod_{i \in I} B_i$ is an po-s-dense monomorphism. Since u'_i and f_i, $i \in I$, are monomorphisms, f is a monomorphism too. Let $b \in \coprod_{i \in I} B_i$, $s \in S$. Then there exists $i \in I$, $b_i \in B_i$ such that $b = u'_i(b_i)$. Since f_i is po-s-dense, there exists $a_i \in A_i$ with $b_i s \leq f_i(a_i)$. So $bs \leq u'_i f_i(a_i) = f u_i(a_i) \in Imf$. Thus f is a po-s-dense monomorphism.

A monomorphism $f : A \to B$ is said to be regular monomorphisms (order-embeddings) in the category **Pos-S** of S-posets, if it is action-preserving monotone map.

Theorem 3.2 *Pushouts transfers po-s-dense monomorphisms, that is, for the following pushout diagram*

$$\begin{array}{ccc} A & \stackrel{f}{\to} & B \\ g \downarrow & & \downarrow h' \\ C & \stackrel{h}{\to} & Q \end{array}$$

in **Pos-S**, *if f is po-s-dense then h is so.*

Proof. Recall that $Q = (B \sqcup C)/\theta$ where $\theta = \rho(H)$ and H consists of all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C$, $u_C : C \to B \sqcup C$ are coproduct injections. And $h = \pi u_C : C \to (B \sqcup C)/\theta$, $h' = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. By [11], pushout transfer regular monomorphism. So h is a monomorphism. We show that h is po-s-dense monomorphism. Let $[x]_\theta \in (B \sqcup C)/\theta$ and $s \in S$. Then, $x = u_C(c)$ for some $c \in C$, or $x = u_B(b)$ for some $b \in B$. In the former case, we have $[x]_\theta s = h(c)s = h(cs) \in Imh$. In the latter case, using that f is po-s-dense, we get $a \in A$ with $bs \leq f(a)$ and hence $[x]_\theta s = [u_B(b)]_\theta s = h'(b)s = h'(bs) \leq h' f(a) = hg(a) \in Imh$. So $[x]_\theta s \in (Imh) \downarrow$.

We say that *multiple pushouts* transfer po-s-dense monomorphisms if in multiple pushout $(P, A_\alpha \stackrel{h_\alpha}{\to} P)$ of a family of po-s-dense monomorphisms $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$, every h_α, $\alpha \in I$, is a po-s-dense monomorphism. In multiple pushout diagram for every $\alpha, \beta \in I$, $h_\beta f_\beta = h_\alpha f_\alpha$ which is called diagonal map.

Theorem 3.3 *Multiple pushouts transfer po-s-dense monomorphisms.*

Proof. Let $(P, A_\alpha \stackrel{h_\alpha}{\to} P)$ be a multiple pushout of the family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of po-s-dense monomorphisms. We know that $P = \coprod A_\alpha / \rho(H)$ where $H = \{(f_\alpha(a), f_\beta(a)) \mid a \in A, \alpha, \beta \in I\}$ (we have taken the image of each element of A_α under coproduct morphisms to be equal to itself). By using [3, Th, 3.5], for every $\alpha \in I$, h_α is a monomorphism. Now let $q \in P$ and $s \in S$. There exist $\beta \in I$ and $p \in A_\beta$ such that $q = h_\beta(p)$. Since f_β is po-s-dense then $ps \leq f_\beta(a)$, for some $a \in A$, and hence $qs = h_\beta(ps) \leq h_\beta(f_\beta(a)) = h_\alpha(f_\alpha(a))$. Thus h_α is po-s-dense.

The following corollary immediately obtained from Corollary 3.1 and Theorem 3.3.

Corollary 3.2 *If $S \subseteq S^2 \downarrow$, then in every multiple pushout diagram of po-s-dense regular monomorphisms the diagonal map is an po-s-dense regular monomorphism.*

Theorem 3.4 *Let $\{h_\alpha : A_\alpha \to B_\alpha | \ \alpha \in I\}$ be a directed family of po-s-dense monomorphisms. Then, the directed colimit homomorphism induced by $h : \varinjlim A_\alpha \to \varinjlim B_\alpha$ is po-s-dense.*

Proof. Let $(\varinjlim A_\alpha, f_\alpha), (\varinjlim B_\alpha, g_\alpha)$ be directed colimits of the directed systems $((A_\alpha), (\psi_{\alpha\beta}))_{\alpha\leq\beta\in I}$ and $((B_\alpha), (\varphi_{\alpha\beta}))_{\alpha\leq\beta\in I}$ and suppose $\{h_\alpha : A_\alpha \to B_\alpha | \ \alpha \in I\}$ is a directed family of po-s-dense monomorphisms such that for every $\alpha \leq \beta, h_\beta\psi_{\alpha\beta} = \varphi_{\alpha\beta}h_\alpha$. Then, for every $\alpha \leq \beta$, $g_\beta h_\beta \psi_{\alpha\beta} = g_\beta\varphi_{\alpha\beta}h_\alpha = g_\alpha h_\alpha$, so $h = \varinjlim h_\alpha$ exists by the universal property of colimits. Consider $\varinjlim A_\alpha = \coprod_{\alpha\in I} A_\alpha/\rho$ and $\varinjlim B_\alpha = \coprod_{\alpha\in I} B_\alpha/\rho'$ as defined in section 1. Let $h[a_\alpha]_\rho = h[a_\beta]_\rho$. Then, $[h_\alpha(a_\alpha)]_{\rho'} = g_\alpha h_\alpha(a_\alpha) = g_\beta h_\beta(a_\beta) = [h_\beta(a_\beta)]_{\rho'}$, and so there exists $\gamma \in I$ with $\gamma \geq \alpha, \beta$ and $\varphi_{\alpha\gamma}h_\alpha(a_\alpha) = \varphi_{\beta\gamma}h_\beta(a_\beta)$ which implies that $h_\gamma\psi_{\alpha\gamma}(a_\alpha) = h_\gamma\psi_{\beta\gamma}(a_\beta)$. Since h_γ is a monomorphism, $[a_\alpha]_\rho = [a_\beta]_\rho$, and so h is a monomorphism. To see that f is po-s-dense, let $s \in S$, $x \in \varinjlim B_\alpha$. So for some $\alpha \in I$, $x = g_\alpha(b_\alpha)$. Since f_α is po-s-dense, there exists $a \in A_\alpha$ with $b_\alpha s \leq h_\alpha(a)$. Then, $xs = g_\alpha(b_\alpha s) \leq g_\alpha h_\alpha(a) = hf_\alpha(a)$.

Theorem 3.5 *The category* **Pos-S** *has $\mathcal{M}_{pd}$-directed colimits.*

Proof. Suppose that $(\varinjlim B_\alpha, g_\alpha)$ is the directed colimit of the directed system $((B_\alpha), (g_{\alpha\beta}))_{\alpha\leq\beta\in I}$, and $\{h_\alpha : A \to B_\alpha \mid \alpha \in I\}$ is a directed family of po-s-dense monomorphisms such that $g_{\alpha\beta}h_\alpha = h_\beta$, for each $\alpha \leq \beta$. Let $h : A \to \varinjlim B_\alpha$ be a directed colimit of $\{h_\alpha\}_{\alpha\in I}$ in **Pos-S**, with the colimit maps $g_\alpha : B_\alpha \to \varinjlim B_\alpha$. Since $h = \varinjlim h_\alpha = g_\alpha h_\alpha$ for each $\alpha \in I$, similar to the argument of Theorem 3.4, h is a monomorphism because of each h_α. Now we show that h is po-s-dense. Let $b \in \varinjlim B_\alpha$ and $s \in S$. Since $b \in \varinjlim B_\alpha$, there exists $x_\beta \in B_\beta$ such that $b = [x_\beta]_\rho$ and since h_β is po-s-dense, there exists an element $a_s \in A$ with $x_\beta s \leq h_\beta(a_s)$. Then $bs = [x_\beta]_\rho s = g_\beta(x_\beta)s = g_\beta(x_\beta s) \leq g_\beta h_\beta(a_s) = h(a_s)$.

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