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Research Article

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Solving Some Initial-Boundary Value Problems Including Non-classical Cases of Heat Equation By Spectral and Countour Integral Methods

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Abstract

In this paper, we consider some initial-boundary value problems which contain one-dimensional heat equation in non-classical case. For this problem, we can not use the classical methods such as Fourier, Laplace transformation and Fourier-Birkhoff methods. Because the eigenvalues of their spectral problems are not strictly and they are repeated or we have no eigenvalue. The presentation of the solution and also satisfying the solution in the given P.D.E and satisfing the given initial and boundary conditions are established by complex analysis theory and Countour integral method.

Keywords : Initial-Boundary Value Problem; Laplace Line; Countor Integral; Heat Equation.

1 Introduction

There are several methods to solve the spectral problems which resulted from initial-boundary value problems. As we know from text books, if the operator of spectral problem is self adjoint, then the eigenvalues are real and distinct. Also, the related eigenfunctions are ortogonal and form a complete system of basis. In this case we can apply Fourier method (seperation of variables) [1]-[5]. If the related operator is not self adjoint, $L \neq L^{\star}$, consequently the eigenvalues are repeated (not simple), then we can not apply Fourier method. In this case, the eigenfunctions can not form a complete basis system and we should use Fourier-Birkhoff method which proposed to use the eigenfunctions of related adjoint operater $L^{\star}$. Because the eigenfunctions of L and the eigenfunctions of $L^{\star}$ are biortogonal [6, 7]. In this paper, we will consider some initial-boundary value problems which their related spectral problems are not self adjoint and their eigenvalues are repeated. Other non-classical case for heat equation $u_t = u_{xx}$, when the related spectral problem has no eigenvalue, and consequently we do not have any eigenfunction. For this, consider the following boundary conditions:

$$\begin{cases} u(0,t) - 2u(1,t) = 0 \\ u_x(x,t)|_{x=0} + 2u_x(x,t)|_{x=1} = 0 \end{cases}, t \geq 0 \tag{1.1}$$

and with initial condition $u(x,0) = \varphi(x)$. Then the related spectral problem is:

$$y'' - \lambda^2 y = 0,$$

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$$y(0) - 2y(1) = 0,$$
$$y'(0) + 2y'(1) = 0$$

For this problem we have no eigenvalue. Therefore for any $\lambda \in \mathbb{C}$ the solution is only trivial solution $y(x) \equiv 0$, hence we can not stablish the series solution.

Other non-classical case is when the eigenvalues of spectral problem fill the complex plane. In other word, any points of complex plane $\mathbb{C}$ is an eigenvalue. See the following conditions for heat equation:

$$\begin{cases} u(0,t) + u(1,t) = 0 \\ \\ u_x(x,t)|_{x=0} - u_x(x,t)|_{x=1} = 0 \end{cases} , t \geq 0 \tag{1.2}$$

and with initial condition $u(x,0) = \varphi(x)$.
Then the related spectral problem is:

$$y'' - \lambda^2 y = 0, \lambda \in \mathbb{C},$$
$$y(0) + y(1) = 0,$$
$$y'(0) - y'(1) = 0$$

For these cases, we can not use classical methods such as Fourier method and Fourier- Birkhoff method and Laplace transformation [10]-[12]. These cases can be considered as unsolved problems, see the final section of the paper.

We are going to consider the one-dimensional heat equation with some initial and boundary conditions. For this problem, at first its spectral problem is constructed. Then by countour integral method, an analytic solution will be given as integral form over a suitable countour. Finaly in section 3, we show that this analytic solution satisfies in the given differential equation and given boundary and initial conditions.

2 Main problem and its spectral problem

We consider the following problem for heat equation:

$$u_t = u_{xx}, \qquad x \in (0,1), \qquad t > 0 \tag{2.3}$$

with boundary conditions and initial condition:

$$\begin{cases} u(0,t) = u(1,t) \\ \\ u_x(x,t)|_{x=0} = 0 \end{cases} , t \geq 0 \tag{2.4}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{2.5}$$

At first we show that the eigenvalues of the spectral problem are repeated. For this, by Fourier method we will have the following spectral problem:

$$X''(x) - \lambda X(x) = 0, \qquad x \in (0,1) \tag{2.6}$$

$$\begin{cases} X(0) = X(1) \\ X'(0) = 0 \end{cases} \tag{2.7}$$

the general solution of equation (2.6) is:

$$X(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x} \tag{2.8}$$

Imposing boundary conditions (2.7) to this solution, yields following algebraic system:

$$\begin{cases} c_1 + c_2 - c_1 e^{-\sqrt{\lambda}} - c_2 e^{\sqrt{\lambda}} = 0 \\ -c_1\sqrt{\lambda} + c_2\sqrt{\lambda} = 0 \end{cases},$$

$$\begin{vmatrix} 1 - e^{-\sqrt{\lambda}} & 1 - e^{\sqrt{\lambda}} \\ -\sqrt{\lambda} & \sqrt{\lambda} \end{vmatrix} = -\sqrt{\lambda}e^{-\sqrt{\lambda}}(e^{\sqrt{\lambda}} - 1)^2 = 0$$

We consider the eigenvalue $\lambda = 0$ is simple and the eigenvalues $\lambda_k = -4k^2\pi^2, k \in \mathbb{N}$ are repeated two times. The eigenvalues and eigenfunctions are as follows:

$$\lambda_0 = 0, \qquad X_0(x) = 1,$$
$$\lambda_k = -4k^2\pi^2, X_k(x) = \frac{1}{\sqrt{2}} \cos 2k\pi x, k \in \mathbb{N}$$

Because of eigenvalues are repeated, the eigenfunctions can not establish a complete basis system. And we should use the generalized vectors as extra eigenfunctions. Therefore we can not continue the Fourier method, and we are going to apply Countour integral method. For this, we construct the spectral problem by making use of Laplace transform:

$$y''(x,\lambda) - \lambda y(x,\lambda) = \varphi(x), \qquad x \in (0,1) \tag{2.9}$$

$$\begin{cases} y(0,\lambda) = y(1,\lambda) \\ y'(0,\lambda) = 0 \end{cases} \tag{2.10}$$

Now, we are going to compute the general solution of non-homogeneous equation (2.9) by Lagrange method (Change variable method), that is :

$$y(x,\lambda) = c_1(x)e^{-\sqrt{\lambda}x} + c_2(x)e^{\sqrt{\lambda}x} \tag{2.11}$$

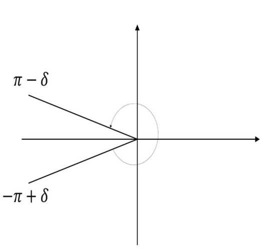
where $c_1(x)$ and $c_2(x)$ are equal to:

$$c_1(x) = c_1 - \int_{x_1}^{x} \frac{e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\varphi(\xi)d\xi,$$

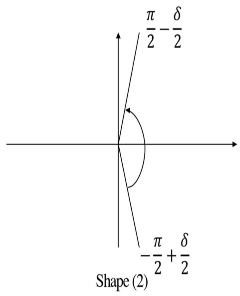
$$c_2(x) = c_2 + \int_{x_2}^{x} \frac{e^{-\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\varphi(\xi)d\xi$$

where c_1, c_2 are real arbitrary constants. Now, by considering the parameter λ in complex plane we suppose,

$\arg\lambda \in (-\pi+\delta, \pi-\delta), \delta > 0$, as is shown in shape 1. Regarding values for $\arg\sqrt{\lambda}$, and



Shape (1)



Shape (2)

supposing $x_1 = 0$ and $x_2 = 1$ for lower limits of above integrals, these integrals will be converge when $|\lambda| \to \infty$, therefore we have:

$$y(x,\lambda) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x} - \int_0^1 \frac{e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}}\varphi(\xi)d\xi \tag{2.12}$$

Now by imposing the boundary conditions (2.10) we have the following amount for c_1, c_2:

$c_1 = \frac{1}{B(\lambda)}$

$$\int_0^1 \frac{e^{-\sqrt{\lambda}\xi} - e^{-\sqrt{\lambda}(1-\xi)} - e^{-\sqrt{\lambda}\xi} + e^{\sqrt{\lambda}(1-\xi)}}{2\sqrt{\lambda}}\varphi(\xi)d\xi,$$

$c_2 = \frac{1}{B(\lambda)}$

$$\int_0^1 \frac{e^{-\sqrt{\lambda}\xi} - e^{-\sqrt{\lambda}(1+\xi)} + e^{-\sqrt{\lambda}\xi} - e^{-\sqrt{\lambda}(1-\xi)}}{2\sqrt{\lambda}}\varphi(\xi)d\xi$$

where $B(\lambda) = -e^{-\sqrt{\lambda}}(e^{\sqrt{\lambda}} - 1)^2$.

By substituting these values for c_1 and c_2 in $y(x,\lambda)$ we have:

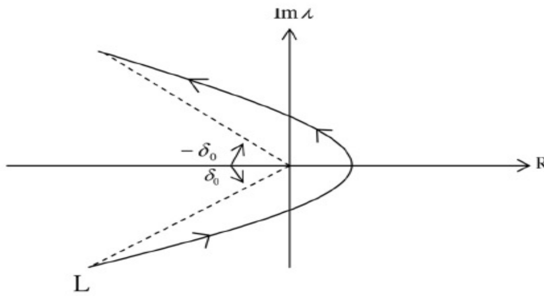
$y(x,\lambda) =$

$$\int_0^1 \frac{e^{\sqrt{\lambda}(x+\xi)} - 2e^{\sqrt{\lambda}(1+x-\xi)} + e^{\sqrt{\lambda}(x-\xi)}}{2\sqrt{\lambda}(e^{\sqrt{\lambda}}-1)^2}\varphi(\xi)d\xi \tag{2.13}$$

$$-\int_0^1 \frac{e^{\sqrt{\lambda}(2-x-\xi)} + e^{-\sqrt{\lambda}(x-\xi)}}{2\sqrt{\lambda}(e^{\sqrt{\lambda}}-1)^2}\varphi(\xi)d\xi$$

$$-\int_0^1 \frac{e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}}\varphi(\xi)d\xi$$

It is easy to see that the exponential expersions in $y(x,\lambda)$ tends to zero when $|\lambda| \to \infty$. Now, we consider the following countour in complex plane such that $\delta_0 > \delta > 0$. Line with in snatches shows the asymptotic line of countour L_ν [8, 9]. By using of inverse transformation of $y(x,\lambda)$ we



Shape (3)

get an analytic solution for the main problem:

$u(x,t) =$

$$-\frac{1}{2\pi i}\int_L e^{\lambda t} y(x,\lambda)d\lambda \tag{2.14}$$

$$= \frac{1}{2\pi i}\int_L e^{\lambda t}d\lambda \int_0^1 \frac{e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}}\varphi(\xi)d\xi$$

$$-\frac{1}{2\pi i}\int_L e^{\lambda t}d\lambda \int_0^1 \frac{e^{\sqrt{\lambda}(x+\xi)} - 2e^{\sqrt{\lambda}(1+x-\xi)} + e^{\sqrt{\lambda}(x-\xi)}}{2\sqrt{\lambda}(e^{\sqrt{\lambda}}-1)^2}\varphi(\xi)d\xi$$

$$-\frac{1}{2\pi i}\int_L e^{\lambda t}d\lambda \int_0^1 \frac{e^{\sqrt{\lambda}(2-x-\xi)} + e^{-\sqrt{\lambda}(x-\xi)}}{2\sqrt{\lambda}(e^{\sqrt{\lambda}}-1)^2}\varphi(\xi)d\xi$$

3 Main Results

Theorem 3.1 *Consider the initial-boundary value problem*

$$u_t = u_{xx}, \qquad x \in (0,1), \qquad t > 0 \tag{3.15}$$

$$\begin{cases} u(0,t) = u(1,t) \\ u_x(x,t)|_{x=0} = 0 \end{cases} \quad , t \geq 0 \tag{3.16}$$

$$u(x,0) = \varphi(x), \qquad x \in [0,1] \tag{3.17}$$

if the function $\varphi(x)$ satisfies the following conditions:

$$\varphi(0) = \varphi(1) = 0, \qquad \varphi \in C^2(0,1) \tag{3.18}$$

then this problem has a solution in form of (2.14).

The proof is stablished in three steps. In first step we show that the solution (2.14) satisfies in the equation (3.15). In the second step we show that the solution (2.14) satisfies in boundary conditions(3.16). In third step we show that the solution (2.14) satisfies in the initial condition(3.17).

Proof:

Step1:

$$u_{xx} - u_t =$$

$$-\frac{1}{2\pi i}\int_L e^{\lambda t}[y''(x,\lambda) - \lambda y(x,\lambda)]d\lambda \tag{3.19}$$

$$= -\frac{1}{2\pi i}\int_L e^{\lambda t}\varphi(x)d\lambda = -\frac{\varphi(x)}{2\pi i}\int_L e^{\lambda t}d\lambda = 0$$

Since the function $e^{\lambda t}$ is an analytic function,

that is:

$$\int_{L_\nu - C_\nu} e^{\lambda t}d\lambda = \int_{L_\nu} e^{\lambda t}d\lambda + \int_{-C_\nu} e^{\lambda t}d\lambda \tag{3.20}$$

Note that the $L_\nu - C_\nu$ is a closed countour and C_ν is a part of circle with radius γ_ν, then we have:

$$\int_{L_\nu} e^{\lambda t}d\lambda = \int_{C_\nu} e^{\lambda t}d\lambda \tag{3.21}$$

$$\int_L e^{\lambda t}d\lambda = \lim_{\nu\to\infty}\int_{L_\nu} e^{\lambda t}d\lambda = \lim_{\nu\to\infty}\int_{C_\nu} e^{\lambda t}d\lambda = 0 \tag{3.22}$$

Step2: For satisfying boundary conditions, we have:

$$u(0,t) - u(1,t) = \frac{1}{2\pi i}\int_L e^{\lambda t}[y(0,\lambda) - y(1,\lambda)]d\lambda = 0 \tag{3.23}$$

$$u_x(x,t)|_{x=0} = \frac{-1}{2\pi i}\int_L e^{\lambda t}y'(0,\lambda)d\lambda = 0 \tag{3.24}$$

Step3: For satisfying the initial condition (3.17) we use the asymptotic expansion of $e^{-\sqrt{\lambda}|x-\xi|}$.

$$u(x,0) = \frac{-1}{2\pi i}\lim_{t\to 0}\int_L y(x,\lambda)d\lambda \tag{3.25}$$

We can write the second term of solution as the following asymptotic expansion:

$$-\int_0^1 \frac{e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}}\varphi(\xi)d\xi = \tag{3.26}$$

$$\frac{-1}{2\sqrt{\lambda}}\int_0^x e^{-\sqrt{\lambda}(x-\xi)}\varphi(\xi)d\xi$$

$$-\frac{1}{2\sqrt{\lambda}}\int_x^1 e^{-\sqrt{\lambda}(\xi-x)}\varphi(\xi)d\xi =$$

$$-\frac{e^{-\sqrt{\lambda}x}}{2\sqrt{\lambda}}\int_0^x e^{\sqrt{\lambda}\xi}\varphi(\xi)d\xi - \frac{e^{\sqrt{\lambda}x}}{2\sqrt{\lambda}}\int_x^1 e^{-\sqrt{\lambda}\xi}\varphi(\xi)d\xi$$

Now by using part rule of integration in above integrals we have:

$$\int_0^x e^{\sqrt{\lambda}\xi}\varphi(\xi)d\xi = \frac{e^{\sqrt{\lambda}\xi}}{\sqrt{\lambda}}\varphi(\xi)|_0^x - \int_0^x \frac{e^{\sqrt{\lambda}\xi}}{\sqrt{\lambda}}\varphi'(\xi)d\xi = \tag{3.27}$$

$$\frac{e^{\sqrt{\lambda}x}}{\sqrt{\lambda}}\varphi(x) - \frac{\varphi(0)}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}}\int_0^x e^{\sqrt{\lambda}\xi}\varphi'(\xi)d\xi$$

For second integral we also have:

$$\int_x^1 e^{-\sqrt{\lambda}\xi}\varphi(\xi)d\xi =$$

$$\frac{e^{-\sqrt{\lambda}\xi}}{-\sqrt{\lambda}}\varphi(\xi)|_x^1 - \int_x^1 \frac{e^{-\sqrt{\lambda}\xi}}{-\sqrt{\lambda}}\varphi'(\xi)d\xi = \tag{3.28}$$

$$-\frac{e^{-\sqrt{\lambda}}}{\sqrt{\lambda}}\varphi(1) + \frac{e^{-\sqrt{\lambda}x}}{\sqrt{\lambda}}\varphi(x) + \frac{1}{\sqrt{\lambda}}\int_x^1 e^{-\sqrt{\lambda}\xi}\varphi'(\xi)d\xi$$

Therefore we get:

$$-\int_0^1 \frac{e^{-\sqrt{\lambda}|x-\xi|}}{2\sqrt{\lambda}}\varphi(\xi)d\xi =$$

$$-\frac{\varphi(x)}{\lambda} + \frac{\varphi(0)}{2\lambda}e^{-\sqrt{\lambda}x} + \frac{\varphi(1)}{2\lambda}e^{-\sqrt{\lambda}(1-x)} + \tag{3.29}$$

$\frac{e^{-\sqrt{\lambda}x}}{2\lambda}\int_0^x e^{\sqrt{\lambda}\xi}\varphi'(\xi)d\xi - \frac{e^{\sqrt{\lambda}x}}{2\lambda}\int_x^1 e^{-\sqrt{\lambda}\xi}\varphi'(\xi)d\xi$

Since eigenvalues in negative part of real axes are distingly and discrete, hence we can choose the countour C_ν such that this countour dose not contain any eigenvalue, therefore:

$$\lim_{\nu\to\infty}\int_{C_\nu} e^{\lambda t}y(x,\lambda)d\lambda = 0 \tag{3.30}$$

finaly we have:

$$u(x,0) = \frac{-1}{2\pi i}\lim_{\nu\to\infty}\lim_{t\to 0}\int_{L_\nu - C_\nu} e^{\lambda t}y(x,\lambda)d\lambda = \tag{3.31}$$

$$\frac{-1}{2\pi i}\lim_{\nu\to\infty}\int_{L_\nu - C_\nu}[-\frac{\varphi(x)}{\lambda}+$$

$$\frac{\varphi(0)}{2\lambda}e^{-\sqrt{\lambda}x} + \frac{\varphi(1)}{2\lambda}e^{-\sqrt{\lambda}(1-x)}$$

$$+\frac{e^{-\sqrt{\lambda}x}}{2\lambda}\int_0^x e^{\sqrt{\lambda}\xi}\varphi'(\xi)d\xi - \frac{e^{\sqrt{\lambda}x}}{2\lambda}\int_x^1 e^{-\sqrt{\lambda}x}\varphi'(\xi)d\xi]d\lambda$$

According to the conditions (3.18), we have:

$$u(x,0) = \varphi(x)\frac{1}{2\pi i}\lim_{\nu\to\infty}\int_{L_\nu - C_\nu}\frac{d\lambda}{\lambda} = \varphi(x) \tag{3.32}$$

4 Unsolved Problems

As mentioned in introduction, when the eigenvalues fill the complex plane or the case that there is no eigenvalue, we should use the countor integral method. Apply this method for the following problems:

Problem1: $u_t = u_{xx}, \quad x \in (0,1), \quad t > 0$

$$(4.1)\begin{cases} u(0,t) - 2u(1,t) = 0 \\ ,t \geq 0 \\ u_x(x,t)|_{x=0} + 2u_x(x,t)|_{x=1} = 0 \end{cases}$$

Problem2: $u_t = u_{xx}, \quad x \in (0,1), \quad t > 0$

$$(4.2)\begin{cases} u(0,t) + u(1,t) = 0 \\ ,t \geq 0 \\ u_x(x,t)|_{x=0} - u_x(x,t)|_{x=1} = 0 \end{cases}$$

With initial condition $u(x,0) = \varphi(x)$

Problem3: $u_{tt} = u_{xx}, \quad x \in (0,1), t \in (0,T)$

$$(4.3)\begin{cases} u(0,t) = u(1,t) \\ ,t \geq 0 \\ u_x(x,t)|_{x=0} = 0 \end{cases}$$

With initial conditions $\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = \varphi_k(x), \quad x \in [0,1], \quad k = 0,1$

5 Conclusion

We applied the Countour integral method to solve non-classical cases of these problems. We shown that the spectral problem had repeated eigenvalues and consequently the eigenfunctions can not stablish a complete basis system. We also shown that the resulted analytic solution satisfied in given differantial equation and given initial and boundary condition. We can apply same method for solving more spectral problems which their eigenvalues are repeated such as one and two dimensional wave equation and Steklov problem for Laplace and Helemholtz equation.

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