Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 9, No. 4, 2017 Article ID IJIM-00796, 5 pages
Research Article

Solving Volterra's Population Model via Rational Christov Functions Collocation Method

K. Parand *†, E. Hajizadeh ‡, A. Jahangiri §, S. Khaleqi ¶

Received Date: 2015-12-07 Revised Date: 2016-04-26 Accepted Date: 2017-09-17

Abstract

The present study is an attempt to find a solution for Volterra's Population Model by utilizing Spectral methods based on Rational Christov functions. Volterra's model is a nonlinear integro-differential equation. First, the Volterra's Population Model is converted to a nonlinear ordinary differential equation (ODE), then researchers solve this equation (ODE).The accuracy of method is tested in terms of RES error and compare the obtained results with some well-known results.The numerical results obtained show that the proposed method produces a convergent solution.

Keywords : Volterra's Population Model; Collocation Method; Rational Christov Functions; Nonlinear ODE.

1 Introduction

Many problems arising in science and engineering are set in unbounded domain. Spectral methods have been successfully applied in the approximation of ordinary differential equations (ODEs) defined in unbounded domains in recent years[17, 18, 13, 15, 20].

1.1 Spectral Methods

Different spectral methods can be applied to solve ODEs in unbounded domains, The first approach using orthogonal functions over the unbounded domains were Sinc, Hermite and Laguerre polynomials [9]. The second approach is to reformulate original problems in unbounded domains to singular problems in bounded domains by variable transformations, and then to use suitable Jacobi polynomials to approximate the resulting singular problems [10]. The third approach is to replacing infinite domain with $[-L, L]$ and semi-infinite interval with $[0, L]$ by choosing L, sufficiently large. This method is named as domain truncation [4].The fourth approach for solving such problems is based on Rational approximations. Christov[6] and Boyd[2, 3] developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [3] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by suitable mapping to the Chebyshev polynomials. Guo and Shen [11] proposed and analyzed a set of Legendre rational functions which are mutually orthogonal in

*Corresponding author. k_parand@sbu.ac.ir, Tel: +982129903001
†Department of Computer Sciences, Faculty of Mathematical, Shahid Beheshti University, Tehran, Iran.
‡Department of Computer Sciences, Faculty of Mathematical, Shahid Beheshti University, Tehran, Iran.
§Department of Computer Sciences, Salman Farsi University of Kazerun, Kazerun, Iran.
¶Department of Computer Sciences, Faculty of Mathematical, Shahid Beheshti University, Tehran, Iran.

$L^2(0,\infty)$.Authors of [1] compared rational and exponential Legendre functions Tau approach to solve the governing equations for the flow of a third grade fluid in a porous half space. Boyd [5] applied pseudo-spectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre and mapped Fourier sine [20].

1.2 Volterra's Population Model

Attempts to explain the balance of nature through mathematics began to appear around the turn of the century. A simple set of differential equations to describe malaria epidemics was proposed by Ross [23]. Martini improved these equations by allowing for the immunity of individuals who had recovered from infection [8]. The Volterra model for population growth of a species within a closed system is given in [24, 26, 28] as:

$$\kappa\frac{du}{dt} = u - u^2 - u\int_0^t u(x)dx, \quad u(0) = u_0 \quad (1.1)$$

where $u(t)$ is the scaled population of identical individuals at a time t, and $\kappa = \frac{c}{ab}$ is a prescribed non-dimensional parameter. Moreover, $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient. The coefficient c indicates the essential behavior of the population evolution before its level falls to zero in the long term. One may show that the only equilibrium solution of (1.1) is the trivial solution $u(t) = 0$. Furthermore, the analytical solution [25]

$$u(t) = u_0\; exp(\frac{1}{\kappa}\int_0^t [1 - u(\tau) - \int_0^\tau u(x)dx]d\tau), \quad (1.2)$$

shows that $u(t) > 0$ for all $u_0 > 0$ [22].

The solution of Eq. (1.1) has been considerable concerned. Although a closed form solution was achieved in [25, 26], it was formally shown that the closed form solution cannot lead to any insight into the behavior of the population evolution [28]. Some approximate and numerical solutions for Volterra's population model have been reported. In Ref. [24], the successive approximations method was offered for the solution of Eq. 1.1, but was not implemented. In [25], the singular perturbation method for Volterra's population model is considered. It is shown in [25] that for the case $\kappa << 1$, where populations are weakly sensitive to toxins, a rapid rise occurs along the logistic curve that will reach a peak and be followed by a slow exponential decay. And, for κ large, where populations are strongly sensitive to toxins, the solution is proportional to $sech^2(t)$. In this case the solution $u(t)$ has a smaller amplitude compared to the amplitude of $u(t)$ for the case $\kappa << 1$. In [26], three numerical algorithms, namely the Euler method, the modified Euler method and a fourth order Rung-Kutta method, have been used for Eq. 1.1. Recently, some researchers employed spectral methods to solve Volterras Population Model for example [22, 16] ; also, some researchers have used the analytical methods for approximating this problem[28, 12].

In the present paper, we utilized Christov functions to solve Volterra's Population Model by collocation method.

The rest of this paper is organized as follows: In Section 2, the researchers have described rational Christov functions. In section 3, Volterra's Population Model has converted to a nonlinear ordinary differential equation (ODE). In section 4, the presented method has applied to solve Volterra's Population Model. The researchers have shown the approximate solutions and compared it with other results. The last part of this study described several concluding remarks.

2 Rational Christov functions

The system

$$\rho_n = \frac{1}{\sqrt{\pi}}\frac{(ix-1)^n}{(ix+1)^{n+1}} \qquad n = 0, 1, 2... \quad , i = \sqrt{-1}, \quad (2.3)$$

was introduced by Wiener as Fourier transform of the Laguerre functions (functions of parabolic cylinder). Higgins defined it also for negative indices n and proved its completeness and orthogonality [7]. Christov invented a new system, the new system is comprised a two real-valued subsequences of odd functions S_n and even functions C_n with asymptotic behavior x^{-1} and x^{-2} respectively, namely [6, 7]:

$$S_n = \frac{\rho_n + \rho_{-n-1}}{i\sqrt{2}}\ , \qquad n = 0, 1, 2...\ , \quad (2.4)$$

$$C_n = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}\ , \qquad n = 0, 1, 2...\ . \quad (2.5)$$

Both sequences are orthonormal and each member of (2.4) is orthogonal to all members of (2.5); each member of (2.5) is also orthogonal to all members of (2.4). it is worth mentioning that (2.4) and (2.5) can be defined for negative n through the relations [6]

$$S_{-n} = S_{n-1} \quad and \quad C_{-n} = -C_{n-1}. \tag{2.6}$$

The functions S_n and C_n can be easily expressed in an explicit way[6]:

$$S_n = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1} x^{2k-1}(-1)^{n+k}\binom{2n+1}{2k-1}}{(x^2+1)^{n+1}} \tag{2.7}$$

$$C_n = \sqrt{\frac{2}{\pi}} \frac{\sum_{k=1}^{n+1} x^{2k-2}(-1)^{n+k+1}\binom{2n+1}{2k-2}}{(x^2+1)^{n+1}} \tag{2.8}$$

For more details and explanation about Christov Functions and its properties, see [6, 17].

2.1 Function Approximation

For any function, f in $L^2(-\infty,\infty)$ can be written as follows:

$$f(x) = \sum_{i=0}^{\infty} a_i \phi_i(x), \tag{2.9}$$

where ϕ_i is the C_n or S_n function. The function f can be expanded as follows [7]:

$$f(x) = \sum_{i=0}^{\infty} (a_i C_i(x) + b_i S_i(x)) \tag{2.10}$$

If the infinite series in Eq (2.9,2.10) is truncated with N terms, then it can be written as follows [27]:

$$f_N(x) \simeq \sum_{i=0}^{N} a_i \phi_i(x) \tag{2.11}$$

$$f_N(x) \simeq \sum_{i=0}^{N} (a_i C_i(x) + b_i S_i(x)) \tag{2.12}$$

where ϕ_i is the C_n or S_n function.

3 Converting Volterra's Population Model to a Nonlinear ODE

The researchers convert Volterra's Population Model in Eq (1.1) to an equivalent nonlinear ordinary differential equation, let:

$$y(x) = \int_0^x u(t)dt, \tag{3.13}$$

this leads to:

$$y'(x) = u(x), \tag{3.14}$$
$$y''(x) = u'(x). \tag{3.15}$$

inserting Eqs. (3.13, 3.14, 3.15) into Eq (1.1) yields the nonlinear differential equation

$$\kappa y''(x) = y'(x) - (y'(x))^2 - y(x)y'(x), \tag{3.16}$$

with initial conditions:

$$y(0) = 0 \quad , \quad y'(0) = u_0 \, , \tag{3.17}$$

obtained via Eqs. (3.13) and (3.14) respectively [22] .

4 Solving Problem

In this section, we try to solve Volterra's Population Model by using collocation method. We multiply series (2.12) to x^2, and also we construct a function $p(x)$ to satisfy the conditions (3.17), given by

$$p(x) = u_0 x \tag{4.18}$$

Therefore, the approximate solution of y(x), to solve Eq (3.16) with conditions, Eq (3.17), become:

$$\widehat{y}(x) = u_0 x + x^2 \sum_{i=0}^{N} (a_i C_i(x) + b_i S_i(x)) \tag{4.19}$$

Hence, $\widehat{y}(0) = 0$ and $\frac{d\widehat{y}(x)}{dx} = u(0)$.

We construct the residual function by substituting y(x) by $\widehat{y}(x)$ in Eq.(3.16)

$$Res(x) = \kappa \frac{d^2\widehat{y}(x)}{dx^2} - \frac{d\widehat{y}(x)}{dx} + (\frac{d\widehat{y}(x)}{dx})^2 + \widehat{y}(x)\frac{d\widehat{y}(x)}{dx}. \tag{4.20}$$

A method for forcing the residual function (4.20) to zero is Collocation algorithm. With collocating $\{x_k\}_{k=0}^{2N+1}$ to residual function (4.20), we have $2N+2$ equations and $2N+2$ unknown coefficients (spectral coefficients), in all of spectral methods, the purpose is to find these coefficients. In shape of algorithmic for solving equation (3.16), we do [19]:

BEGIN

1. Input N.

Table 1: A comparison of present method with the exact value for u_{max}

κ	Exact u_{max}	N=4 Presented method	N=4 $\|Res(x)\|^2$	N=5 Presented method	N=5 $\|Res(x)\|^2$
0.04	0.873719	0.7998379	$1.81e-3$	0.829863	$1.80e-4$
0.10	0.769741	0.7569857	$5.72e-4$	0.768735	$4.55e-7$
0.20	0.659050	0.6590674	$3.64e-7$	0.659057	$9.84e-8$
0.50	0.485190	0.4851861	$2.90e-7$	0.485190	$1.55e-7$
		N=6 Presented method	N=6 $\|Res(x)\|^2$	N=7 Presented method	N=7 $\|Res(x)\|^2$
		0.872953	$5.18e-5$	0.873408	$7.64e-7$
		0.768926	$3.45e-7$	0.769488	$6.37e-8$
		0.659050	$2.18e-8$	0.659050	$7.23e-10$
		0.485190	$1.01e-9$	0.485190	$1.72e-11$

Table 2: A comparison of method in [22, 28, 14] and the present method with the exact values for u_{max}.

κ	N	Exact u_{max}	Present method	ADM[28]	TAU[22]	GA-IRBF [14]
0.04	7	0.873719	0.873408	0.861240	0.873708	0.873719
0.10	7	0.769741	0.769488	0.765113	0.769734	0.769741
0.20	6	0.659050	0.659050	0.657912	0.659045	0.659050
0.50	5	0.485190	0.485190	0.485282	0.485188	0.485190

2. Construct the series (2.12)

3. Construct the Eq.(4.19) to satisfy conditions (3.17)

4. Construct the Residual function (4.20) by substituting y(x) by Eq.(4.19) in Eq.(3.16)

5. Choice $\{x_i\}$, $i = 0, 1, ..., 2N+1$ as collocation points.

6. By substituting collocation points in $Res(x; a_0, a_1, ..., a_n, b_0, b_1, ..., b_n)$, we construct a system containing $2N+2$ equations.

7. By solving obtained system of equations in step 6, via Newton's method and gain the a_n, b_n $n = 0, 1, ..., N$.

END

The researchers solve the Eq. (3.16) with $u_0 = 0.1$ and $\kappa = 0.04, 0.1, 0.2, 0.5$, and also evaluate important values u_{max}. Table 1 represents the obtained values by present method with results obtained in [28, 22], and compare u_{max} with exact values obtained by [26]:

$$u_{max} = 1 + \kappa \ln(\frac{\kappa}{1 + \kappa - u_0}). \tag{4.21}$$

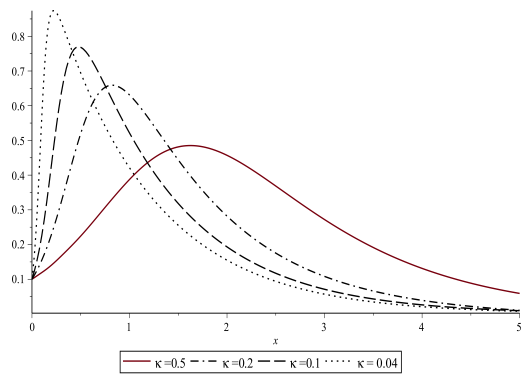

Figure 1: Graph of the approximated $y(x)$ of Volterra's Population Model (ODE), with data in Table 1, and rational Christov functions basis function.

5 Conclusion

In this paper, the researchers converted integro-ordinary differential equation to a non-linear ordinary differential equation (ODE). The method used in this study was rational Christov functions. This method solved the problem on the infinite domain without truncating it to a finite domain and transforming domain of the problem to a finite domain. Additionally, through the comparison with other methods, we have shown that the both of the presented approaches have good reliability and efficiency.

References

[1] F. Baharifard, S. Kazem, K. Parand, Rational and Exponential Legendre Tau Method on Steady Flow of a Third Grade Fluid in a Porous Half Space, *International Journal of Applied and Computational Mathematics* 2 (2016) 679698

[2] J. P. Boyd, Spectral methods using rational basis functions on an infinite interval, *Journal of Computational Physics* 69 (1987) 112-142.

[3] J. P. Boyd, Orthogonal rational functions on a semi-infinite interval, *Journal of Computational Physics* 70 (1987) 63-88.

[4] J. P. Boyd, Chebyshev and Fourier spectral methods. 2nd ed, *New York Dover,* (2000).

[5] J. P. Boyd, C. Rangan, P. H. Bucksbaum, Pseudospectral methods on a semi-infinite interval with application to the Hydrogen atom: a comparison of the mapped Fourier-sine method with Laguerre series and rational Chebyshev expansions, *Journal of Computational Physics* 188 (2003) 56-74.

[6] C. Christov, A complete orthonormal system of functions in $L^2(-\infty, \infty)$ space, *SIAM Journal on Applied Mathematics* 42 (1982) 1337-1342.

[7] M. A. Christou, C. I. Christov, Interacting localized waves for the regularized long wave equation via a Galerkin spectral method, *Mathematics and Computers in Simulation* 69 (2005) 257-268.

[8] U. DAncona, La Lotta per lEsistenza, *G. Einaudi, Torino*, (1942).

[9] D. Funaro, Polynomial Approximation of Differential Equations, *Springer-Verlag, Berlin Heidelberg,* (1992).

[10] B. Y. Guo, Jacobi spectral approximation and its applications to differential equations on the half line, *Journal of Computational Mathematics* 18 (2000) 95-112.

[11] B. Y. Guo, J. Shen, ZQ. Wang, A rational approximation and its applications to differential equations on the half line, *Journal of Scientific Computing* 15 (2000) 117-147.

[12] S. Momani, R. Qaralleh, Numerical approximations and Pade approximations for a fractional population growth model, *Applied Mathematics and Computation* 31 (2007) 1907-1914.

[13] K. Parand, S. Abbasbandy, S. Kazem, A. R. Rezaei, An improved numerical method for a class of astrophysics problems based on radial basis functions, *Physica Scripta* 83 (2011)

[14] K. Parand, S. Abbasbandy, S. Kazem, J. A. Rad, A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation, *Communications in Nonlinear Science and Numerical Simulation* 16 (2011) 4250-4258.

[15] K. Parand, M. Dehghan, A. Taghavi, Modified generalized Laguerre function Tau method for solving laminar viscous flow: The Blasius equation, *International Journal of Numerical Methods for Heat & Fluid Flow* 20 (2010) 728-743.

[16] K. Parand, Z. Delafkar, N. Pakniat, A. Pirkhedri, M. K. Haji, Collocation method using Sinc and Rational Legendre function for solving Volterra's population model, *Communications in Nonlinear Science and Numerical Simulation* 16 (2011) 1811-1819.

[17] K. Parand, E. Hajizadeh, Solving Steady Flow of a Third-Grade Fluid in a Porous Half Space via Normal and Modified Rational Christov Functions Collocation Method, *Zeitschrift fr Naturforschung A* 69 (2014) 188-194.

[18] K. Parand, S. Khaleqi, Rational Chebyshev of Second Kind Collocation Method for Solving a Class of Astrophysics Problems, *The European Physical Journal Plus* 131 (2016) 1-24 .

[19] K. Parand, M. Nikarya, J. A. Rad, F. Baharifard, A new Reliable Numerical Algorithm

Based on the First Kind of Bessel Functions to Solve PrandtlBlasius Laminar Viscous Flow over a Semi-Infinite Flat Plate, *Zeitschrift fr Naturforschung A* 67 (2012) 665-673.

[20] K. Parand, M. Shahini, M. Dehghan, Solution of a laminar boundary layer flow via a numerical method, *Communications in Nonlinear Science and Numerical Simulation* 15 (2010) 360-367.

[21] K. Parand, J. A. Rad, M. Nikarya, A new numerical algorithm based on the first kind of modified Bessel function to solve population growth in a closed system, *International Journal of Computer Mathematics* 91 (2014) 1239-1254.

[22] K. Parand, M. Razzaghi, Rational Chebyshev Tau method for solving Volterra's population model, *Applied Mathematics and Computation* 149 (2004) 893-900.

[23] R. Ross, The Prevention of Malaria, *New York, E. P. Dutton and company, London,* (1911).

[24] F. Scudo, Vito Volterra and theoretical ecology, *Theoretical Population Biology* 2 (1971) 1-23.

[25] R. D. Small, Population Growth in a Closed Systemand Mathematical Modeling, *SIAM Society for Industrial and Applied Mathematics* 25 (1983) 93-95.

[26] K. G. TeBeest, Numerical and analytical solutions of Volterra,s population model, *SIAM Journal Review* 39 (1997) 484-493.

[27] T. Tajvidi, M. Razzaghi, M. Dehghan, Modified rational Legendre approach to laminar viscous flow over a semi-infinite flat plate,*Chaos, Solitons & Fractals* 35 (2008) 59-66.

[28] A. M. Wazwaz, Analytical approximation and Pade approximation for Volterra's population model, *Applied Mathematics and Computation* 100 (1999) 13-25.

Kourosh Parandhas got PhD degree fromAmirkabir University of Technology in 2003 and now he is a professor in Department ofComputer Sciences, Faculty of Mathematical Sciences, ShahidBeheshti University, Tehran,Iran. He has published more than 100 papers in international journals and conferences.Main research interest includes Spectralmethod, Optimum control, Orthogonal functions, Operation research and Data envelopment analysis.

Emran Hajizadeh has got BsD degree from Sistan and BaluchestanUniversity in 2010 and also got MsD degree in Department of Computer Science, ShahidBeheshti University of Tehran, Iran. Main research interest includes Spectral methods and rational Christov functions.

Abdolhamid Jahangiri has got BsD degree from ShahidBahonar University in 2011 and also got MsD degree in Department of Computer Science, ShahidBeheshti University of Tehran, Iran. Main research interest includes Spectral methods and Barycentric interpolation.

Sajjad Khaleghi has got BsD degree from DamghanUniversity in 2010 and also got MsD degree in Department of Computer Science, ShahidBeheshti University of Tehran, Iran. Main research interest includes Spectral methods and Chebyshev of second kind and Sinc functions.