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Research Article

G-frames in Hilbert Modules Over Pro-C*-algebras

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Abstract

G-frames are natural generalizations of frames which provide more choices on analyzing functions from frame expansion coefficients. First, they were defined in Hilbert spaces and then generalized on C*-Hilbert modules. In this paper, we first generalize the concept of g-frames to Hilbert modules over pro-C*-algebras. Then, we introduce the g-frame operators in such spaces and show that they share many useful properties with their corresponding notions in Hilbert spaces. We also show that, by having a g-frame and an invertible operator in this spaces, we can produce the corresponding dual g-frame. Finally we introduce the canonical dual g-frames and provide a reconstruction formula for the elements of such Hilbert modules.

Keywords : Pro-C*-algebra; Hilbert modules; G-frames; Frame operators.

1 Introduction

FRames that are a generalization of bases in Hilbert space, were introduced by Duffin and Schaeffer [9] in 1952. They have many applications, such as: study and characterization of function spaces [8], signal and image processing, wireless communications, transceiver design, data compression and so on. we refer to [2, 3, 6, 7, 11, 12, 21] for an introduction to the frame theory and its applications. Diverse applications of frame theory in science and engineering, led to the theory, should be extended to diferent forms. G-frames are natural generalizations of frames in Hilbert space [20]. In this paper, we generalize the concept of g-frame into a general space which is called, *Hilbert module over a Pro-C*-algebra*. We also introduce the g-frame transforms and study their properties. we show that many of the properties and the main results of frame theory in the Hilbert space, in this case is also true. Finally, we introduce the canonical dual g-frames and provide a reconstruction formula of the elements of such spaces.

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2 Hilbert pro-C*-modules

In this section, we recall some of the basic definitions and properties of pro-C*-algebras and Hilbert modules over them from [13, 18, 19].

A pro-C*-algebra is a complete Hausdorff complex topological $*$-algebra A whose topology is determined by its continuous C*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $p(a_\lambda) \to 0$ for any continuous C*-seminorm p on A and we have:

1) $p(ab) \leq p(a)p(b)$

2) $p(a^*a) = p(a)^2$

for all C*-seminorm p on A and $a, b \in A$.

If the topology of a pro-C*-algebra is determined by only countably many C*-seminorms, then it is called a σ-C*-algebra.

Let A be a unital pro-C*-algebra with unit 1_A and let $a \in A$. Then, the spectrum $sp(a)$ of $a \in A$ is the set $\{\lambda \in \mathbb{C} \ : \ \lambda 1_A - a$ is not invertible$\}$. If A is not unital, then the spectrum is taken with respect to its unitization $\tilde{A}$.

If A^+ denotes the set of all positive elements of A, then A^+ is a closed convex cone such that $A^+ \cap (-A^+) = 0$. We denote by $S(A)$, the set of all continuous C*-seminorms on A. For $p \in S(A)$, we put $\ker(p) = \{a \in A : p(a) = 0\}$; which is a closed ideal in A. For each $p \in S(A)$, $A_p = A/\ker(p)$ is a C*-algebra in the norm induced by p which defined as;

$$\|a + \ker(p)\|_{A_p} = p(a) \quad , \quad p \in S(A).$$

We have $A = \varprojlim_p A_p$ (see [19]).

The canonical map from A onto A_p for $p \in S(A)$, will be denoted by π_p and the image of $a \in A$ under π_p will be denoted by a_p. Hence $l^2(A_p)$ is a Hilbert A_p-module (see [14]), with the norm, defined as:

$$\|(\pi_p(a_i))_{i\in\mathbb{N}}\|_p = [\ p(\sum_{i\in\mathbb{N}} a_i a_i{}^*)]^{1/2} \ ,$$
$$p \in S(A) \ , \ (\pi_p(a_i))_{i\in\mathbb{N}} \in l^2(A_p).$$

Example 2.1 *Every C*-algebra is a pro-C*-algebra.*

Example 2.2 *A closed ∗-subalgebra of a pro-C*-algebra is a pro-C*-algebra.*

Example 2.3 *([19]) Let X be a locally compact Hausdorff space and let $A = C(X)$ denotes all continuous complex-valued functions on X with the topology of uniform convergence on compact subsets of X. Then A is a pro-C*-algebra.*

Example 2.4 *([19]) A product of C*-algebras with the product topology is a pro-C*-algebra.*

Remark 2.1 *$a \geq 0$ denotes $a \in A^+$ and $a \leq b$ denotes $a - b \geq 0$.*

Proposition 2.1 *([13]) Let A be a unital pro-C*-algebra with an identity 1_A. Then for any $p \in S(A)$, we have:*

1. $p(a) = p(a^*)$ *for all* $a \in A$
2. $p(1_A) = 1$
3. *If* $a, b \in A^+$ *and* $a \leq b$, *then* $p(a) \leq p(b)$
4. $a \leq b$ *iff* $a_p \leq b_p$
5. *If* $1_A \leq b$, *then* b *is invertible and* $b^{-1} \leq 1_A$
6. *If* $a, b \in A^+$ *are invertible and* $0 \leq a \leq b$, *then* $0 \leq b^{-1} \leq a^{-1}$
7. *If* $a, b, c \in A$ *and* $a \leq b$, *then* $c^*ac \leq c^*bc$
8. *If* $a, b \in A^+$ *and* $a^2 \leq b^2$, *then* $0 \leq a \leq b$.

Definition 2.1 *A pre-Hilbert module over pro-C*-algebra A is a complex vector space E which is also a left A-module compatible with the complex algebra structure, equipped with an A-valued inner product $\langle .,. \rangle : E \times E \to A$ which is $\mathbb{C}$-and A-linear in its first variable and satisfies the following conditions:*

1. $\langle x, y\rangle^* = \langle y, x\rangle$
2. $\langle x, x\rangle \geq 0$
3. $\langle x, x\rangle = 0$ *iff* $x = 0$

for every $x, y \in E$. We say that E is a Hilbert A-module (or Hilbert pro-C*-module over A) if E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(x) = \sqrt{p(\langle x, x\rangle)} \qquad x \in E \ , \ p \in S(A).$$

Let E be a pre-Hilbert A-module. By ([22], Lemma 2.1), for every $p \in S(A)$ and for all $x, y \in E$, the following Cauchy-Bunyakovskii inequality holds

$$p(\langle x, y\rangle)^2 \leq p(\langle x, x\rangle) p(\langle y, y\rangle).$$

Consequently, for each $p \in S(A)$, we have:

$$\bar{p}_E(ax) \leq p(a)\bar{p}_E(x) \qquad a \in A \ , \ x \in E.$$

If E is a Hilbert A-module and $p \in S(A)$, then $\ker(\bar{p}_E) = \{x \in E \ : \ p(\langle x, x\rangle) = 0\}$ is a closed submodule of E and $E_p = E/\ker(\bar{p}_E)$ is a Hilbert A_p-module with scalar product

$$a_p.(x + \ker(\bar{p}_E)) = ax + \ker(\bar{p}_E) \quad , \quad a \in A \ , \ x \in E$$

and inner product

$$\langle \ x + \ker(\bar{p}_E) \ , \ y + \ker(\bar{p}_E) \ \rangle = \langle x, y\rangle_p \quad , \quad x, y \in E.$$

By ([19], Proposition 4.4), we have $E \cong \varprojlim_p E_p$.

Example 2.5 *If A is a pro-C^*-algebra, then it is a Hilbert A-module with respect to the inner product defined by :*

$$\langle a,b\rangle = ab^* \qquad a,b\in A\ .$$

Example 2.6 *(See [19], Remark 4.8) Let $l^2(A)$ be the set of all sequences $(a_n)_{n\in\mathbb{N}}$ of elements of a pro-C^*-algebra A such that the series $\sum_{i=1}^{\infty} a_i a_i{}^*$ is convergent in A. Then $l^2(A)$ is a Hilbert module over A with respect to the pointwise operations and inner product defined by:*

$$\langle\ (a_i)_{i\in\mathbb{N}}\ ,\ (b_i)_{i\in\mathbb{N}}\ \rangle = \textstyle\sum_{i=1}^{\infty} a_i b_i{}^*.$$

Example 2.7 *Let E_i for $i\in\mathbb{N}$, be a Hilbert A-module with the topology induced by the family of continuous seminorms $\{\bar{p}_i\}_{p\in S(A)}$ defined as:*

$$\bar{p}_i(x) = \sqrt{p(\langle x,x\rangle)}\quad ,\quad x\in E_i.$$

Direct sum of $\{E_i\}_{i\in\mathbb{N}}$ is defined as follows:

$$\textstyle\bigoplus_{i\in\mathbb{N}} E_i = \{(x_i)_{i\in\mathbb{N}}\ :\ x_i\in E_i\ ,\ \sum_{i=1}^{\infty}\langle x_i,x_i\rangle \text{ is convergent in } A\}.$$

It has been shown (see [17], Example 3.2.3) that the direct sum $\bigoplus_{i\in\mathbb{N}} E_i$ is a Hilbert A-module with A-valued inner product $\langle x,y\rangle = \sum_{i=1}^{\infty}\langle x_i,y_i\rangle$, where $x=(x_i)_{i\in\mathbb{N}}$ and $y=(y_i)_{i\in\mathbb{N}}$ are in $\bigoplus_{i\in\mathbb{N}} E_i$, pointwise operations and a topology determined by the family of seminorms:

$$\bar{p}(x) = \sqrt{p(\langle x,x\rangle)}\quad ,\quad x\in \textstyle\bigoplus_{i\in\mathbb{N}} E_i\ ,\ p\in S(A).$$

The direct sum of a countable copies of a Hilbert module E is denoted by $l^2(E)$.

We recall that an element a in A (x in E) is bounded, if

$$\|a\|_\infty = \sup\{p(a)\ ;\ p\in S(A)\} < \infty,$$

$$(\|x\|_\infty = \sup\{\bar{p}_E(x)\ ;\ p\in S(A)\} < \infty).$$

The set of all bounded elements in A (in E) will be denoted by $b(A)$ ($b(E)$). We know that $b(A)$ is a C*-algebra in the C*-norm $\|.\|_\infty$ and $b(E)$ is a Hilbert $b(A)$-module.([19], Proposition 1.11) and ([22], Theorem 2.1)

Let $M\subset E$ be a closed submodule of a Hilbert A-module E and let

$$M^\perp = \{y\in E\ :\ \langle x,y\rangle = 0\quad for\ all\ x\in M\}.$$

Note that the inner product in a Hilbert modules is separately continuous, hence $M^\perp$ is a closed submodule of the Hilbert A-module E. Also, a closed submodule M in a Hilbert A-module E is called orthogonally complementable if $E = M\oplus M^\perp$. A closed submodule M in a Hilbert A-module E is called topologically complementable if there exists a closed submodule N in E such that $M\oplus N = E$, $N\cap M = \{0\}$.

Let A be a pro-C*-algebra and let E and F be two Hilbert A-modules. An A-module map $T: E\to F$ is said to bounded if for each $p\in S(A)$, there is $C_p>0$ such that:

$$\bar{p}_F(Tx)\le C_p.\bar{p}_E(x) \qquad (x\in E),$$

where $\bar{p}_E$, respectively $\bar{p}_F$, are continuous seminorms on E, respectively F. A bounded A-module map from E to F is called an operator from E to F. We denote the set of all operators from E to F by $Hom_A(E,F)$, and we set $Hom_A(E,E) = End_A(E)$.

Let $T\in Hom_A(E,F)$. We say T is adjointable if there exists an operator $T^*\in Hom_A(F,E)$ such that:

$$\langle Tx,y\rangle = \langle x,T^*y\rangle$$

holds for all $x\in E$, $y\in F$.

We denote by $Hom_A^*(E,F)$, the set of all adjointable operators from E to F and $End_A^*(E) = Hom_A^*(E,E)$.

By a little modification in the proof of ([22], Lemma 3.2), we have the following result:

Proposition 2.2 *Let $T: E\to F$ and $T^*: F\to E$ be two maps such that the equality*

$$\langle x,T^*y\rangle = \langle Tx,y\rangle$$

holds for all $x\in E, y\in F$. Then $T\in Hom_A^*(E,F)$.

It is easy to see that for any $p\in S(A)$, the map defined by:

$$\hat{p}_{E,F}(T) = \sup\{\ \bar{p}_F(Tx)\ :\ x\in E,\ \bar{p}_E(x)\le 1\}\quad ,\ T\in Hom_A(E,F),$$

is a seminorm on $Hom_A(E,F)$. Moreover $Hom_A(E,F)$ with the topology determined by the family of seminorms $\{\hat{p}_{E,F}\}_{p\in S(A)}$ is a complete locally convex space (see [15], Proposition 3.1). Moreover using ([22], Lemma 2.2), for each $y \in F$ and $p \in S(A)$, we can write:

$$\begin{aligned}\bar{p}_E(T^*y) &= \sup\{p\langle T^*y, x\rangle \,:\, \bar{p}_E(x) \leq 1\} \\ &= \sup\{p\langle Tx, y\rangle \,:\, \bar{p}_E(x) \leq 1\} \\ &\leq \sup\{\bar{p}_F(Tx) \,:\, \bar{p}_E(x) \leq 1\}.\bar{p}_F(y) \\ &= \hat{p}(T)\bar{p}_F(y).\end{aligned}$$

Thus for each $p \in S(A)$, we have $\hat{p}_{F,E}(T^*) \leq \hat{p}_{E,F}(T)$ and since $T^{**} = T$, by replacing T with T^*, for each $p \in S(A)$, we obtain:

$$\hat{p}_{F,E}(T^*) = \hat{p}_{E,F}(T). \tag{2.1}$$

By ([19], Proposition 4.7), we have the canonical isomorphism

$$Hom_A(E,F) \cong \varprojlim_p Hom_{A_p}(E_p, F_p).$$

Consequently, $End^*_A(E)$ is a pro-C*-algebra for any Hilbert A-module E and its topology is obtained by $\{\hat{p}_E\}_{p\in S(A)}$ ([22]). By ([22], Proposition 3.2), T is a positive element of $End^*_A(E)$ if and only if $\langle Tx, x\rangle \geq 0$ for any $x \in E$.

Definition 2.2 *Let E and F be two Hilbert modules over pro-C*-algebra A. Then the operator $T : E \to F$ is called uniformly bounded(below), if there exists $C > 0$ such that for each $p \in S(A)$ and $x \in E$,*

$$\bar{p}_F(Tx) \leq C\bar{p}_E(x). \tag{2.2}$$
$$(C\bar{p}_E(x) \leq \bar{p}_F(Tx)) \tag{2.3}$$

The number C is called an upper bound for T and we set:

$$\|T\|_\infty = \inf\{C \;:\; \text{C is an upper bound for T}\}.$$

Clearly, in this case we have:

$$\hat{p}(T) \leq \|T\|_\infty \quad , \quad \forall p \in S(A).$$

Let T be an invertible element in $End^*_A(E)$ such that both are uniformly bounded. Then by ([1], Proposition 3.2), for each $x \in E$ we have the following inequality:

$$\|T^{-1}\|_\infty^{-2}\langle x, x\rangle \leq \langle Tx, Tx\rangle \leq \|T\|_\infty^2\langle x, x\rangle. \tag{2.4}$$

The following proposition will be used in the next section.

Proposition 2.3 *Let T be an uniformly bounded below operator in $Hom_A(E,F)$. then T is closed and injective.*

Proof. Let $Tx = 0$, then by (2.2) we have $\bar{p}_E(x) = 0$, for all $p \in S(A)$. Therefore $x = 0$. It follows that T is injective.
Now we show that T is closed. Let M be a closed subset of E and $\{Tx_\alpha\}_\alpha$ a net in TM such that converges to $y \in F$ and so is a Cauchy net. By assumptions of the theorem, there exists $C > 0$ such that for each $p \in S(A)$,

$$C\bar{p}_E(x_\beta - x_\alpha) \leq \bar{p}_F(Tx_\beta - Tx_\alpha).$$

Hence $\{x_\alpha\}_\alpha$ is a Cauchy net in the closed subset M and so converges to $x \in M$. Since T is cotinuous, $\{Tx_\alpha\}_\alpha$ converges to Tx. But F is a Hausdorff space and the convergent net in these spaces has a unique limit.Thus we have $y = Tx$. Therefore TM is closed in F. Consequently T is closed.

3 G-frames in Hilbert modules

Throughout this section, A is a pro-C*-algebra, X and Y are two Hilbert A-modules. also $\{Y_i\}_{i\in I}$ is a countable sequence of closed submodules of Y.

Definition 3.1 *A sequence $\Lambda = \{\Lambda_i \in Hom^*_A(X, Y_i)\}_{i\in I}$ is called a g-frame for X with respect to $\{Y_i\}_{i\in I}$ if there are two positive constants C and D such that for every $x \in X$,*

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq D\langle x, x\rangle.$$

The constants C and D are called g-frame bounds for Λ. The g-frame is called tight if $C = D$ and called a Parseval if $C = D = 1$. If in the above we only need to have the upper bound, then Λ is called a g-Bessel sequence. Also if for each $i \in I$, $Y_i = Y$, we call it a g-frame for X with respect to Y.

Example 3.1 *Let $\{x_i\}_{i\in I}$ be a frame for X with bounds, C and D. Then by definition for each $x \in X$,*

$$C\langle x, x\rangle \leq \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle \leq D\langle x, x\rangle.$$

Now for $i \in I$ define the operator Λ_{x_i} as follows:

$$\Lambda_{x_i} : X \to A \qquad , \qquad \Lambda_{x_i}(x) = \langle x, x_i\rangle.$$

Clearly Λ_{x_i} is a bounded operator in $Hom_A(X, A)$ and has adjoint as follows:

$$\Lambda_{x_i}^* : A \to X \quad , \quad \Lambda_{x_i}^*(a) = ax_i.$$

Hence $\Lambda_{x_i} \in Hom_A^*(X, A)$, $i \in I$. Also for each $x \in X$,

$$C\langle x, x\rangle \le \sum_{i\in I}\langle x, x_i\rangle\langle x_i, x\rangle = \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \le D\langle x, x\rangle.$$

Therefore $\Lambda = \{\Lambda_{x_i}\}_{i\in I}$ is a g-frame for X with respect to A.

Let $\Lambda = \{\Lambda_i \in Hom_A^*(X, Y_i)\}_{i\in I}$ be a g-frame for X with respect to $\{Y_i\}_{i\in I}$ and bounds C , D. We define the corresponding g-frame transform as follows:

$$T_\Lambda : X \to \bigoplus_{i\in I} Y_i \quad , \quad T_\Lambda(x) = \{\Lambda_i x\}_{i\in I} \ .$$

Since Λ is a g-frame, hence for each $x \in X$ we have:

$$C\langle x, x\rangle \le \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \le D\langle x, x\rangle \ .$$

So T_Λ is well-defined. Also for any $p \in S(A)$ and $x \in X$ the following inequality is obtained:

$$\sqrt{C}\ \bar{p}_X(x) \le \bar{p}_{\oplus_i Y_i}(T_\Lambda x) \le \sqrt{D}\ \bar{p}_X(x) \ .$$

From the above, it follows that the g-frame transform is an uniformly bounded below operator in $Hom_A(X, \bigoplus_{i\in I} Y_i)$. Thus by Proposition 2.2, T_Λ is closed and injective.
Also, we define the synthesis operator for g-frame Λ as follows:

$$T_\Lambda^* : \bigoplus_{i\in I} Y_i \to X \quad , \quad T_\Lambda^*(\{y_i\}_i) = \sum_{i\in I}\Lambda_i^*(y_i) \tag{3.5}$$

where Λ_i^* is the adjoint operator of Λ_i.

Proposition 3.1 *The synthesis operator defined by (3.5) is well-defined, uniformly bounded and adjoint of the transform operator.*

Proof. Since $\Lambda = \{\Lambda_i \ : \ i \in I\}$ is a g-frame for X with respect to $\{Y_i\}_{i\in I}$, there exist positive constants C and D such that for any $x \in X$,

$$C\langle x, x\rangle \le \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \le D\langle x, x\rangle \ .$$

Let J be an arbitrary finite subset of I. Using Cauchy-Bunyakovskii inequality and ([22], Lemma 2.2), for any $p \in S(A)$ and $(y_i)_i \in \oplus_{i\in I} Y_i$ we have:

$$\begin{aligned}
&\bar{p}_X(\sum_{i\in J}\Lambda_i^*(y_i)) \\
&= \sup\{p\langle \sum_{i\in J}\Lambda_i^*(y_i), x\rangle \ : \ x \in X\, ,\ \bar{p}_X(x) \le 1\} \\
&= \sup\{p(\sum_{i\in J}\langle y_i, \Lambda_i x\rangle) \ : \ x \in X\, ,\ \bar{p}_X(x) \le 1\} \\
&\le \sup_{\bar{p}_X(x)\le 1} p(\sum_{i\in J}\langle y_i, y_i\rangle)^{0.5} p(\sum_{i\in J}\langle \Lambda_i x, \Lambda_i x\rangle)^{0.5} \\
&\le \sup_{\bar{p}_X(x)\le 1}\left(\sqrt{D}\ \bar{p}_X(x)(p\sum_{i\in J}\langle y_i, y_i\rangle)^{1/2}\right) \\
&\le \sqrt{D}\left(p(\sum_{i\in J}\langle y_i, y_i\rangle)\right)^{1/2} .
\end{aligned}$$

Now, since the series $\sum_{i\in I}\langle y_i, y_i\rangle$ converges in A, the above inequality shows that $\sum_{i\in I}\Lambda_i^*(y_i)$ is convergent. Hence T_Λ^* is well-defined. On the other hand for any $x \in X$ and $(y_i)_i \in \oplus_{i\in I} Y_i$, we have:

$$\begin{aligned}
\langle T_\Lambda(x), (y_i)_i\rangle &= \langle (\Lambda_i x)_i, (y_i)_i\rangle \\
&= \sum_{i\in I}\langle \Lambda_i x, y_i\rangle \\
&= \sum_{i\in I}\langle x, \Lambda_i^* y_i\rangle \\
&= \langle x, \sum_{i\in I}\Lambda_i^* y_i\rangle \\
&= \langle x, T_\Lambda^*(y_i)_i\rangle \ .
\end{aligned}$$

Therefore by Proposition 2.2 it follows that the synthesis operator is adjoint of the transform operator. Also, for any $p \in S(A)$ we have:

$$\bar{p}_X(T_\Lambda^*(y)) \le \sqrt{D}\ \bar{p}_{\oplus_{i\in I} Y_i}(y) \ , \quad y = (y_i)_i \in \oplus_{i\in I} Y_i$$

Hence the synthesis operator is uniformly bounded.

Let $\Lambda = \{\Lambda_i \ , \ i \in I\}$ be a g-frame for X with respect to $\{Y_i\}_{i\in I}$. Define the corresponding g-frame operator S_Λ as follows:

$$S_\Lambda = T_\Lambda^* T_\Lambda : X \to X \quad , \quad S_\Lambda(x) = \sum_{i\in I}{\Lambda_i}^*\Lambda_i x$$

Since S_Λ is a combination of two bounded operators, it is a bounded operator.

Theorem 3.1 *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for X with respect to $\{Y_i\}_{i\in I}$ and with bounds C, D. Then S_Λ is invertible positive operator. Also it is a self-adjoint operator such that:*

$$CI_X \leq S_\Lambda \leq DI_X \ . \tag{3.6}$$

Here I_X is the identity function on X.

Proof. According to the definition of the transform operator, for any $x \in X$ we can write:

$$\begin{aligned}\langle T_\Lambda(x), T_\Lambda(x)\rangle &= \langle \{\Lambda_i x\}_{i\in I}, \{\Lambda_i x\}_{i\in I}\rangle \\ &= \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \ .\end{aligned}$$

Since Λ is a g-frame for X with bounds C and D, for each $x \in X$ it follows that:

$$C\langle x, x\rangle \leq \langle T_\Lambda(x), T_\Lambda(x)\rangle \leq D\langle x, x\rangle \ .$$

On the other hand,

$$\begin{aligned}\langle S_\Lambda(x), x\rangle &= \langle T_\Lambda^* T_\Lambda(x), x\rangle = \langle T_\Lambda(x), T_\Lambda(x)\rangle \\ &= \langle x, T_\Lambda^* T_\Lambda(x)\rangle = \langle x, S_\Lambda(x)\rangle \ .\end{aligned}$$

Consequently, S_Λ is a self-adjoint operator. Also for any $x \in X$, we obtain:

$$C\langle x, x\rangle \leq \langle S_\Lambda(x), x\rangle \leq D\langle x, x\rangle \ .$$

From the above, it follows that the g-frame operator is positive and (3.6) is obtained too. Moreover by Proposition it follows that S_Λ is invertible.

By previous discussions, we have the following useful result.

Remark 3.1 *According to (3.6) and Proposition 2.1. it follows that:*

$$D^{-1}I_X \leq S_\Lambda^{-1} \leq C^{-1}I_X \ .$$

Hence the g-frame operator and its inverse belong to $End_A^*(X)$

Now we are able to generalize ([4], Theorem 3.2), to g-frames in Hilbert modules.

Theorem 3.2 *For each $i \in I$ let $\Lambda_i \in Hom_A^*(X, Y_i)$ and $\{x_{ij} : j \in J_i\}$ be a frame in Y_i with frame bounds C_i and D_i. Suppose that:*

$$0 < C = \inf_i C_i \leq D = \sup_i D_i < \infty$$

Then the following conditions are equivalent.

1. *$\{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$ is a frame for X.*
2. *$\{\Lambda_i : i \in I\}$ is a g-frame for X with respect to $\{Y_i\}_{i\in I}$.*

Proof. Since for each $i \in I$, $\{x_{ij} : j \in J_i\}$ is a frame for Y_i with bounds C_i and D_i , we obtain:

$$\begin{aligned}C_i\langle \Lambda_i x, \Lambda_i x\rangle &\leq \sum_{j\in J_i}\langle \Lambda_i x, x_{ij}\rangle\langle x_{ij}, \Lambda_i x\rangle \\ &\leq D_i\langle \Lambda_i x, \Lambda_i x\rangle \ .\end{aligned}$$

Therefore for each $x \in X$ we have:

$$\begin{aligned}C\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle &\leq \sum_{i\in I} C_i\langle \Lambda_i x, \Lambda_i x\rangle \\ &\leq \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i x, x_{ij}\rangle\langle x_{ij}, \Lambda_i x\rangle \\ &\leq \sum_{i\in I} D_i\langle \Lambda_i x, \Lambda_i x\rangle \\ &\leq D\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \ .\end{aligned}$$

Since each Λ_i is adjointable, the above inequality can be summarized as follows:

$$C\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \leq \sum_{i\in I}\sum_{j\in J_i}\langle x, \Lambda_i^* x_{ij}\rangle\langle \Lambda_i^* x_{ij}, x\rangle \tag{3.7}$$

$$\leq D\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \ , \tag{3.8}$$

which shows that $\{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$ is a frame for X if and only if $\{\Lambda_i : i \in I\}$ is a g-frame for X. Our next result is analog to ([20], Theorem 3.1).

Corollary 3.1 *For each $i \in I$ let $\Lambda_i \in Hom_A^*(X, Y_i)$ and $\{x_{ij} : j \in J_i\}$ be a Parseval frame for Y_i . Then we have the followings:*

1. *$\{\Lambda_i : i \in I\}$ is a g-frame (resp. g-Bessel sequence, tight g-frame) for X iff $\{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$ is a frame (resp. Bessel sequence, tight frame) for X.*
2. *The g-frame operator of $\Lambda = \{\Lambda_i : i \in I\}$ is the frame operator of $\mathcal{F} = \{\Lambda_i^* x_{ij} : j \in J_i, i \in I\}$.*

Proof. In the previous Theorem, let $C_i = D_i = 1$. Then (3.8) will be as follows,

$$\sum_{i\in I}\sum_{j\in J_i}\langle x, \Lambda_i^* x_{ij}\rangle\langle \Lambda_i^* x_{ij}, x\rangle = \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle \ .$$

From this, we conclude the first result. For the second result, let S_Λ and $S_\mathcal{F}$ be the frame operators for Λ and $\mathcal{F}$ respectively. Then by definition, for any $x \in X$,

$$S_\Lambda(x) = \sum_{i\in I} \Lambda_i^* \Lambda_i x \quad ,$$
$$S_\mathcal{F}(x) = \sum_{i\in I}\sum_{j\in J_i} \langle x, \Lambda_i^* x_{ij}\rangle \Lambda_i^* x_{ij} \ .$$

On the other hand for any $i \in I$ and $x \in X$ we have:

$$\Lambda_i x = \sum_{j\in J_i} \langle \Lambda_i x, x_{ij}\rangle x_{ij} \ ,$$

because $\Lambda_i x \in Y_i$ and the above equality is the recostruction formula for $\Lambda_i x$ with respect to Parseval frame $\{x_{ij} : j \in J_i\}$. So for each $x \in X$,

$$\begin{aligned} S_\mathcal{F}(x) &= \sum_{i\in I}\sum_{j\in J_i} \langle x, \Lambda_i^* x_{ij}\rangle \Lambda_i^* x_{ij} \\ &= \sum_{i\in I}\sum_{j\in J_i} \langle \Lambda_i x, x_{ij}\rangle \Lambda_i^* x_{ij} \\ &= \sum_{i\in I} \Lambda_i^* \left(\sum_{j\in J_i} \langle \Lambda_i x, x_{ij}\rangle x_{ij} \right) \\ &= \sum_{i\in I} \Lambda_i^* \Lambda_i x \\ &= S_\Lambda(x) \ . \end{aligned}$$

The proof is complete.

The next result is a generalization of ([16], Theorem 3.5), to Hilbert Pro-C*-modules.

Theorem 3.3 *Let $\Lambda = \{\Lambda_i \in Hom_A^*(X, Y_i)\}_{i\in I}$ be a g-frame for X with bounds C, D and g-frame operator S_Λ. If $T \in End_A^*(X)$ is an invertible operator such that both are uniformly bounded then $\{\Lambda_i T : i \in I\}$ is also a g-frame for X with respect to $\{Y_i : i \in I\}$ and with g-frame operator $T^* S_\Lambda T$.*

Proof. Note that $\Lambda_i T \in Hom_A^*(X, Y_i)$. Also by (2.4), for each $x \in X$ we have:

$$\|T^{-1}\|_\infty^{-2} \langle x, x\rangle \le \langle Tx, Tx\rangle \le \|T\|_\infty^2 \langle x, x\rangle \ .$$

Since $\{\Lambda_i : i \in I\}$ is a g-frame with bounds C and D, for each $x \in X$ we can write:

$$\begin{aligned} C\|T^{-1}\|_\infty^{-2} \langle x, x\rangle &\le C\langle Tx, Tx\rangle \\ &\le \sum_{i\in I} \langle \Lambda_i T x, \Lambda_i T x\rangle \\ &\le D\langle Tx, Tx\rangle \\ &\le D\|T\|_\infty^2 \langle x, x\rangle \ . \end{aligned}$$

Therefore the sequence $\{\Lambda_i T : i \in I\}$ is a g-frame for X with respect to $\{Y_i : i \in I\}$ and bounds $C\|T^{-1}\|_\infty^{-2}$, $D\|T\|_\infty^2$. Also for any $x \in X$ we have:

$$T^* S_\Lambda T(x) = T^* \sum_{i\in I} \Lambda_i^* \Lambda_i T(x)$$
$$= \sum_{i\in I} T^* \Lambda_i^* \Lambda_i T(x) = \sum_{i\in I} (\Lambda_i T)^* (\Lambda_i T) x \ ,$$

which shows that $T^* S_\Lambda T$ is the g-frame operator for $\{\Lambda_i T : i \in I\}$.

As a result we can introduce a reconstruction formula for elements of a Hilbert pro-C*-module.

Corollary 3.2 *Let $\Lambda = \{\Lambda_i \in Hom_A^*(X, Y_i)\}_{i\in I}$ be a g-frame for X with bounds C, D and g-frame operator S_Λ. For each $i \in I$, let $\widetilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$. Then $\widetilde{\Lambda} = \{\widetilde{\Lambda}_i : i \in I\}$ is a g-frame for X with respect to $\{Y_i : i \in I\}$ and bounds C/D^2 , D/C^2 and g-frame operator S_Λ^{-1}. Also for each $x \in X$ we have the following reconstruction formula:*

$$x = \sum_{i\in I} (\widetilde{\Lambda}_i)^* \Lambda_i x = \sum_{i\in I} \Lambda_i^* \widetilde{\Lambda}_i x \ .$$

$\widetilde{\Lambda}$ is called the canonical dual g-frame of Λ.

Proof. In the theorem 3.3 let $T = S_\Lambda^{-1}$. So we conclude that $\{\widetilde{\Lambda_i} = \Lambda_i S_\Lambda^{-1} : i \in I\}$ is a g-frame for X with respect to $\{Y_i : i \in I\}$ and g-frame operator as follows:

$$T^* S_\Lambda T = S_\Lambda^{-1} S_\Lambda S_\Lambda^{-1} = S_\Lambda^{-1} \ .$$

Moreover by Remark 3.1. we have:

$$D^{-1} I_X \le S_\Lambda^{-1} \le C^{-1} I_X \ .$$

Here I_X is the identity operator on X. Hence we obtain:

$$D^{-2} I_X \le S_\Lambda^{-2} \le C^{-2} I_X \ .$$

According to this and that Λ is a g-frame, for each $x \in X$ we have:

$$\begin{aligned} \sum_{i\in I} \langle \widetilde{\Lambda}_i x, \widetilde{\Lambda}_i x\rangle &= \sum_{i\in I} \langle \Lambda_i S_\Lambda^{-1} x, \Lambda_i S_\Lambda^{-1} x\rangle \\ &\le D\langle S_\Lambda^{-1} x, S_\Lambda^{-1} x\rangle \\ &\le D\langle S_\Lambda^{-2} x, x\rangle \\ &\le DC^{-2} \langle x, x\rangle \ . \end{aligned}$$

Similarly, for each $x \in X$ it follows:

$$CD^{-2}\langle x, x\rangle \leq \sum_{i\in I}\langle \widetilde{\Lambda}_i x, \widetilde{\Lambda}_i x\rangle \ .$$

Therefore C/D^2 and D/C^2 are the bounds for $\widetilde{\Lambda}$. Moreover for any $x \in X$ we can write:

$$x = S_\Lambda^{-1} S_\Lambda x = S_\Lambda^{-1} \sum_{i\in I} \Lambda_i^* \Lambda_i x$$
$$= \sum_{i\in I} S_\Lambda^{-1} \Lambda_i^* \Lambda_i x = \sum_{i\in I} (\widetilde{\Lambda}_i)^* \Lambda_i x \ ,$$

Similarly,

$$x = S_\Lambda S_\Lambda^{-1} x = \sum_{i\in I} \Lambda_i^* \Lambda_i (S_\Lambda^{-1} x) = \sum_{i\in I} \Lambda_i^* \widetilde{\Lambda}_i x$$

This completes the proof.

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