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A New Iterative Method For Solving Fuzzy Integral Equations

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Abstract

In the present work, by applying known Bernstein polynomials and their advantageous properties, we establish an efficient iterative algorithm to approximate the numerical solution of fuzzy Fredholm integral equations of the second kind. The convergence of the proposed method is given and the numerical examples illustrate that the proposed iterative algorithm are valid.

Keywords : Fuzzy Fredholm Integral Equation; Modulus of continuity; Partial modulus of continuity; fuzzy Bernstein polynomials.

1 Introduction

THe concept of fuzzy integral was initiated by Dubois and Prade [11] and then investigated by Kaleva [21], Goetschel and Voxman [20], Nanda [23] and others. In [33], the Henstock integral of fuzzy-valued functions is defined, while the fuzzy Riemann integral and its numerical integration was investigated by Wu in [34]. In [7], the authors introduced some quadrature rules for the integral of fuzzy-number-valued mappings. Kaleva [21] proposed the existence and uniqueness of the solution of fuzzy differential equations using the Banach fixed point principle. Mordeson and Newman (see [22]) started the study of the subject of fuzzy integral equations. The Banach fixed point principle is the powerful tool to investigate of the existence and uniqueness of the solution of fuzzy integral equations. The existence and uniqueness of the solution of fuzzy integral equations can be found in [5, 6, 16, 27, 28, 29]. In [19, 24], sufficient conditions are given, which under those conditions, solutions of fuzzy integral equations are bounded. In [14], the authors gave one of the applications of fuzzy integral for solving fuzzy Fredholm integral equation of the second kind. The iterative techniques are applied to fuzzy Fredholm integral equation of the second kind in [7, 15, 26]. Friedman et al. [16] presented a numerical algorithm to solve fuzzy Fredholm integral equations of the second kind based on successive approximations method. Also, Friedman et al. [17] investigated numerical procedures for solving fuzzy Fredholm integral equation of the second kind using the embedding method. Babolian et al. [4] used the Adomian decomposition method (ADM) to solve fuzzy Fredholm integral equation of the second kind. Abbssbandy et al. [1] obtained the solution of fuzzy Fredholm integral equations of the second kind by using the Nystrom method. In [8], the successive approximations method is used for nonlinear fuzzy Fredholm integral equations. Recently, Bica et al. [9] developed an iterative numerical method to

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solve nonlinear fuzzy Hammerstein–Volterra integral equations with constant delay. In [10], the same method has been applied to the solutions that take values in the set of right-sided fuzzy numbers for a fuzzy Volterra integral equation with constant delay arising in epidemiology.
Recently, the authors used Bernstein polynomials (see [12]), Lagrange interpolation (see [13]), divided and finite differences (see [25]), Legendre wavelets (see [31]) and predictor-corrector procedures (see [32]) for fuzzy integral equations.
Here, we propose a numerical approach for solving linear fuzzy Fredholm integral equations of the second kind and obtain the error estimate in the approximation of the solution of such fuzzy integral equations. The rest of this paper is organized as follows: In Section 2 , we review some elementary concepts of the fuzzy set theory and modulus of continuity. In Section 3, we drive the proposed method to obtain numerical solution of linear fuzzy Fredholm integral equations based on an iterative procedure. The error estimation of the proposed method is obtained in Section 4 in terms of uniform and partial modulus of continuity, proving the convergence of the method. Section 5 includes two numerical examples for the proposed method. Finally, Section 6 gives our concluding remarks.

2 Preliminaries

Definition 2.1 *[2]. A fuzzy number is a function $u : R \to [0,1]$ having the properties:*

(1) *u is normal , that is $\exists\ x_0 \in R$ such that $u(x_0) = 1$,*

(2) *u is fuzzy convex set i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}\ \forall x, y \in R, \lambda \in [0,1]$),*

(3) *u is upper semi-continuous on R,*

(4) *the $\overline{\{\ x \in R : u(x) > 0\}}$ is compact set.*

The set of all fuzzy numbers is denoted by R_F. An alternative definition which yields the same R_F is given by [21].

Definition 2.2 *[17]. An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, which satisfy the following requirements:*

(1) *$\underline{u}(r)$ is a bounded left continuous nondecreasing function over [0,1],*

(2) *$\overline{u}(r)$ is a bounded left continuous nonincreasing function over [0,1],*

(3) $\underline{u}(r) \leq \overline{u}(r)\quad ,\quad 0 \leq r \leq 1.$

The addition and scaler multiplication of fuzzy numbers in R_F are defined as follows:

(1) $(u \oplus v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,

(2) $(\lambda \odot u)(r) = \begin{cases} (\lambda\underline{u}(r), \lambda\overline{u}(r)) & \lambda \geq 0, \\ (\lambda\overline{u}(r), \lambda\underline{u}(r)) & \lambda < 0. \end{cases}$

Definition 2.3 *[3]. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ the quantity $D(u, v) = \sup_{r\in[0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|\,,\ |\overline{u}(r) - \overline{v}(r)|\,\}$ is the distance between u and v.*
The following properties are hold [7]:

(1) *(R_F, D) is a complete metric space,*

(2) $D(u \oplus w, v \oplus w) = D(u, v) \quad \forall\ u, v, w \in R_F$,

(3) $D(k \odot u, k \odot v) = |k|\, D(u, v) \quad \forall\ u, v \in R_F \quad \forall\ k \in R$,

(4) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \quad \forall\ u, v, w, e \in R_F$.

Theorem 2.1 *[2, 8].*

(1) *The pair $(R_F, \oplus)$ is a commutative semigroup with $\tilde{0} = \chi_0$ zero element.*

(2) *For fuzzy numbers which are not crisp, there is no opposite element (that is, $(R_F, \oplus)$ cannot be a group).*

(3) *For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and for any $u \in R_F$, we have*
$(a + b) \odot u = a \odot u \oplus b \odot u$.
For arbitrary $a, b \in R$, this property is not fulfilled.

(4) *For any $\lambda, \mu \in R$ and $u \in R_F$, we have*
$\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot u$.

(5) *For any $\lambda \in R$ and $u, v \in R_F$, we have*
$\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$.

(6) *The function of* $\|.\|_F$: $R_F \to R$ *by* $\|u\|_F = D(u, \tilde{0})$ *has the usual properties of the norm, that is,* $\|u\|_F = 0$ *if and only if* $u = \tilde{o}, \|\lambda \odot u\|_F = |\lambda|\|u\|_F$ *and* $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$.

(7) $|\|u\|_F - \|v\|_F| \leq D(u, v)$ *and* $D(u, v) \leq |u\|_F + \|v\|_F$ *for any* $u, v \in R_F$.

Definition 2.4 *[3]. Let* $f : [a, b] \to R_F$ *be a fuzzy real valued function, then function* $\omega_{[a,b]}(f, .) : R_+ \cup \{0\} \to R_+$ *defined by*

$$\omega_{[a,b]}(f, \delta) = \sup\{D(f(x), f(y))|\ x, y \in [a, b], |x - y| \leq \delta\}, \quad (2.1)$$

where R_+ *is the set of positive real numbers, is called the modulus of continuity of* f *on* $[a, b]$.

Some properties of the modulus of continuity are given in below.

Theorem 2.2 *[7]. The following properties hold:*

(1) $D(f(x), f(y)) \leq \omega_{[a,b]}(f, |x - y|)$ *for any* $x, y \in [a, b]$,

(2) $\omega_{[a,b]}(f, \delta)$ *is increasing function of* δ,

(3) $\omega_{[a,b]}(f, 0) = 0$,

(4) $\omega_{[a,b]}(f, \delta_1 + \delta_2) \leq \omega_{[a,b]}(f, \delta_1) + \omega_{[a,b]}(f, \delta_2)$ *for any* $\delta_1, \delta_2 \geq 0$,

(5) $\omega_{[a,b]}(f, n\delta) \leq n\omega[a, b](f, \delta)$ *for any* $\delta \geq 0$ $n \in N$,

(6) $\omega_{[a,b]}(f, \lambda\delta) \leq (\lambda + 1)\omega_{[a,b]}(f, \delta)$ *for any* $\delta, \lambda \geq 0$,

(7) *If* $[c, d] \subseteq [a, b]$ *then* $\omega_{[c,d]}(f, \delta) \leq \omega_{[a,b]}(f, \delta)$.

Definition 2.5 *[21]. A fuzzy real number valued function* $f : [a, b] \to R_F$ *is said to be continuous in* $x_0 \in [a, b]$, *if for each* $\varepsilon > 0$ *there exist* $\delta > 0$ *such that* $D(f(x), f(x_0)) < \varepsilon$, *whenever* $x \in [a, b]$ *and* $|x - x_0| < \delta$. *We say that* f *is fuzzy continuous on* $[a, b]$ *if* f *is continuous at each* $x_0 \in [a, b]$, *and denote the space of all such functions by* $C_F[a, b]$.

Definition 2.6 *[3]. Let* $f : [a, b] \to R_F$. f *is fuzzy-Riemann integrable to* $I(f) \in R_F$ *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any division* $P = \{[u, v]; \xi\}$ *of* $[a, b]$ *with the norms* $\Delta(p) < \delta$, *we have,*

$$D\left(\sum_P{}^{*}(v - u) \odot f(\xi), I(f)\right) < \varepsilon, \qquad (2.2)$$

where $\sum^*$ *denotes the fuzzy summation. In this case, it is denoted by*

$$I(f) = (FR)\int_a^b f(t)dt.$$

In [20], the authors proved that if $f \in C_F[a, b]$, *its definite integral exists, and also,*

$$\underline{(FR)\int_a^b f(t;r)dt} = \int_a^b \underline{f}(t, r)dt,$$

$$\overline{(FR)\int_a^b f(t;r)dt} = \int_a^b \overline{f}(t, r)dt.$$

Lemma 2.1 *[18]. If* $f, g : [a, b] \subseteq R \to R_F$ *are fuzzy continuous functions, then the function* $F : [a, b] \to R_+$ *by* $F(t) = D(f(t), g(t))$ *is continuous on* $A = [a, b]$, *and*

$$D\left((FR)\int_a^b f(t)dt, (FR)\int_a^b g(t)dt\right) \leq \int_a^b D(f(t), g(t))dt. \quad (2.3)$$

Definition 2.7 *(see [3], [18]) For* $f \in C_F[0, 1]$, *the Bernstein-type fuzzy polynomials for all* $x \in [0, 1]$ *is as follows*

$$B_n^{(F)}(f)(x) = \sum_{k=0}^{n}{}^{*} f\left(\frac{k}{n}\right) \odot p_{n,k}(x)\ ,\ n \in N$$

where $P_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k(1 - x)^{n-x}$ *and* $\sum^*$ *means addition with respect to* $\oplus \in R_F$.
It is obvious that $P_{n,k}(x) \geq 0, \quad \forall \quad x \in [0, 1]$ *and* $P_{n,0}(x), P_{n,1}(x), ..., P_{n,n}(x)$ *are linearly independent algebraic polynomials of degree* $\leq n$ *and*

$$\sum_{k=0}^{n} P_{n,k}(x) = 1$$

Theorem 2.3 *(see [3], [18]). If* $f \in C_F[0, 1]$, *then*

$$\left(B_n^{(F)}(f)(x), f(x)\right) \leq \frac{3}{2}\omega_{[0,1]}(f, \frac{1}{\sqrt{n}}), \quad n \in N, \quad x \in [0, 1]; \quad (2.4)$$

i.e.,

$$\lim_{n\to\infty} D\Big(B_n^{(F)}(f)(x), f(x)\Big) = 0,$$

uniformly with respect to $x \in [0,1]$.

3 Fuzzy integral equations

The linear fuzzy Fredholm integral equations of the second kind is as follows:

$$F(t) = f(t) \oplus \lambda \odot (FR) \int_a^b k(s,t) \odot F(s)ds, \quad (3.5)$$

where $k(s,t)$ is an arbitrary crisp kernel function over the square $a \le s, t \le b$, $\lambda \ge 0$, and $F(t)$ is a fuzzy real valued function.
In [13], the authors presented sufficient conditions for the existence and unique solution of (2.4) as follows:

Theorem 3.1 *[17]. Let $k(s,t)$ be continuous for $a \le s, t \le b, \lambda > 0$, and $f(t)$ a fuzzy continuous of $t, a \le t \le b$. If*

$$\lambda < \frac{1}{M(b-a)},$$

where

$$M = \max_{a \le s,t \le b} |k(s,t)|,$$

then the iterative procedure

$$F_0(t) = f(t),$$

$$F_m(t) = f(t) \oplus \lambda \odot (FR) \int_a^b k(s,t) \odot F_{m-1}(s)ds, \qquad m \ge 1, \quad (3.6)$$

converges to the unique solution of (2.4). Specifically,

$$D^*(F, F_m) \le \frac{L^m}{1-L} D^*(F_0, F_1), \quad (3.7)$$

where $L = \lambda M(b-a)$, and $D^*(f,g) = \sup_{a \le t \le b} D(f(t), g(t))$ denotes the uniform distance between fuzzy-number-valued functions.

Throughout this paper, we consider fuzzy Fredholm integral equation (4.11) with $a = 0, b = 1$. Here, we consider the linear fuzzy Fredholm integral equation (2.4) and uniform partition of the interval $[0,1]$:

$$\Delta : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1, \quad (3.8)$$

with $t_i = ih$ where $h = \frac{1}{n}$, then the following iterative procedure gives the approximate solution of (2.4) in point t

$$y_0(t) = f(t),$$

$$y_m(t) = f(t) \oplus \lambda \odot \sum_{i=0}^{n}{}^* k(t_i, t) \odot y_{m-1}(t_i) \int_0^1 p_{n,i}(s)ds, \qquad m \ge 1, \quad (3.9)$$

where

$$p_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}.$$

Remark 3.1 *Since*

$$\int_0^1 t^n (1-t)^m dx = \frac{m!\, n!}{(m+n+1)!}$$

we get

$$\int_0^1 p_{n,k}(s)ds = \frac{1}{n+1}.$$

4 Error estimation

Now, we obtain error estimate for given linear fuzzy Fredholm integral equations of the second kind (2.4).

Theorem 4.1 *Consider the linear fuzzy Fredholm integral equation (2.4) with continuous kernel $k(s,t)$ having constant sign on $[0,1] \times [0,1]$, f continuous on $[0,1]$ and also, if $L = \lambda M \le 1$ where $M = \max_{s,t \in [0,1]} |k(s,t)|$, then iterative procedure (3.8) converges to the unique solution of (2.4), F, and its error estimate is as follows:*

$$D^*(F, y_m) \le \frac{3L}{2(1-L)} \omega_{[0,1]}(f, \frac{1}{\sqrt{n}}) + \Bigg[\frac{L^{m+1}}{1-L} + \frac{L}{2M(1-L)^2}\Big(3\omega_s(k,h) + 2L\omega_t(k, \frac{1}{\sqrt{n}})\Big)\Bigg] \| f \|_F,$$

where

$$\| f \|_F = \sup_{a \le t \le b} \| f(t) \|_F$$

and

$$\omega_s(k,h) = \sup_{a \le t \le b} \{\sup |k(s_1,t) - k(s_2,t)| : |s_1 - s_2| \le h\},$$

and

$$\omega_t(k,h) = \sup_{a \le s \le b} \{\sup |k(s,t_1) - k(s,t_2)| : |t_1 - t_2| \le h\}.$$

Proof. Since $F_m(t) = f(t) \oplus \lambda \odot (FR)\int_0^1 k(s,t) \odot F_{m-1}(s)ds$, $\forall t \in [0,1]$, we have:

$$D(F_m(t), y_m(t)) =$$

$$D(f(t), f(t)) + \lambda D\Big((FR)\int_0^1 k(s,t)$$

$$\odot F_{m-1}(s)ds, \sum_{i=0}^{n}{}^{*} k(t_i,t) \odot y_{m-1}(t_i) \odot \int_0^1 p_{n,i}(s)ds\Big)$$

$$= \lambda D\Bigg((FR)\int_0^1 k(s,t) \odot F_{m-1}(s)ds, \sum_{i=0}^{n}{}^{*} k(t_i,t)$$

$$\odot y_{m-1}(t_i) \odot \int_0^1 p_{n,i}(s)ds\Bigg)$$

$$\leq \lambda D\Big(\int_0^1 k(s,t) \odot F_{m-1}(s)ds, \int_0^1 \sum_{i=0}^{n}{}^{*} p_{n,i}(s)k(s,t)$$

$$\odot F_{m-1}(t_i)ds\Big) + \lambda \int_0^1 \sum_{i=0}^{n} D\Big(p_{n,i}(s)k(s,t) \odot F_{m-1}(t_i),$$

$$p_{n,i}(s)k(s,t) \odot y_{m-1}(t_i)\Big)ds$$

$$+\lambda \int_0^1 \sum_{i=0}^{n} D\Big(p_{n,i}(s)k(s,t) \odot y_{m-1}(t_i), p_{n,i}(s)k(t_i,t)$$

$$\odot y_{m-1}(t_i)\Big)ds.$$

With suppose that $M = \max_{s,t \in [0,1]} |k(s,t)|$ we have:

$$D(F_m(t), y_m(t)) \leq \lambda M \int_0^1 D\Big(F_{m-1}(s),$$

$$\sum_{i=0}^{n}{}^{*} p_{n,i}(s)F_{m-1}(t_i)\Big)ds + \lambda M \int_0^1 \sum_{i=0}^{n} |p_{n,i}(t)|$$

$$D\Big(F_{m-1}(t_i), y_{m-1}(t_i)\Big)ds$$

$$+\lambda \int_0^1 \sum_{i=0}^{n} |p_{n,i}(t)||k(s,t) - k(t_i,t)|$$

$$D\Big(y_{m-1}(t_i), \tilde{0}\Big)ds.$$

Regarding to Theorem 2.3 and taking into account that $\sum_{i=0}^{n} p_{n,i}(t) = 1$ and $L = \lambda M$ and $\| y_{m-1}\|_F = \sup_{0\leq t\leq 1} D(y_{m-1}(t), \tilde{0})$ we have:

$$D(F_m(t), y_m(t)) \leq \frac{3L}{2}\omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+L.D\Big(F_{m-1}(t_i), y_{m-1}(t_i)\Big)$$

$$+\lambda \| y_{m-1}\|_F \omega_s(k,h).$$

Taking the supremum for $0 \leq t \leq 1$ from above inequality we have:

$$D^*(F_m, y_m) \leq \frac{3L}{2}\omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+L.D^*(F_{m-1}, y_{m-1}) + \frac{L}{M} \| y_{m-1}\|_F \omega_s(k,h),$$

where $\omega_s(k,h)$ is the partial modulus of continuity with respect to s.
Considering inequality (3.9) we rewrite the following inequalities:

$$D^*(F_m, y_m) \leq \frac{3L}{2}\omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+L.D^*(F_{m-1}, y_{m-1}) + \frac{L}{M} \| y_{m-1}\|_F \omega_s(k,h),$$

$$D^*(F_{m-1}, y_{m-1}) \leq \frac{3L}{2}\omega_{[0,1]}(F_{m-2}, \frac{1}{\sqrt{n}})$$

$$+L.D^*(F_{m-2}, y_{m-2}) + \frac{L}{M} \| y_{m-2}\|_F \omega_s(k,h),$$

$$\vdots$$

$$D^*(F_1, y_1) \leq \frac{3L}{2}\omega_{[0,1]}(f, \frac{1}{\sqrt{n}})$$

$$+ \frac{L}{M} \| f\|_F \omega_s(k,h). \tag{4.10}$$

Multiplying the above inequalities by $1, L, L^2 \ldots, L^{m-1}$, respectively and summing them we have

$$D^*(F_m, y_m) \leq \frac{3L}{2}\Big(\omega_{[0,1]}(F_{m-1}, \frac{1}{\sqrt{n}})$$

$$+L\omega_{[0,1]}(F_{m-2}, \frac{1}{\sqrt{n}}) + \ldots + L^{m-1}\omega_{[0,1]}(f, \frac{1}{\sqrt{n}})\Big)$$

$$+\frac{L}{M}\omega_s(k,h)\Big(\| y_{m-1}\|_F + L \| y_{m-2}\|_F$$

$$+ \ldots + L^{m-1} \| f\|_F\Big). \tag{4.11}$$

Taking norm from (3.8) and considering the Remark 3.1 we obtain

$$\|y_i(t)\|_F \leq \|f(t)\|_F + L\|y_{i-1}(t)\|_F, \quad 1 \leq i \leq m.$$

Taking supremum the above inequality and then by successive substitutions on the obtained inequality and taking into account $L = \lambda M \leq 1$ we have:

$$\| y_i\|_F \leq \frac{1}{1-L} \| f\|_F \qquad 1 \leq i \leq m-1. \tag{4.12}$$

Also, according to the proof of Theorem (11) in [7] we have the following inequalities:

$$\omega_{[0,1]}(F_i, h) \leq \omega_{[0,1]}(f, h) + \frac{L}{M}\omega_t(k, h) \| F_{i-1}\|_F$$
$$1 \leq i \leq m-1. \tag{4.13}$$

Also, taking norm from (3.5) we get

$$\|F_i(t)\|_F \leq \|f(t)\|_F + L\|F_{i-1}(t)\|_F, \quad 1 \leq i \leq m.$$

Taking supremum the above inequality and then by successive substitutions on the obtained inequality and taking into account $L = \lambda M \leq 1$ we have:

$$\| F_i\|_F \leq \frac{1}{1-L} \| f\|_F \qquad 1 \leq i \leq m-2. \tag{4.14}$$

Thus, substituting (4.13) into (4.14) we obtain:

$$\omega_{[0,1]}(F_i, h) \leq \omega_{[0,1]}(f, h) + \frac{L}{M(1-L)} \| f\|_F \omega_t(k, h)$$
$$1 \leq i \leq m-1. \tag{4.15}$$

Finally, by substituting (4.12) and (4.15) into (4.11), we obtain the following inequality:

$$D^*(F_m, y_m) \leq \frac{3L}{2(1-L)}\omega_{[0,1]}(f, \frac{1}{\sqrt{n}})$$
$$+ \frac{L\| f\|_F}{2M(1-L)^2}\Big(3\omega_s(k, h) + 2L\omega_t(k, \frac{1}{\sqrt{n}})\Big).$$

Considering the inequality (3.6) we obtain

$$D^*(F, y_m) \leq D^*(F, F_m) + D^*(F_m, y_m)$$
$$\leq \frac{L^m}{1-L}D^*(F_1, F_0) + D^*(F_m, y_m).$$

Since

$$D(F_1(t), F_0(t)) = D\bigg(f(t) \oplus \lambda(FR)\int_0^1 k(s, t)$$
$$\odot F_0(s)ds, F_0(t)\bigg)$$
$$\leq \lambda D\bigg((FR)\int_0^1 k(s, t) \odot F_0(s)ds, \tilde{0}\bigg),$$

we conclude that

$$D^*(F_1, F_0) \leq L \sup_{0\leq s\leq 1} D(F_0(s), \tilde{0}) \leq L \| f\|_F.$$

Hence, we have:

$$D^*(F, y_m) \leq \frac{3L}{2(1-L)}\omega_{[0,1]}(f, \frac{1}{\sqrt{n}})$$
$$+ \Bigg[\frac{L^{m+1}}{1-L} + \frac{L}{2M(1-L)^2}\Big(3\omega_s(k, h)$$
$$+2L\omega_t(k, \frac{1}{\sqrt{n}})\Big)\Bigg] \| f\|_F. \quad \square$$

Remark 4.1 *Since $L < 1$, it follows that*

$$\lim_{m\to\infty} L^{m+1} = 0.$$

In addition,

$$\lim_{h\to 0} \omega_{[0,1]}(f, h) = 0,$$
$$\lim_{h\to 0} \omega_s(k, h) = 0,$$
$$\lim_{h\to 0} \omega_t(k, h) = 0.$$

So,

$$\lim_{m\to\infty, h\to 0} D^*(F, y_m) = 0$$

that shows the convergence of the method.

5 Numerical examples

To illustrate the efficiency of the presented method in the previous section, we give two examples. Also, we compare the numerical solution obtained by using the proposed method with the exact solution.

Example 5.1 *Consider the following linear fuzzy Fredholm integral equations of the second kind:*

$$F(t) = \Big(e^t - \frac{0.2(e^{t+1}-1)}{t+1}\Big)\odot\gamma\oplus 0.2\odot\int_0^1 e^{st}\odot F(s)ds,$$

where $\gamma = (r, 2-r)$.
The exact solution in this case is given by

$$F(t) = e^t \odot \gamma.$$

To compare the error with $n = 10$, $m = 12$ and $n = 100$, $m = 9$, see Table 1.

Table 1: The accuracy on the level sets for Example 5.1 in $t = 0.5$

	n=10, m=12		n=100, m=9	
r-level	$\lvert\underline{F} - \underline{y}_m\rvert$	$\lvert\overline{F} - \overline{y}_m\rvert$	$\lvert\underline{F} - \underline{y}_m\rvert$	$\lvert\overline{F} - \overline{y}_m\rvert$
0.00	0.000000	0.222423	0.000000	0.027002
0.25	0.027803	0.194620	0.003375	0.023627
0.50	0.055606	0.166817	0.006751	0.020252
0.75	0.083409	0.139014	0.010126	0.016876
1.00	0.111211	0.111211	0.013501	0.013501

Table 2: The accuracy on the level sets for Example 5.2 in $t = 1.5$

	n=10, m=9		n=100, m=9	
r-level	$\lvert\underline{F} - \underline{y}_m\rvert$	$\lvert\overline{F} - \overline{y}_m\rvert$	$\lvert\underline{F} - \underline{y}_m\rvert$	$\lvert\overline{F} - \overline{y}_m\rvert$
0.00	0.000000	0.163788	0.000000	0.019717
0.25	0.020473	0.143314	0.002465	0.017253
0.50	0.040947	0.122841	0.004930	0.014788
0.75	0.061420	0.102367	0.007394	0.012323
1.00	0.081894	0.081894	0.009859	0.009859

Example 5.2 *Consider the following linear fuzzy Fredholm integral equations of the second kind:*

$$\underline{f}(t,r) = rt - \frac{3}{26}rt^2 - \frac{3}{52}r$$

$$\overline{f}(t,r) = 2t - rt - \frac{3}{13}t^2 - \frac{3}{26} + \frac{3}{26}rt^2 + \frac{3}{52}r$$

and kernel

$$k(s,t) = \frac{(s^2 + t^2 - 2)}{13}$$

and $a = 1, b = 2$. The exact solution in this case is given by

$$\underline{F}(t,r) = rt$$

$$\overline{F}(t,r) = (2 - r)t.$$

Since the the fuzzy Fredholm integral equations with this method is defined only for $t \in [0,1]$, so the transformation $z = (b-a)t+a$ must be done. To compare the error with $n = 10,\ m = 9$ and $n = 100,\ m = 9$, see Table 2.

6 Conclusions

In this paper, we proposed a numerical method to solve linear fuzzy Fredholm integral equations of the second kind based on iterative procedure using fuzzy Bernstein polynomials. Also, we have presented the error estimation for approximate solution of linear fuzzy Fredholm integral equations of the second kind, in terms of modulus of continuity. Illustrative numerical examples are included to demonstrate the accuracy of the proposed method. In the above presented numerical examples we see that the proposed method well perform for linear fuzzy integral equations and the convergence result, Theorem 4.1, is confirmed.

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