Planarity of Intersection Graph of submodules of a Module

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Abstract

Let \( R \) be a commutative ring with identity and \( M \) be a unitary \( R \)-module. The intersection graph of an \( R \)-module \( M \), denoted by \( \Gamma(M) \), is a simple graph whose vertices are all non-trivial submodules of \( M \) and two distinct vertices \( N_1 \) and \( N_2 \) are adjacent if and only if \( N_1 \cap N_2 \neq 0 \). In this article, we investigate the concept of a planar intersection graph and maximal submodules of an \( R \)-module. In particular, we show that if \( \Gamma(M) \) is a planar graph, then \( M \cong M_1 \oplus M_2 \) for a multiplication \( R \)-module \( M \) with \( |\text{Max}(M)| \neq 1 \).

Keywords: Interval methods; Multiplication modules; Planar Graph; Module Theory; Torsion Graphs.

1 Introduction

It is well known that graph is a very useful tool to model problems originated in all most all areas of our life. In this article, we concentrate our discussion on intersection graphs. Let \( S = \{ S_i : i \in I \} \) be an arbitrary family of sets. The intersection graph \( \Gamma(S) \) of \( S \) is the graph whose vertices are \( S_i, i \in I \) and there is an edge between two distinct vertices \( S_i \) and \( S_j \) if and only if \( S_i \cap S_j \neq \emptyset \). It is more interesting to study the intersection graphs \( \Gamma(S) \) when the elements of \( S \) have an algebraic structure. These studies allow us to obtain characterization and representation of the classes of algebraic structure in terms of graphs and vice versa.

Let \( R \) be a commutative ring with identity and \( M \) be a unitary \( R \)-module. The idea of the intersection graph of semigroups was introduced by Bosak in [5]. Inspired by his work, Csókány and Pollák in [8], studied the graph of subgroups of a finite group. The intersection graph of ideals of a ring, was considered by Chakrabarty, Ghosh, Mukherjee and Sen in [7]. Recently, Akbari, Tavallaee and Khaiaashi in [1], introduced and investigated the intersection graph of submodules of a module.

In this paper, we investigate the concept of intersection graph of a module. The intersection graph of an \( R \)-module \( M \), denoted by \( \Gamma(M) \), is defined to be the undirected simple graph whose vertices are all non-trivial submodules of \( M \) and two distinct vertices are adjacent if and only if the corresponding submodules of \( M \) have nonzero intersection. This study helps to illuminate the structure of \( M \), for example, if \( \Gamma(M) \) is a planar graph, then \( M \) is both Noetherian and Artinian.

Recall that a simple graph is finite if its vertices set is finite, and we use the symbol \( |\Gamma(M)| \) to denote the number of vertices in graph \( \Gamma(M) \). Also, a graph \( G \) is connected if there is a path between any two distinct vertices. The distance, \( d(x, y) \) between connected vertices \( x, y \) is the length of the shortest path from \( x \) to \( y \), \( d(x, y) = \infty \) if there is no such path). An isolated vertex is a
vertex that has no edges incident to it. A complete \( r \)-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes \( m \) and \( n \) is denoted by \( K_{m,n} \). A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use \( K_n \) for the complete graph with \( n \) vertices. The complement \( \overline{G} \) of \( G \) is the graph with vertex set \( V(\overline{G}) = V(G) \), and \( E(\overline{G}) = \{ uv : uv \notin E(G) \} \). The complement of a complete graph is the null graph. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A remarkably simple characterization of planar graphs was given by Kuratowski in [5], p.153. Kuratowski’s Theorem says that a graph is planar if and only if it contains no subdivision of \( K_5 \) or \( K_{3,3} \).

An \( R \)-module \( M \) is a multiplication module if for every \( R \)-submodule \( K \) of \( M \) there is an ideal \( I \) of \( R \) such that \( K = IM \). Note that \( I \subseteq [N : M] \), hence \( N = IM \subseteq [N : M]M \subseteq N \). So \( N = [N : M]M \). An \( R \)-module \( M \) is called a cancellation module if \( IM = JIM \) for any ideals \( I \) and \( J \) of \( R \) implies that \( I = J \). Also, an \( R \)-module \( M \) is a weak-cancellation module if \( IM = JIM \) for any ideals \( I \) and \( J \) of \( R \) implies that \( I + Ann(M) = J + Ann(M) \). Finitely generated multiplication modules are weak cancellation, Theorem 3 [2]. Let \( P \) be a maximal ideal of \( R \). An \( R \)-module \( M \) is called \( P \)-torsion if for each \( m \in M \) there exists \( p \in P \) such that \((1 - p)m = 0 \). On the other hand, \( M \) is called \( P \)-cyclic if there exists \( x \in M \) and \( q \in P \) such that \((1 - q)M \subseteq Rx \). Theorem 1.2 [6] showed that an \( R \)-module \( M \) is multiplication if and only if for every maximal ideal \( P \) of \( R \) either \( M \) is \( P \)-torsion or \( P \)-cyclic.

In this paper, we study the number of maximal and minimal prime submodule of multiplication modules. It is shown that if \( \Gamma(M) \) is a planar graph, then \(|Max(M)| \leq 4 \) and \(|Min(M)| \leq 4 \). Also, we show that, if \( M \) is a multiplication \( R \)-module with \(|Max(M)| \neq 1 \) and \( \Gamma(M) \) is a planar graph, then \( M \cong M_1 \oplus M_2 \).

Throughout the paper, \( Max(M) \) is a set of the maximal submodules \( H \) of \( M \), we use symbol \( |Max(M)| \) to denote the number of maximal submodule of \( M \). As a consequence of Theorem 2.5 [6], for any non-zero multiplication \( R \)-module \( Max(M) \neq \emptyset \). Also, \( Min(M) \) is a set of the minimal prime submodules \( N \) of \( M \). let \( J(R) \) be the Jacobson radical of \( R \) and

\[
J(M) := \cap_{H \in Max(M)} H.
\]

We follow standard notation and terminology from graph theory [5] and module theory [3].

2 Planar intersection graph

This section is concerned with some basic and important results in the theory of planar torsion graphs over a module.

Lemma 2.1 Let \( M \) be an \( R \)-module. If \( \Gamma(M) \) is a planar graph, then \( M \) is both Noetherian and Artinian.

Proof. Let \( N_1 \subseteq N_2 \subseteq N_3 \subseteq N_4 \subseteq N_5 \subseteq \ldots \) be a chine of nontrivial proper submodule of \( M \). Then vertices \( N_i \), \( 1 \leq i \leq 5 \) form \( K_5 \) as an induced subgraph, which is a contradiction. So every chain of nontrivial proper submodule of \( M \) is stationary. Therefore \( M \) is both Noetherian and Artinian.

Lemma 2.2 Let \( M \) be a multiplication \( R \)-module and \( N \) be a prime submodule of \( M \). If \( \bigcap_{i=1}^{n} N_i \subseteq N \), where \( N_i \) be a submodule of \( M \), then there is \( 1 \leq i \leq n \) such that \( N_i \subseteq N \).

Proof. Let \( \bigcap_{i=1}^{n} N_i \subseteq N \), where \( N_i \) be a submodule of \( M \). Then \( [N_1 : M][N_2 : M] \ldots [N_n : M]M \subseteq N \). Since \( N \) is a prime submodule of \( M \), there is \( 1 \leq i \leq n \) such that \( [N_i : M] \subseteq [N : M] \). Therefore \( N_i \subseteq N \).

Lemma 2.3 Let \( M \) be a \( Q \)-cyclic \( R \)-module for all maximal ideal \( Q \) of \( R \). Then \( [N : M] \) is a prime ideal of \( R \) for any proper submodule \( N \) of \( M \) if and only if \( [N : M]M \) is a prime submodule of \( M \).

Proof. Let \( [N : M] \) be a prime ideal of \( R \). Clearly \( [N : M]M \) is a proper submodule of \( M \). Suppose \( ax \in [N : M]M \) such that \( a \not\in [N : M] \), for some \( a \in R \) and \( x \in M \). Let \( k = \{ r \in R | rx \in [N : M]M \} \). If \( k \neq R \), then there is a maximal ideal \( Q \) of \( R \) such that \( k \subseteq Q \). Since \( M \) is a \( Q \)-cyclic \( R \)-module, \((1 - q)M \subseteq Rm \) for some \( q \in Q \) and \( m \in M \). Hence \((1 - q)ax \in (1 - q)[N : M]M \subseteq [N : M]m \). So \((1 - q)x = sm \) and \((1 - q)ax = \alpha m \) for some \( s \in R \) and \( \alpha \in [N : M] \). Thus \((\alpha - sa)m = 0 \). It is clear that \((1 -
Therefore vertices $H$ imply that $H$. Proposition 2.1

Let $M$ be a prime submodule of $M$. Hence $(1 - q) = \kappa \subseteq [N : M]$. Therefore $[N : M]$ is a prime submodule of $M$.

Conversely, let $N$ be a prime submodule of $M$. Thus $[N : M]$ is a proper ideal of $R$. Suppose $st \in [N : M]$. So $sM \subseteq N$ or $tM \subseteq N$. Therefore $[N : M]$ is a prime ideal of $R$.

**Theorem 2.1** Let $M$ be a $Q$-cyclic $R$-module for all maximal ideal $Q$ of $R$. If $\Gamma(M)$ is a planar graph, then $|\text{Min}(M)| \leq 3$.

**Proof.** Let $\Gamma(M)$ be a planar graph. Suppose $|\text{Min}(M)| \geq 4$ and $N_1, N_2, \ldots, N_4$ be distinct minimal submodules of $M$, such that $N_1 \cap N_2 \cap N_3 = 0$. Then $[N_1 : M] = [N_2 : M] = [N_3 : M] = [N_4 : M]$. Hence $[N_1 : M][N_2 : M][N_3 : M] \subseteq [N_4 : M]$. It is clear that $[N_1 : M]$ is a prime ideal of $R$. So $[N_i : M]M \subseteq [N_i : M]M \subseteq N$, for some $1 \leq i \leq 3$. By Lemma 2.3, $[N : M]M$ is a prime submodule of $M$. Also, since $N$ is a minimal prime submodule of $M$, $[N : M]M = N$. Therefore $N_i = N$ for some $1 \leq i \leq 3$, which is a contradiction. Hence $N_1 \cap N_2 \cap N_3 = 0$. Therefore, vertices $N_1 \cap N_2, N_1 \cap N_3, N_2 \cap N_3, N_1, N_2$ and $N_3$ form $K_5$ as an induced subgraph, which is a contradiction. Consequently $|\text{Min}(M)| \leq 3$.

**Corollary 2.1** Let $M$ be a multiplication $R$-module. If $\Gamma(M)$ is a planar graph, then $\bigcap_{N \in \text{Min}(M)} N = 0$.

**Proposition 2.1** Let $M$ be a multiplication $R$-module. If $\Gamma(M)$ is a planar graph, then $1 \leq |\text{Max}(M)| \leq 3$.

**Proof.** Let $\Gamma(M)$ be a planar graph. Suppose $|\text{Max}(M)| \geq 4$ and $H_1, H_2, \ldots, H_4$ be distinct maximal submodules of $M$, such that $H_1 \cap H_2 \cap H_3 = 0$. Then $H_1 \cap H_2 \cap H_3 \subseteq H_4$. Since every maximal submodule of multiplication modules is prime, by Lemma 2.2, $H_i \subseteq H_4$, for some $1 \leq i \leq 3$. But $H_4$ is a maximal submodule of $M$ implies that $H_4 = H_4$ for some $1 \leq i \leq 3$, which is a contradiction, hence $H_1 \cap H_2 \cap H_3 = 0$. Therefore, vertices $H_1 \cap H_2, H_1 \cap H_3, H_2 \cap H_3, H_1, H_2$ and $H_3$ form $K_6$ as an induced subgraph, which is a contradiction. Consequently $1 \leq |\text{Max}(M)| \leq 3$.

**Corollary 2.2** Let $M$ be a multiplication $R$-module. If $\Gamma(M)$ is a planar graph, then $J(M) = 0$.

**Proposition 2.2** Let $M = M_1 \times M_2$ be an $R$-module. Then $\Gamma(M)$ is planar if and only if $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty and another is null.

**Proof.** Let $\Gamma(M)$ be a planar graph. Suppose that $\Gamma(M_1)$ and $\Gamma(M_2)$ are not empty. So there exist nontrivial proper submodules $N_1$ of $M_1$ and $N_2$ of $M_2$. Therefore $0 \times N_2, 0 \times M_2, N_1 \times M_2, N_1 \times N_2$ and $M_1 \times M_2$ form $K_5$ as an induced subgraph, which is a contradiction. Hence one of $\Gamma(M_1)$ or $\Gamma(M_2)$ is empty. Let $\Gamma(M_2)$ be empty. Now we show that $\Gamma(M_1)$ is null. If $N_1$ is a proper nontrivial submodule of $M_1$ such that it is adjacent to $H_1$ for some $H_1 \in V(\Gamma(M_1))$, then $N_1 \cap H_1 \neq 0$. So $N_1 \times 0, H_1 \times 0, M_1 \times 0, N_1 \times M_2$ and $H_1 \times M_2$ form $K_5$ as an induced subgraph, which is a contradiction. This contradiction implies that $\Gamma(M_1)$ is null.

**Corollary 2.3** $\Gamma(M_1 \times M_2 \times M_3)$ is planar if and only if $M_i$ is a simple $R_i$-module for $i \in \{1, 2, 3\}$.

**Proposition 2.3** Let $M$ be a multiplication $R$-module with $|\text{Max}(M)| = 3$. If $\Gamma(M)$ is planar, then $M \cong M_1 \oplus M_2$ where $M_1$ and $M_2$ are simple.

**Proof.** Let $|\text{Max}(M)| = 3$ and $M_i, 1 \leq i \leq 3$ be distinct maximal submodules of $M$. By Corollary 2.2, $H_1 \cap H_2 \cap H_3 = 0$. If $H_2 \cap H_3 = 0$, then $H_2 \cap H_3 \subseteq H_1$ and by Lemma 2.2, $H_1 \cap H_2$ or $H_1 \cap H_3$, which is a contradiction. Hence $M = H_1 \oplus H_2 \cap H_3$. By Proposition 2.2, one of $\Gamma(H_1)$ or $\Gamma(H_2 \cap H_3)$ is null another is empty. Suppose that $\Gamma(H_1)$ be null. If $H_1$ is not a simple submodule of $M$. Then there is a nontrivial submodule $N_1$ of $H_1$ such that $N_1$ is adjacent to $N_1 \cap H_2 \cap H_3$. So $\Gamma(H_1)$ is not null, which is a contradiction. Thus $H_1$ and $H_2 \cap H_3$ are simple.

**Lemma 2.4** Let $M$ be a faithfully finitely generated multiplication $R$-module. Then $J(R)M = J(M)$.
Proof. Let $M$ be a faithful finitely generated multiplication $R$-module and $H$ be a maximal submodule of $M$. By Theorem 3.1 of [6], $hM \neq M$ for all maximal ideal $h$ of $M$. Also, by Theorem 2.5 of [6], $H = hM$ for some maximal ideal $h$ of $M$. On the other hand by Theorem 1.6 of [6], $J(M) = \bigcap_{h \in \text{Max}(R)} hM = \bigcap_{h \in \text{Max}(R)} (hM) = (\bigcap_{h \in \text{Max}(R)} h)M = J(R)M$

Theorem 2.2 Let $M$ be a faithful multiplication $R$-module with $|\text{Max}(M)| = 2$. Then $\Gamma(M)$ is a planar graph if and only if $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ such that $\Gamma([H_1 : M]^4M)$ or $\Gamma([H_1 : M]^4M)$ is empty another is null, where $H_1, H_2$ are maximal submodule of $M$.

Proof. Let $H_1$ and $H_2$ be distinct maximal submodules of $M$. Suppose that $[H_1 : M]^4M + [H_2 : M]^4M \neq M$. By Theorem 2.5 of [6], there is a maximal submodule $H$ of $M$ such that $[H_1 : M]^4M + [H_2 : M]^4M \subseteq H$. Since $|\text{Max}(M)| = 2$, we have $H = H_1$ or $H = H_2$. It follows that $[H_1 : M]^4M \subseteq H_2$ or $[H_2 : M]^4M \subseteq H_1$. Thus $H_1 = H_2$, which is a contradiction. So $M = [H_1 : M]^4M + [H_2 : M]^4M$. Assume $[H_1 : M]^4M \cap [H_2 : M]^4M \neq 0$. Hence $H_1 \cap H_2 \neq 0$. On the other hand By Theorem 1.6 [6], $[H_1 : M]^4M \cap [H_2 : M]^4M = ([H_1 : M]^4 \cap [H_2 : M]^4)M$, for all positive integer $i$. Since $M$ is a cyclic faithful multiplication module, by Lemma 2.4, we have $J(R)M = J(M)$. Now Nakayama’s lemma follows that $([H_1 : M]^4 \cap [H_2 : M]^4)M \subset \ldots \subset ([H_1 : M] \cap [H_2 : M])M \subset H_1$. Hence $\Gamma(M)$ contains an induced subgraph $K_5$, which is a contradiction. Therefore $[H_1 : M]^4M \cap [H_2 : M]^4M = 0$. Consequently $M \cong [H_1 : M]^4M \oplus [H_2 : M]^4M$ and by Proposition 2.2, the result follows.

Proposition 2.4 Let $M$ be a multiplication $R$-module with $|\text{Max}(M)| = 1$. If $\Gamma(M)$ is a planar graph, then $|M| \leq 5$ or $[H : M]^5M = 0$ where $H$ is a maximal submodule of $M$.

Proof. Suppose $M$ be a faithful multiplication $R$-module. If $\Gamma(M)$ is a planar graph, then by Lemma 2.1, $M$ is finitely generated and by Lemma 2.4, $R$ is a local ring with unique maximal ideal $[H : M]$. By Nakayama’s lemma, we have $[H : M]^iM \neq [H : M]^jM$ for all positive integer $i \neq j$. Since $\Gamma(M)$ is a planar graph, then $[H : M]^5M = 0$. If $M$ is not faithful, then $\Gamma(M)$ is a complete graph. Hence $|M| \leq 5$.

Now we obtain the central results of this section.

Corollary 2.4 Let $M$ be a multiplication $R$-module with $|\text{Max}(M)| \neq 1$. If $\Gamma(M)$ is a planar graph, then $M \cong M_1 \oplus M_2$.

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References


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